Palestine Journal of Mathematics

Vol 14(3)(2025) , 772–807

Knowledge-Based Accuracy Measures in Standard Complex-Valued Fuzzy Environments for Pattern Recognition and Clustering

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Communicated by Nasreen Kausar

MSC 2010 Classifications: 03B52; 94A17; 62H30; 68T10.

Keywords and phrases: Complex Fuzzy set, Knowledge measure, Accuracy measure, Pattern Recognition, Clustering.

Abstract: Complex Fuzzy sets in the polar coordinates are represented by complex membership degrees. The complex membership degree contains a magnitude function and a crisp phase function. Compared to an ordinary fuzzy set, the complex value can be more informative and enhance reasoning. Since the phase function in any Complex Fuzzy set takes only crisp values, and the complement of the phase function does not affect the phase function, a standard Complex-valued Fuzzy set is more informative and hence proposed in this manuscript. In this manuscript, a novel standard complex-valued fuzzy knowledge measure is proposed and is checked for its credibility. The proposed measure's main properties are explored with their proofs. The proposed measure is compared with some previously known measures in the form of structured linguistic and criteria weight computation. Besides this, Novel Accuracy, Similarity, and Distance measures are deducted from the proposed measure and checked for credibility. These deducted measures have been successfully applied to handle Clustering and Pattern recognition issues. The derived measures are compared with previously known ones in solving pattern recognition problems.

1 Introduction

Numerous theoretical frameworks of real-world phenomena that just offer the option of choosing between falsity & truth are not able to capture the true reality of the issues. The complexities within the framework are the cause of this, which is why a system needs to be created to deal with the framework's ill-defined scenarios. These scenarios can be handled with two alternatives. They are- The 1^{st} alternate is to determine the numerical solutions to the issues, & the 2^{nd} is to build a model numerically. In both scenarios, a numerical solution is obtained. In 2^{nd} alternate, there exists a fuzzy set [1, 2] theory which deals with the ambiguity and inaccuracies of all the real-world issues. It includes the theory of Interval-valued fuzzy ($\mathfrak{I}^V$) set [3–5], Rough sets [6], Intuitionistic fuzzy ($\mathfrak{I}$) sets [7], Soft sets [8], Complex Fuzzy ($\mathcal{C}_{\mathcal{F}}$) sets [9], Fuzzy Soft sets [10], and Pythagorean fuzzy ($\mathfrak{P}$) sets [11]. Each theory has distinct characteristics, positive effects, and restrictions. These theories are applied to uncertain contexts, including recognition of patterns, medical diagnosis, engineering, social media, computer science, and decision-making ($\mathfrak{DM}$) issues. The (3, 2)-fuzzy set's uses were given in optimal choices and topology by Ibrahim et al. [12]. A $\mathfrak{DM}$ approach was created by Turk et al. [13] to choose an ideal site for electric charge in interval type-2 fuzzy context. Bulut & Ozcan [14] presented a novel approach for battery energy storage systems evaluation in the fuzzy context. In interval type-2 trapezoid fuzzy context, Meng et al. [15] introduced a unique $\mathfrak{DM}$ approach. The assessment of Sponge City development was carried out by this methodology. Mishra et al. [16] expanded the use of the ARAS approach to manage complex hesitant fuzzy $\mathfrak{DM}$ issues. Singh & Kumar [17] extended the R-norm entropy to fuzzy knowledge measure and an accuracy-based VIKOR method is proposed. Aggregation operators are generalized for Fermatean normal vague set to treat cancer patients [18]. A Diophantine spherical vague MADM method for micro-technology robots is proposed by Palanikumar et al. [19]. Many researchers [20–28] extended the concept of fuzzy set to higher extensions.

The idea of $\mathfrak{I}^V$-sets, in which fuzzy values are intervals, was first out by Gehrke et al. [29]. There are several uses for $\mathfrak{I}^V$-sets in various scientific domains. The $\mathfrak{I}^V$-distance measures were used in medical diagnostics by Dutta [30]. Pkekala et al. [31] proposed $\mathfrak{I}^V$-similarity & inclusion measures. Jeevaraj [32] introduced an interval-valued Fermatean fuzzy $\mathfrak{DM}$ system. The idea of convexity of $\mathfrak{I}^V$-sets was established and used to $\mathfrak{DM}$ issues by Huidobro et al. [33]. Based on the notion of interval embedding & aggregation operators, Bouchet et al. [34] proposed interval-valued embedding. Depending upon the cumulative prospect theory, a $\mathfrak{DM}$ method was suggested by Chai et al. [35] in $\mathfrak{I}^V$ & $\mathfrak{I}$-context to handle supplier selection issues. Singh & Kumar [36] proposed $\mathfrak{I}^V$-knowledge measure and used it in Clustering. The ranking of the doctors in online healthcare was carried out by a $\mathfrak{DM}$ method proposed by Zhang & Hu [37] in multi $\mathfrak{I}^V$-context. In the diophantine fuzzy environment, Jayakumar et al. [38] proposed MARCOS method is used to solve multicriteria group $\mathfrak{DM}$. A $\mathfrak{DM}$ method for solving real-life problems is proposed by Hasan et al. [39] and Surya et al. [40].

Atanassov [41] was the first person who gave the concept of $\mathfrak{I}$-sets. Many scientific domains find great applications for the $\mathfrak{I}$-set context. The $\mathfrak{I}$-$\mathfrak{DM}$ approach was introduced by Xue & Deng [42]. The two potential solutions to the issue of creating a shadowing set from an $\mathfrak{I}$-set were examined by Yang & Yao [43]. Four categories of $\mathfrak{I}$-similarities were identified by Duan & Li [44] using the implication operator and matching logical metric spaces. Wu et al. [45] proposed a novel $\mathfrak{I}$-knowledge measure to handle $\mathfrak{DM}$ issues. A three-way approximation of an $\mathfrak{I}$-set was proposed by Yang et al. [46]. An $\mathfrak{I}$-dissimilarity measure was proposed by Gohain et al. [47] and its applications were given in Clustering as well as pattern recognition issues. Singh & Kumar [48] developed a novel $\mathfrak{I}$-knowledge measure based on R-norm entropy and presented a $\mathfrak{DM}$ model based on it. A Moderator Intuitionistic Fuzzy TOPSIS approach was given by Joshi et al. [49] to select a source for Renewable Energy. Ngan [50] proposed an extension framework for $\mathfrak{I}$-sets. A circular $\mathfrak{I}$-entropy & similarity measure was proposed by Alreshidi et al. [51] and a case study for the selection of the site for an epidemic hospital was carried out.

The $\mathfrak{P}$-set was first proposed by Yager & Abbasov [52] as an extension of the $\mathfrak{I}$-set. In uncertain issues, these sets have been applied extensively. For $\mathfrak{P}$-sets, Ejegwa & Awolola [53] presented a few new measures of distance. They spoke about how they may be used for pattern recognition issues. Lin et al. [54] proposed some $\mathfrak{P}$-directional correlation coefficient measures and spoke about how they may be used in clustering and medical diagnosis. Farhadinia [55] introduced similarity measures based on $\mathfrak{P}$-distance measure and proposed s & t-norm's concepts. He demonstrated their usefulness in pattern recognition & medical diagnosis. A pandemic hospital $\mathfrak{DM}$ issue was solved by Boyaci & Sisman [56] in $\mathfrak{P}$-context for Atakum city. Ejegwa et al. [57] suggested a 3-way method for calculating $\mathfrak{P}$-correlation coefficient depending upon variance and covariance. Kumar & Chen [58] proposed a $\mathfrak{P}$-entropy measure and $\mathfrak{P}$-mean aggregation operator for handling $\mathfrak{DM}$ issues. Alkan & Kahraman [59] proposed decomposed $\mathfrak{P}$-sets to extend the CODAS method and used it in strategy selection issues. Jaccard index-based similarity measure was proposed by Hussain et al. [60] and its applications were given in solving clustering issues. Aggregation Operators are provided by Palanikumar et al. [21] for Solving medical diagnosis issues.

All these extensions of fuzzy sets cannot regulate imprecise, incomplete, and inconsistent data. These extensions are unable to explain two-dimensional occurrences, although these extensions are highly helpful in many ambiguous situations. To address this shortcoming, Ramot et al. [9] presented the idea of $\mathcal{C}_{\mathcal{F}}$-sets. The membership function belongs to the interval $[0, 1]$ in the case of a fuzzy set. But it lies within a unit circle in a complex plane (Ramot et al. [9]), in the case of $\mathcal{C}_{\mathcal{F}}$-set. The membership function in a $\mathcal{C}_{\mathcal{F}}$-set contains two functions. They are the magnitude function & Phase function. A key component in characterizing the features of the model in $\mathcal{C}_{\mathcal{F}}$-context is the phase function. This is the main difference in the $\mathcal{C}_{\mathcal{F}}$-models and other previously known models. $\mathcal{C}_{\mathcal{F}}$-sets are better at handling intuitive and ambiguous data, which are common in time-periodic phenomena due to their potential for capturing two-dimensional phenomena. Applications like advanced control systems and periodic event prediction rely heavily on $\mathcal{C}_{\mathcal{F}}$-sets and their classes. Since the Fourier transform has many applications in a variety of fields, including optics, geology, astronomy, communication, and signals, etc., a $\mathcal{C}_{\mathcal{F}}$-set is the same as a Fourier transform with a unit circle range. As a result, a $\mathcal{C}_{\mathcal{F}}$-set may be applied to the

Fourier transform and other models. Several additional real-world events are uncertain and cannot be represented with one-dimensional variables. $\mathcal{C}_{\mathcal{F}}$-sets have useful uses in recurring events and advanced control systems. Jia et al. [61] explored the applicability of a novel Z-number solution in $\mathcal{C}_{\mathcal{F}}$-context and $\mathfrak{DM}$ issues. The $\mathcal{C}_{\mathcal{F}}$-distance measure was introduced by Song et al. [62] and used to $\mathfrak{DM}$ scenarios. Complex valued migration was introduced into $\mathcal{C}_{\mathcal{F}}$-operations by Xu et al. [63]. Zeeshan et al. [64] created $\mathcal{C}_{\mathcal{F}}$-distance measure and used it in signal issues. Liu et al. [65] proposed cross entropy & distance measures in $\mathcal{C}_{\mathcal{F}}$-context. By using $\mathcal{C}_{\mathcal{F}}$-distance measure, they solved several issues in several disciplines. Complex linguistic fuzzy sets were examined by Dai et al. [66] and their uses in $\mathfrak{DM}$ were discussed. Ali [67] established $\mathcal{C}_{\mathcal{F}}$-trigonometric similarity measures, which quantify the similarity or dissimilarity of $\mathcal{C}_{\mathcal{F}}$-sets. Yousafzai et al. [68] established a link between $\mathcal{C}_{\mathcal{F}}$-sets & AG-groupoids and used this model in system analysis for suitable signal selection. Jan et al. [69] combined a soft set with a complex T-spherical fuzzy set to introduce an integrated model and proposed its applications in quantum computing & energy resources. In all these theories, only the magnitude function represents fuzzy information. This manuscript expands the notions of $\mathcal{C}_{\mathcal{F}}$-set (Ramot et al. [9]) by introducing a cartesian $\mathcal{C}_{\mathcal{F}}$-membership function that contains real as well as imaginary parts. However, the polar representation can also be used to express ambiguous information. The new notion is more rigorous. The examples of stock market and solar activity demonstrate the fragility of traditional $\mathcal{C}_{\mathcal{F}}$-sets [70]. In the definition of $\mathcal{C}_{\mathcal{F}}$-set [70], the membership function contains two parts- real part & imaginary part. The range of the membership function of this $\mathcal{C}_{\mathcal{F}}$-set lies in the first quadrant with real and imaginary parts belonging to the interval $[0, 1]$. This new notion of traditional $\mathcal{C}_{\mathcal{F}}$-set is called Standard Complex-valued fuzzy ($\mathcal{C}_{\mathcal{F}}^{V}$) set. Some main objectives of this manuscript are-

- A novel knowledge measure in $\mathcal{C}_{\mathcal{F}}^{V}$-context is defined.
- The Perfection of the proposed measure over previously known measures is verified numerically.
- Deduced Similarity, Dissimilarity, and Accuracy measure in $\mathcal{C}_{\mathcal{F}}^{V}$-context from the proposed measure.
- The uses of the deducted measures are given in Pattern Recognition & Clustering.

This research article has the following structure: Section 2 introduces some basic definitions related to the $\mathcal{C}_{\mathcal{F}}^{V}$-sets. A novel knowledge measure for $\mathcal{C}_{\mathcal{F}}^{V}$-sets is proposed and checked for credibility in Section 3. A comparative study with the previously known measures is taken in Section 4. In Section 5, some novel measures are deducted for $\mathcal{C}_{\mathcal{F}}^{V}$-sets from the proposed measure. The applications of the deducted measures are given in Pattern Recognition and Clustering issues are taken in Section 6. Some advantages of the proposed measures and the proposed methods are given in Section 7. Finally, the findings and prospective research directions are presented in Section 8.

2 Definitions

This section provides various definitions for $\mathcal{C}_{\mathcal{F}}^{V}$-sets. Throughout this paper, $\mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$ is used to symbolize the collection of $\mathcal{C}_{\mathcal{F}}^{V}$-sets defined on non-empty universal set $\mathfrak{R} = \{\mathfrak{r}_\gamma\}_{\gamma=1}^{\Gamma}$ with $\Gamma \geq 2$.

Definition 2.1. (Zadeh [1]) Let $\mathfrak{R} = \{\mathfrak{r}_\gamma\}_{\gamma=1}^{\Gamma}$ is a finite universe for $\Gamma \geq 2$. For this set $\mathfrak{R}$, a fuzzy set $\mathfrak{F}$ defined as follows

$$\mathfrak{F} = \left\{\langle \mathfrak{r}_\gamma, \mathcal{C}_{\mathfrak{F}}^{\mu}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}, \tag{2.1}$$

where $\mathcal{C}_{\mathfrak{F}}^{\mu}$: $\mathfrak{R} \to [0, 1]$ is the membership function.

Definition 2.2. (Ramot et al. [9]; Tamir et al. [70]) Let $\mathfrak{R} = \{\mathfrak{r}_\gamma\}_{\gamma=1}^{\Gamma}$ is a finite universe for $\Gamma \geq 2$. For set $\mathfrak{R}$, Ramot et al. [9] defined a $\mathcal{C}_{\mathcal{F}}$-set $\mathfrak{C}$ in polar coordinates as follows

$$\mathfrak{C} = \left\{\left\langle \mathfrak{r}_\gamma, M_{\mathfrak{C}}(\mathfrak{r}_\gamma) \cdot e^{jP_{\mathfrak{C}}(\mathfrak{r}_\gamma)}\right\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}, \tag{2.2}$$

where functions $M_{\mathfrak{C}} : \mathfrak{R} \to [0,1]$ and $P_{\mathfrak{C}} : \mathfrak{R} \to [0, 2\pi]$ represents the magnitude and the phase functions, respectively for $j = \sqrt{-1}$.

However, the phase function is not fuzzy. Also, the complement of a $\mathcal{C}_{\mathcal{F}}$-set does not affect the phase function. So, the above definition cannot handle the $\mathcal{C}_{\mathcal{F}}$-problems in some situations (Tamir et al. [70]). For set $\mathfrak{R}$, Tamir et al. [70] defined a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ in cartesian coordinates as follows

$$\mathfrak{C} = \left\{ \left\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}, \tag{2.3}$$

where $\alpha_{\mathfrak{C}}^{\mu} : \mathfrak{R} \to [0,1]$ and $\beta_{\mathfrak{C}}^{\mu} : \mathfrak{R} \to [0,1]$ are Real and Imaginary membership function respectively. The polar and cartesian coordinates definitions are connected by the relations-$M_{\mathfrak{C}}(\mathfrak{r}_\gamma) = \sqrt{(\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2 + (\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2}$, and $P_{\mathfrak{C}}(\mathfrak{r}_\gamma) = \tan^{-1}\left(\frac{\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)}{\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)}\right) \forall \mathfrak{r}_\gamma \in \mathfrak{R}$. A $\mathcal{C}_{\mathcal{F}}^{V}$-number can also be represented by $\langle \alpha_{\mathfrak{C}}^{\mu}, \beta_{\mathfrak{C}}^{\mu} \rangle$.

Definition 2.3. (Tamir et al. [70]) Let $\mathfrak{C}_1 = \left\{ \left\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$, & $\mathfrak{C}_2 = \left\{ \left\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$ are two $\mathcal{C}_{\mathcal{F}}^{V}$-sets. Some basic operations for these $\mathcal{C}_{\mathcal{F}}^{V}$-sets are defined as follows

(a). $\mathfrak{C}_1 \cup \mathfrak{C}_2 = \left\{ \left\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}_1 \cup \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}_1 \cup \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$,
where $\alpha_{\mathfrak{C}_1 \cup \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) = \max\left\{ \alpha_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma), \alpha_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\}$ and $\beta_{\mathfrak{C}_1 \cup \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) = \max\left\{ \beta_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\}$.

(b). $\mathfrak{C}_1 \cap \mathfrak{C}_2 = \left\{ \left\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}_1 \cap \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}_1 \cap \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$,
where $\alpha_{\mathfrak{C}_1 \cap \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) = \min\left\{ \alpha_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma), \alpha_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\}$ and $\beta_{\mathfrak{C}_1 \cap \mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) = \min\left\{ \beta_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma) \right\}$.

(c). ${\mathfrak{C}_1}^* = \left\{ \left\langle \mathfrak{r}_\gamma, \left(1 - \alpha_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma)\right) + j \cdot \left(1 - \beta_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma)\right) \right\rangle \middle| \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$, where * is used for complement.

(d). $\mathfrak{C}_1 \subseteq \mathfrak{C}_2$ iff $\alpha_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma) \le \alpha_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma)$ and $\beta_{\mathfrak{C}_1}^{\mu}(\mathfrak{r}_\gamma) \le \beta_{\mathfrak{C}_2}^{\mu}(\mathfrak{r}_\gamma)$ $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$.

Bi et al. [71] gave the definitions Type-A and Type-B Information measures for $\mathcal{C}_{\mathcal{F}}$-sets. The phase function $P_{\mathfrak{C}}$ was ignored by the Type-A Information measure. As a result, this definition was reduced to the definition of the Fuzzy information measures. However, the Type-B Information measures definition is considered as a definition for valid information measures in $\mathcal{C}_{\mathcal{F}}$-context. Both these definitions are used if any $\mathcal{C}_{\mathcal{F}}$-set contains magnitude and phase functions. The fuzzy set concept was therefore expanded to include the $\mathcal{C}_{\mathcal{F}}^{V}$-set, and the definitions of a credible $\mathcal{C}_{\mathcal{F}}^{V}$-information, as well as knowledge measures, are defined below axiomatically.

Definition 2.4. For a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C} = \left\{ \langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \rangle | \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$, a credible $\mathcal{C}_{\mathcal{F}}^{V}$-Information measure $\varrho^e : \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \to [0,1]$ should meet the following requirements:

(I1) $\varrho^e(\mathfrak{C}) = 1 \Leftrightarrow \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 0.5 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma), \forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(I2) $\varrho^e(\mathfrak{C}) = 0$ if $\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 0 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)$, or $\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 1 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma), \forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(I3) $\varrho^e(\mathfrak{C}) \le \varrho^e(\mathfrak{D})$ if
$$\begin{cases} \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \le \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \le \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) & \text{if } \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \le \frac{1}{2}, \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \le \frac{1}{2} \\ \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \ge \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \ge \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) & \text{if } \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \ge \frac{1}{2}, \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \ge \frac{1}{2} \end{cases} \forall \mathfrak{r}_\gamma \in \mathfrak{R}, \text{ for some}$$
$\mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$;

(I4) $\varrho^e(\mathfrak{C}) = \varrho^e(\mathfrak{C}^*)$, where * is used for complement.

Definition 2.5. For a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C} = \left\{ \langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \rangle | \mathfrak{r}_\gamma \in \mathfrak{R} \right\}$, a credible $\mathcal{C}_{\mathcal{F}}^{V}$-Knowledge measure $\varpi_\kappa : \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \to [0,1]$ should meet the following requirements:

(K1) $\varpi_\kappa(\mathfrak{C}) = 0 \Leftrightarrow \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 0.5 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma), \forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(K2) $\varpi_\kappa(\mathfrak{C}) = 1$ if $\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 0 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)$, or $\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) = 1 = \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma), \forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(K3) $\varpi_\kappa(\mathfrak{C}) \ge \varpi_\kappa(\mathfrak{D})$ if
$$\begin{cases} \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \le \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \le \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) & \text{if } \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \le \frac{1}{2}, \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \le \frac{1}{2} \\ \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \ge \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma), \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) \ge \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) & \text{if } \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \ge \frac{1}{2}, \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) \ge \frac{1}{2} \end{cases} \forall \mathfrak{r}_\gamma \in \mathfrak{R}, \text{ for some}$$
$\mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$;

(*K4*) $\varpi_\kappa(\mathfrak{C}) = \varpi_\kappa(\mathfrak{C}^*)$, where * is used for complement.

Definition 2.6. For $\mathcal{C}^V_\mathcal{F}$-sets $\mathfrak{C}_1 = \left\{\langle \mathfrak{r}_\gamma, \alpha^\mu_{\mathfrak{C}_1}(\mathfrak{r}_\gamma) + j \cdot \beta^\mu_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$, and $\mathfrak{C}_2 = \left\{\langle \mathfrak{r}_\gamma, \alpha^\mu_{\mathfrak{C}_2}(\mathfrak{r}_\gamma) + j \cdot \beta^\mu_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$, a credible $\mathcal{C}^V_\mathcal{F}$-Accuracy measure $A^M : \mathcal{C}^V_\mathcal{F}(\mathfrak{R}) \times \mathcal{C}^V_\mathcal{F}(\mathfrak{R}) \to [0,1]$ should meet the following requirements:

(*A1*) For all $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}^V_\mathcal{F}(\mathfrak{R})$, the value of $A^M(\mathfrak{C}_1, \mathfrak{C}_2)$ lies in between 0 & 1;

(*A2*) The value of $A^M(\mathfrak{C}_1, \mathfrak{C}_2)$ is equals to 0 if both $\mathcal{C}^V_\mathcal{F}$-sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equals to a $\mathcal{C}^V_\mathcal{F}$-set $\mathfrak{C} = \left\{\langle \mathfrak{r}_\gamma, \frac{1}{2} + j \cdot \frac{1}{2}\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$;

(*A3*) The value of $A^M(\mathfrak{C}_1, \mathfrak{C}_2(A3))$ is equals to 1 if both $\mathcal{C}^V_\mathcal{F}$-sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are either equals to a $\mathcal{C}^V_\mathcal{F}$-set $\mathfrak{C} = \{\langle \mathfrak{r}_\gamma, 0 + j \cdot 0\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$, or equals to a $\mathcal{C}^V_\mathcal{F}$-set $\mathfrak{C} = \{\langle \mathfrak{r}_\gamma, 1 + j \cdot 1\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$;

(*A4*) $A^M(\mathfrak{C}_1, \mathfrak{C}_2) = \varpi_\kappa(\mathfrak{C}_1)$ if $\mathfrak{C}_1 = \mathfrak{C}_2$, where ϖ_κ is the knowledge measure.

Only a few authors proposed Information measure in $\mathcal{C}^V_\mathcal{F}$-context. But no author proposed any knowledge measure in $\mathcal{C}^V_\mathcal{F}$-context. So, a Novel Knowledge measure in $\mathcal{C}^V_\mathcal{F}$-context is proposed in the next section.

3 A Novel Standard Complex-valued Fuzzy Knowledge measure

Let us take a collection of complete probability distribution $\mathfrak{Z}_Y = \left\{\mathfrak{Z} = (\mathfrak{z}_1, \mathfrak{z}_2, \ldots, \mathfrak{z}_Y) \mid \sum_{y=1}^{Y} \mathfrak{z}_y = 1; \mathfrak{z}_y \in [0,1]\ \forall\, y\right\}$ for $Y \in \mathbb{N} - \{1\}$. For some $\mathfrak{Z} \in \mathfrak{Z}_Y$, Shannon [72] defined the information measure as follows

$$\varrho^e(\mathfrak{z}) = -\sum_{y=1}^{Y} \mathfrak{z}_y \log \mathfrak{z}_y. \tag{3.1}$$

Renyi [73]; Tsallis [74]; Havrda and Charvat [75]; Boekee and Vander Lubbe [76]; and many other authors extended the Shannon entropy in literature. A new method of measuring uncertainty emerged with the characterization of the Fuzzy set theory. A fuzzy entropy based on Shannon entropy [72] was put up by Zadeh [1], yet it was unable to account for decision makers' hesitation degree. The entropy proposed by Boekee and Vander Lubbe [76] was extended to fuzzy context by many authors. A well-known extension of the entropy proposed by Boekee and Vander Lubbe [76] was given by Hooda [77] as follows

$$\varrho^e_\eta(\mathfrak{F}) = \frac{\eta}{\eta - 1} \sum_{\gamma=1}^{\Gamma} \left[1 - \left(\left(\mathcal{C}^\mu_\mathfrak{F}(\mathfrak{r}_\gamma)\right)^\eta + \left(1 - \mathcal{C}^\mu_\mathfrak{F}(\mathfrak{r}_\gamma)\right)^\eta\right)^{\frac{1}{\eta}}\right],\ \eta \in \mathbb{R}^+ - \{0,1\}. \tag{3.2}$$

A measure has been proposed by Singh & Kumar [17] in fuzzy context, which depends on the entropy measure [77] provided in Eq. (3.1). By taking the help of measure proposed by Singh & Kumar [17], a Novel $\mathcal{C}^V_\mathcal{F}$-Knowledge measure is provided in the next part.

3.1 Definition & Properties

For $\mathfrak{C} \in \mathcal{C}^V_\mathcal{F}(\mathfrak{R})$, define a $\mathcal{C}^V_\mathcal{F}$-knowledge measure as follows

$$\varpi_\kappa(\mathfrak{C}) = \frac{1}{\Gamma(\sqrt[3]{4} - 1)} \sum_{\gamma=1}^{\Gamma} \left[\sqrt[3]{4\left[\begin{array}{c}\left((\alpha^\mu_\mathfrak{C}(\mathfrak{r}_\gamma))^2 + (\beta^\mu_\mathfrak{C}(\mathfrak{r}_\gamma))^2\right)^{\frac{3}{2}} \\ +\left((1 - \alpha^\mu_\mathfrak{C}(\mathfrak{r}_\gamma))^2 + (1 - \beta^\mu_\mathfrak{C}(\mathfrak{r}_\gamma))^2\right)^{\frac{3}{2}}\end{array}\right]} - 1\right]. \tag{3.3}$$

Now, the credibility of the measure ϖ_κ is checked.

Theorem 3.1. *For a $\mathcal{C}^V_\mathcal{F}$-set $\mathfrak{C} = \left\{\langle \mathfrak{r}_\gamma, \alpha^\mu_\mathfrak{C}(\mathfrak{r}_\gamma) + j \cdot \beta^\mu_\mathfrak{C}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$, a credible $\mathcal{C}^V_\mathcal{F}$-Knowledge measure $\varpi_\kappa : \mathcal{C}^V_\mathcal{F}(\mathfrak{R}) \to [0,1]$ proposed in Eq. (3.3) should meet the following requirements:*

(K1) $\varpi_\kappa(\mathfrak{C}) = 0 \Leftrightarrow \alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) = 0.5 = \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(K2) $\varpi_\kappa(\mathfrak{C}) = 1$ *if* $\alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) = 0 = \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, *or* $\alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) = 1 = \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$;

(K3) $\varpi_\kappa(\mathfrak{C}) \geq \varpi_\kappa(\mathfrak{D})$ *if* $\begin{cases} \alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) \leq \alpha^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma), \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) \leq \beta^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) & \text{if } \alpha^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) \leq \frac{1}{2}, \beta^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) \leq \frac{1}{2} \\ \alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) \geq \alpha^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma), \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) \geq \beta^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) & \text{if } \alpha^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) \geq \frac{1}{2}, \beta^\mu_{\mathfrak{D}}(\mathfrak{r}_\gamma) \geq \frac{1}{2} \end{cases}$
$\forall \mathfrak{r}_\gamma \in \mathfrak{R}$, *for some* $\mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$;

(K4) $\varpi_\kappa(\mathfrak{C}) = \varpi_\kappa(\mathfrak{C}^*)$, *where* * *is used for complement.*

Proof. ***(K1)***. Let $0 < \rho, \sigma < 1$. For these independent variable ρ & σ, consider a function Ω as follows

$$\Omega(\rho, \sigma) = \frac{\sqrt[3]{\sqrt{2}\left[\left(\rho^2 + \sigma^2\right)^{\frac{3}{2}} + \left((1-\rho)^2 + (1-\sigma)^2\right)^{\frac{3}{2}}\right]} - 1}{\sqrt[3]{4} - 1}. \tag{3.4}$$

Apply the differential operator $\frac{d}{d\rho}$ & $\frac{d}{d\sigma}$ on the function Ω, we have

$$\frac{d\Omega}{d\rho} = \frac{P\left[3\rho\left(\rho^2 + \sigma^2\right)^{\frac{1}{2}} - 3\left(1-\rho\right)\left((1-\rho)^2 + (1-\sigma)^2\right)^{\frac{1}{2}}\right]}{\left[\left(\left(\rho^2 + \sigma^2\right)^{\frac{3}{2}} + \left((1-\rho)^2 + (1-\sigma)^2\right)^{\frac{3}{2}}\right)\right]^{\frac{2}{3}}}, \tag{3.5}$$

$$\frac{d\Omega}{d\sigma} = \frac{P\left[3\sigma\left(\rho^2 + \sigma^2\right)^{\frac{1}{2}} - 3\left(1-\sigma\right)\left((1-\rho)^2 + (1-\sigma)^2\right)^{\frac{1}{2}}\right]}{\left[\left(\left(\rho^2 + \sigma^2\right)^{\frac{3}{2}} + \left((1-\rho)^2 + (1-\sigma)^2\right)^{\frac{3}{2}}\right)\right]^{\frac{2}{3}}}, \tag{3.6}$$

where $P = \frac{(\sqrt{2})^{\frac{1}{3}}}{\sqrt[3]{4}-1}$. Put $\frac{d\Omega}{d\rho} = 0$, and $\frac{d\Omega}{d\sigma} = 0$ for stationary points. From this, it is obtained that $\rho = \sigma = \frac{1}{2}$. Now, to find the nature of function, we compute the values of expressions $\frac{d^2\Omega}{d\rho^2}$, $\frac{d^2\Omega}{d\sigma^2}$, and $\frac{d^2\Omega}{d\rho d\sigma}$ for $\rho = \sigma = \frac{1}{2}$. On solving, it obtained that $\frac{d^2\Omega}{d\rho^2}\frac{d^2\Omega}{d\sigma^2} - \left(\frac{d^2\Omega}{d\rho d\sigma}\right)^2 > 0$, and $\frac{d^2\Omega}{d\rho^2} > 0$ for $\rho = \sigma = \frac{1}{2}$, i.e., function Ω attaints its least value for $\rho = \sigma = \frac{1}{2}$. Therefore, $\varpi_\kappa(\mathfrak{C})$ attains its minimum value (i.e. 0) for $\rho = \sigma = \frac{1}{2}$ $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$, i.e., $\varpi_\kappa(\mathfrak{C}) = 0$ for $\rho = \sigma = \frac{1}{2}$ $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$.

(K2). Let $\mathfrak{C}$ be a $\mathcal{C}_{\mathcal{F}}^{V}$-set s.t. either $\alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) = 0 = \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, or $\alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma) = 1 = \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$. On evaluating Eq. (3.3) for this $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$, we get $\varpi_\kappa(\mathfrak{C}) = 1$.

(K3). Eq. (3.5) and Eq. (3.6) implies that

$$\frac{d\Omega}{d\rho} = \begin{cases} Positive & \text{for } \rho > \frac{1}{2} \\ Negative & \text{for } \rho < \frac{1}{2} \end{cases}, \quad \& \quad \frac{d\Omega}{d\sigma} = \begin{cases} Positive & \text{for } \sigma > \frac{1}{2} \\ Negative & \text{for } \sigma < \frac{1}{2} \end{cases}, \tag{3.7}$$

i.e., the function Ω increases for $\rho > \frac{1}{2}$ & $\sigma > \frac{1}{2}$, and the function Ω decreases for $\rho < \frac{1}{2}$ & $\sigma < \frac{1}{2}$. Thus, from Eq. (3.3), it is clear that $\varpi_\kappa(\mathfrak{C}) \geq \varpi_\kappa(\mathfrak{D})$.

(K4). For $\mathfrak{C} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, Definition 2.3 implies that $\mathfrak{C}^* = \left\{\langle \mathfrak{r}_\gamma, (1 - \alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)) + j \cdot (1 - \beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma))\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$. From the definition of the measure ϖ_κ, it is obtained that $\varpi_\kappa(\mathfrak{C}) = \varpi_\kappa(\mathfrak{C}^*)$.

Hence, measure ϖ_κ is a credible measure. □

Example 3.2. Take a singleton set $\mathfrak{R} = \{\mathfrak{r}\}$. For different values of $\alpha^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$, $\beta^\mu_{\mathfrak{C}}(\mathfrak{r}_\gamma)$ lies in $[0, 1]$, the graphical representation of the knowledge supplied by the measure ϖ_κ is shown in Figure 1 & 2.

Now, in the upcoming theorems proposed measure's properties are proved.

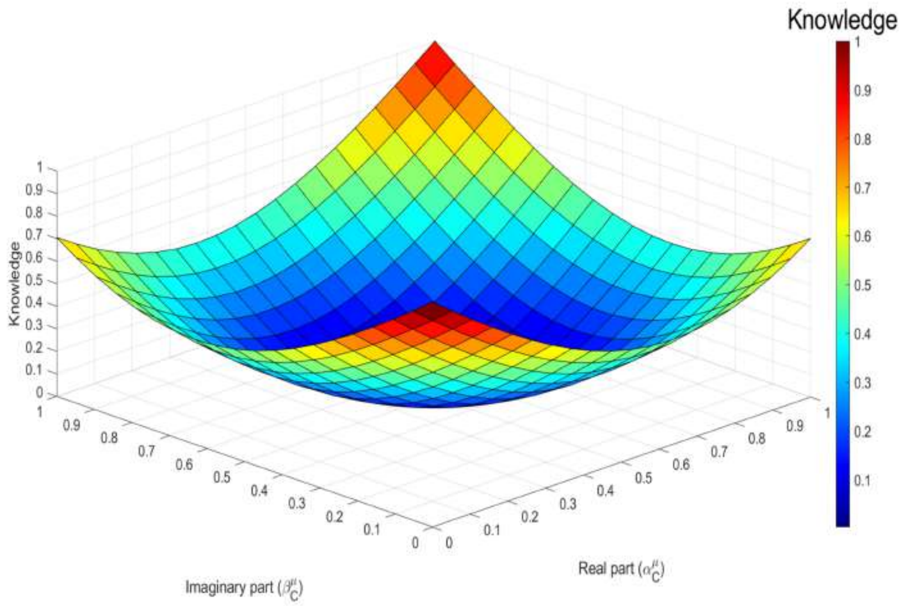


Figure 1: Graphical representation of ϖ_κ measure.

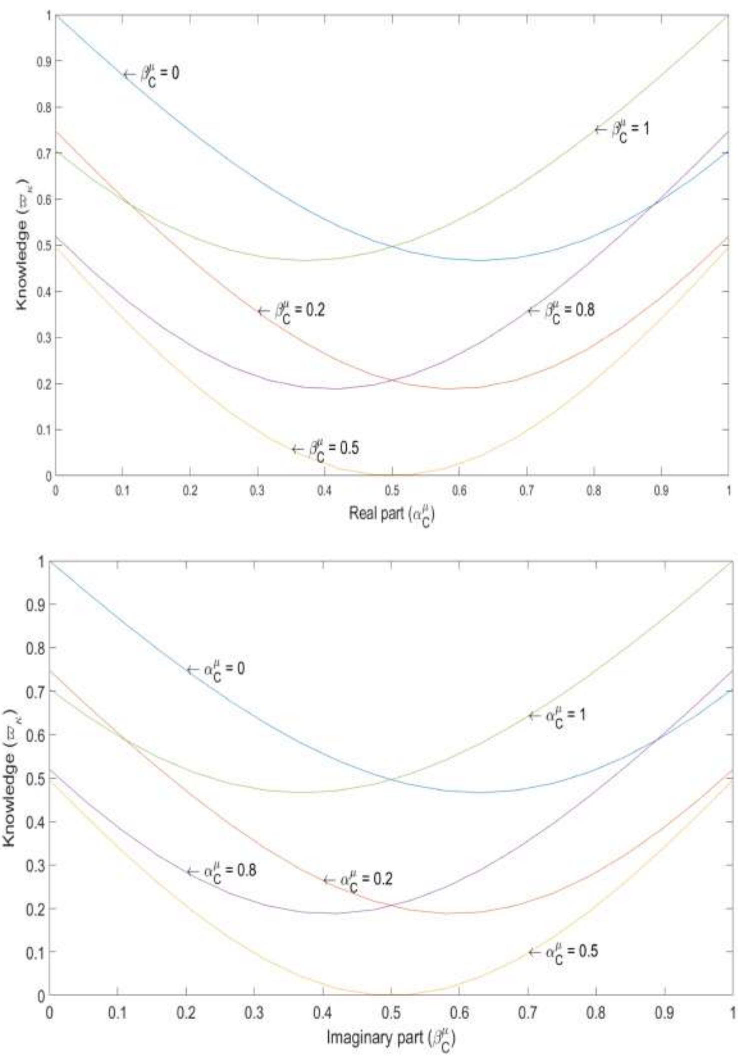


Figure 2: Graphical representation of ϖ_κ measure for fixed Real & Imaginary values.

Theorem 3.3. *For $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, the following property is satisfied by the measure ϖ_κ:*

$$\varpi_\kappa(\mathfrak{C}_1 \cup \mathfrak{C}_2) + \varpi_\kappa(\mathfrak{C}_1 \cap \mathfrak{C}_2) = \varpi_\kappa(\mathfrak{C}_1) + \varpi_\kappa(\mathfrak{C}_2).$$

Proof. Consider

$$\varpi_\kappa(\mathfrak{C}_1 \cup \mathfrak{C}_2) = \frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\gamma=1}^{\Gamma} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_1 \cup \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_1 \cup \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_1 \cup \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_1 \cup \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right],$$

and

$$\varpi_\kappa(\mathfrak{C}_1 \cap \mathfrak{C}_2) = \frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\gamma=1}^{\Gamma} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_1 \cap \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_1 \cap \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_1 \cap \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_1 \cap \mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right].$$

Now, the set $\mathfrak{R}$ is divided into two subsets R^1 & R^2 in such a way that the set $\mathfrak{C}_1$ is contained in the set $\mathfrak{C}_2$ for the subset R^1 and the set $\mathfrak{C}_2$ is contained in the set $\mathfrak{C}_1$ for the subset R^2. For this subset, above both equations become

$$\begin{aligned} \varpi_\kappa(\mathfrak{C}_1 \cup \mathfrak{C}_2) = &\frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\mathfrak{r}_\gamma \in \mathfrak{R}^1} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right] \\ &+ \frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\mathfrak{r}_\gamma \in \mathfrak{R}^2} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right]. \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \varpi_\kappa(\mathfrak{C}_1 \cap \mathfrak{C}_2) = &\frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\mathfrak{r}_\gamma \in \mathfrak{R}^1} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_1}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right] \\ &+ \frac{1}{\Gamma(\sqrt[3]{4}-1)} \sum_{\mathfrak{r}_\gamma \in \mathfrak{R}^2} \left[\sqrt[3]{4 \left[\begin{array}{c} \left(\left(\alpha^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(\beta^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \\ + \left(\left(1-\alpha^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 + \left(1-\beta^{\mu}_{\mathfrak{C}_2}(\mathfrak{r}_\gamma)\right)^2 \right)^{\frac{3}{2}} \end{array} \right]} - 1 \right]. \end{aligned} \tag{3.9}$$

On adding Eq. (3.8) & Eq. (3.9), it is obtained that

$$\varpi_\kappa(\mathfrak{C}_1 \cup \mathfrak{C}_2) + \varpi_\kappa(\mathfrak{C}_1 \cap \mathfrak{C}_2) = \varpi_\kappa(\mathfrak{C}_1) + \varpi_\kappa(\mathfrak{C}_2).$$

□

4 Comparative study

We now compare the measure ϖ_κ to the other existing measures in $\mathcal{C}_{\mathcal{F}}^{V}$-context. This investigates the advantages of the measure ϖ_κ. We investigate these advantages concerning Structured linguistic comparison, in the estimate of ambiguity content, and criteria weight estimation in MCDM problems. There are some $\mathcal{C}_{\mathcal{F}}^{V}$-entropy measure given below

$$\varrho_1^e(\mathfrak{C}) = \frac{-1}{\Gamma}\sum_{\gamma=1}^{\Gamma}\left[\left|M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right| \cdot \log\left|M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right| + \left(1-\left|M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right|\right)\cdot\log\left(1-\left|M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right|\right)\right]. \tag{4.1}$$

$$\varrho_2^e(\mathfrak{C}) = \frac{4}{\Gamma}\sum_{\gamma=1}^{\Gamma}\left[\left|M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right| \cdot \left|1-M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\right|\right]. \tag{4.2}$$

$$\begin{aligned}\varrho_3^e(\mathfrak{C}) = \frac{-1}{2\Gamma}\sum_{\gamma=1}^{\Gamma}\Bigg[& M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\cdot\log M_{\mathfrak{C}}(\mathfrak{r}_\gamma) + (1-M_{\mathfrak{C}}(\mathfrak{r}_\gamma))\cdot\log(1-M_{\mathfrak{C}}(\mathfrak{r}_\gamma))\\ &+\frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi}\cdot\log\frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi} + \left(1-\frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi}\right)\cdot\log\left(1-\frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi}\right)\Bigg].\end{aligned} \tag{4.3}$$

$$\varrho_4^e(\mathfrak{C}) = \frac{2}{\Gamma}\sum_{\gamma=1}^{\Gamma}\left[M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\cdot(1-M_{\mathfrak{C}}(\mathfrak{r}_\gamma)) + \frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi}\cdot\left(1-\frac{P_{\mathfrak{C}}(\mathfrak{r}_\gamma)}{2\pi}\right)\right]. \tag{4.4}$$

where $M_{\mathfrak{C}}(\mathfrak{r}_\gamma) = \sqrt{(\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2 + (\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2}$, and $P_{\mathfrak{C}}(\mathfrak{r}_\gamma) = \tan^{-1}\left(\frac{\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)}{\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)}\right)$ $\forall \mathfrak{r}_\gamma \in \mathfrak{R}$.

4.1 Structured linguistic comparison

A $\mathcal{C}_{\mathcal{F}}^{V}$-set is used to denote a linguistic variable. All the operations that are carried out on $\mathcal{C}_{\mathcal{F}}^{V}$-sets are called linguistic hedges. The are many operations that are carried out on these linguistic variables. With the help of the measure ϖ_κ, these hedges are computed. Let $\mathfrak{R} = \{\mathfrak{r}_\gamma\}_{\gamma=1}^{\Gamma}$ is a finite universal set for $\Gamma \geq 2$. For this set $\mathfrak{R}$, let $\mathfrak{C} = \left\{\langle \mathfrak{r}_\gamma, M_{\mathfrak{C}}(\mathfrak{r}_\gamma)\cdot e^{zP_{\mathfrak{C}}(\mathfrak{r}_\gamma)}\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}$ is a $\mathcal{C}_{\mathcal{F}}^{V}$-set used to represent linguistic variable "Fast". For any number $\Lambda > 0$, the modifier for the $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ is defined as follows

$$\mathfrak{C}^{\Lambda} = \left\{\langle \mathfrak{r}_\gamma, (\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^{\Lambda} + j\cdot(\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^{\Lambda}\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\right\}. \tag{4.5}$$

Dilatation & Concentration of a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ is represented by $\mathfrak{C}^{\frac{1}{2}}$ & $\mathfrak{C}^2$ respectively. The following Linguistic hedges are used for this $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$: More/Less Fast is written as $\mathfrak{C}^{\frac{1}{2}}$, Fast is written as $\mathfrak{C}$, A little Fast is written as $\mathfrak{C}^{1.3}$, Slightly Fast is written as $\mathfrak{C}^{1.7}$, Very Fast is written as $\mathfrak{C}^2$, Extremely Fast is written as $\mathfrak{C}^3$, and Very-Very Fast is written as $\mathfrak{C}^4$. In the theory of $\mathcal{C}_{\mathcal{F}}^{V}$-sets, as we move from $\mathfrak{C}^{\frac{1}{2}}$ to $\mathfrak{C}^4$, the fuzziness/knowledge associated with the $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ decreases/increases, i.e.,

$$\begin{aligned}&\varrho^e(\mathfrak{C}^{\frac{1}{2}}) > \varrho^e(\mathfrak{C}) > \varrho^e(\mathfrak{C}^{1.3}) > \varrho^e(\mathfrak{C}^{1.7}) > \varrho^e(\mathfrak{C}^2) > \varrho^e(\mathfrak{C}^3) > \varrho^e(\mathfrak{C}^4),\\ &\varpi_\kappa(\mathfrak{C}^{\frac{1}{2}}) < \varrho^e(\mathfrak{C}) < \varpi_\kappa(\mathfrak{C}^{1.3}) < \varpi_\kappa(\mathfrak{C}^{1.7}) < \varpi_\kappa(\mathfrak{C}^2) < \varpi_\kappa(\mathfrak{C}^3) < \varpi_\kappa(\mathfrak{C}^4),\end{aligned} \tag{4.6}$$

where ϖ_κ & ϱ^e is used for knowledge and entropy measure respectively.

For the Structured linguistic comparison, take the following examples:

Example 4.1. Take a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ given by

$$\mathfrak{C} = \left\{\begin{array}{c}\langle 13,(0.9+0.1\,j)\rangle, \langle 11,(0.1+0.7\,j)\rangle, \langle 9,(0.1+0.6\,j)\rangle, \langle 7,(0.4+0.1\,j)\rangle,\\ \langle 5,(0.3+0.5\,j)\rangle, \langle 3,(0.1+0.4\,j)\rangle, \langle 1,(0.5+0.1\,j)\rangle\end{array}\right\}, \tag{4.7}$$

defined on a universal set $\mathfrak{R} = \{13, 11, 9, 7, 5, 3, 1\}$. Take this $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ as 'Fast' as a linguistic variable and define the linguistic hedges as given above for the set $\mathfrak{C}$. The information/knowledge conveyed in this case is calculated by measures given in Eq. (3.3) and Eqs. (4.1) - (4.4). The comparison of the information/knowledge conveyed is given in Table 1.

Table 1: Information/Knowledge conveyed by $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ and its linguistic hedges.

Measures	$\mathfrak{C}^{\frac{1}{2}}$	$\mathfrak{C}$	$\mathfrak{C}^{1.3}$	$\mathfrak{C}^{1.7}$	$\mathfrak{C}^{2}$	$\mathfrak{C}^{3}$	$\mathfrak{C}^{4}$
ϱ_1^e	0.5900	0.8892	0.8928	0.8454	0.7944	0.6150	0.4715
ϱ_2^e	0.5225	0.8620	0.8624	0.7987	0.7349	0.5331	0.3934
ϱ_3^e	0.5592	0.6942	0.6892	0.6594	0.6310	0.5377	0.4658
ϱ_4^e	0.4775	0.6419	0.6403	0.6075	0.5757	0.4763	0.4081
ϖ_κ	0.1784	0.3385	0.4500	0.5616	0.6236	0.7534	0.8223

From Table 1, it is obtained that only measure ϖ_κ satisfy Eq. (4.6). Also, the following output is obtained from Table 1:

$$\begin{aligned}
&\varrho_1^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_1^e(\mathfrak{C}) < \varrho_1^e(\mathfrak{C}^{1.3}) > \varrho_1^e(\mathfrak{C}^{1.7}) > \varrho_1^e(\mathfrak{C}^{2}) > \varrho_1^e(\mathfrak{C}^{3}) > \varrho_1^e(\mathfrak{C}^{4}),\\
&\varrho_2^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_2^e(\mathfrak{C}) < \varrho_2^e(\mathfrak{C}^{1.3}) > \varrho_2^e(\mathfrak{C}^{1.7}) > \varrho_2^e(\mathfrak{C}^{2}) > \varrho_2^e(\mathfrak{C}^{3}) > \varrho_2^e(\mathfrak{C}^{4}),\\
&\varrho_3^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_3^e(\mathfrak{C}) > \varrho_3^e(\mathfrak{C}^{1.3}) > \varrho_3^e(\mathfrak{C}^{1.7}) > \varrho_3^e(\mathfrak{C}^{2}) > \varrho_3^e(\mathfrak{C}^{3}) > \varrho_3^e(\mathfrak{C}^{4}),\\
&\varrho_4^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_4^e(\mathfrak{C}) > \varrho_4^e(\mathfrak{C}^{1.3}) > \varrho_4^e(\mathfrak{C}^{1.7}) > \varrho_4^e(\mathfrak{C}^{2}) > \varrho_4^e(\mathfrak{C}^{3}) > \varrho_4^e(\mathfrak{C}^{4}),\\
&\varpi_\kappa(\mathfrak{C}^{\frac{1}{2}}) < \varrho^e(\mathfrak{C}) < \varpi_\kappa(\mathfrak{C}^{1.3}) < \varpi_\kappa(\mathfrak{C}^{1.7}) < \varpi_\kappa(\mathfrak{C}^{2}) < \varpi_\kappa(\mathfrak{C}^{3}) < \varpi_\kappa(\mathfrak{C}^{4}).
\end{aligned} \tag{4.8}$$

Eq. (4.8) implies that the measure ϖ_κ performs better than other measures. Thus, it implies the reliability of the proposed measure ϖ_κ.

Further, one more example is given below.

Example 4.2. Take a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ given by

$$\mathfrak{C} = \left\{ \begin{array}{c} \langle 15,(0.6+0.3\,j)\rangle, \langle 13,(0.2+0.7\,j)\rangle, \langle 11,(0.1+0.6\,j)\rangle, \langle 9,(0.4+0.5\,j)\rangle, \\ \langle 7,(0.9+0.1\,j)\rangle, \langle 5,(0.2+0.5\,j)\rangle, \langle 3,(0.5+0.3\,j)\rangle, \langle 1,(0.3+0.1\,j)\rangle \end{array} \right\}, \tag{4.9}$$

defined on a universal set $\mathfrak{R} = \{15,13,11,9,7,5,3,1\}$. Take this $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ as 'Fast' as a linguistic variable and define the linguistic hedges as given above for the set $\mathfrak{C}$. The information/knowledge conveyed in this case is calculated by measures given in Eq. (3.3) and Eqs. (4.1) - (4.4). The comparison of the information/knowledge conveyed is given in Table 2.

Table 2: Information/Knowledge conveyed by $\mathcal{C}_{\mathcal{F}}^{V}$-set $\mathfrak{C}$ and its linguistic hedges.

Measures	$\mathfrak{C}^{\frac{1}{2}}$	$\mathfrak{C}$	$\mathfrak{C}^{1.3}$	$\mathfrak{C}^{1.7}$	$\mathfrak{C}^{2}$	$\mathfrak{C}^{3}$	$\mathfrak{C}^{4}$
ϱ_1^e	0.4494	0.8742	0.9011	0.8670	0.8203	0.6425	0.4923
ϱ_2^e	0.3731	0.8404	0.8742	0.8299	0.7714	0.5653	0.4113
ϱ_3^e	0.4882	0.6880	0.6944	0.6693	0.6407	0.5392	0.4567
ϱ_4^e	0.3998	0.6266	0.6399	0.6135	0.5815	0.4720	0.3918
ϖ_κ	0.1855	0.2399	0.3510	0.4831	0.5615	0.7281	0.8139

From Table 2, it is obtained that only measure ϖ_κ satisfy Eq. (4.6). Also, the following output is obtained from Table 2:

$$\begin{aligned}
&\varrho_1^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_1^e(\mathfrak{C}) < \varrho_1^e(\mathfrak{C}^{1.3}) > \varrho_1^e(\mathfrak{C}^{1.7}) > \varrho_1^e(\mathfrak{C}^{2}) > \varrho_1^e(\mathfrak{C}^{3}) > \varrho_1^e(\mathfrak{C}^{4}),\\
&\varrho_2^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_2^e(\mathfrak{C}) < \varrho_2^e(\mathfrak{C}^{1.3}) > \varrho_2^e(\mathfrak{C}^{1.7}) > \varrho_2^e(\mathfrak{C}^{2}) > \varrho_2^e(\mathfrak{C}^{3}) > \varrho_2^e(\mathfrak{C}^{4}),\\
&\varrho_3^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_3^e(\mathfrak{C}) < \varrho_3^e(\mathfrak{C}^{1.3}) > \varrho_3^e(\mathfrak{C}^{1.7}) > \varrho_3^e(\mathfrak{C}^{2}) > \varrho_3^e(\mathfrak{C}^{3}) > \varrho_3^e(\mathfrak{C}^{4}),\\
&\varrho_4^e(\mathfrak{C}^{\frac{1}{2}}) < \varrho_4^e(\mathfrak{C}) < \varrho_4^e(\mathfrak{C}^{1.3}) > \varrho_4^e(\mathfrak{C}^{1.7}) > \varrho_4^e(\mathfrak{C}^{2}) > \varrho_4^e(\mathfrak{C}^{3}) > \varrho_4^e(\mathfrak{C}^{4}),\\
&\varpi_\kappa(\mathfrak{C}^{\frac{1}{2}}) < \varrho^e(\mathfrak{C}) < \varpi_\kappa(\mathfrak{C}^{1.3}) < \varpi_\kappa(\mathfrak{C}^{1.7}) < \varpi_\kappa(\mathfrak{C}^{2}) < \varpi_\kappa(\mathfrak{C}^{3}) < \varpi_\kappa(\mathfrak{C}^{4}).
\end{aligned} \tag{4.10}$$

Eq. (4.8) implies that the measure ϖ_κ performs better than other measures. As a result, it is a credible measure. Also, all the defined measures are suitable if the values assigned to the elements of $\mathcal{C}^V_{\mathcal{F}}$-set lie within the unit circle. But the proposed measure works well for those values, that are outside the unit circle. As a result, the proposed measure performs exceptionally well and shows tremendous potential.

4.2 Estimation of Criteria weights

The Criteria weight estimation is an important task in any $\mathfrak{DM}$ issue. Criteria weights are used in $\mathfrak{DM}$ issues to find the correct solution. A little bit of modification in it may modify the desired result. So, their importance is high in any $\mathfrak{DM}$ issue. Chen & Li [78] proposed 2 methods to find criteria weights- The entropy method and the Knowledge method. For 'Γ' alternatives and 'Δ' criterion, the formulas $w_\delta = (1 - \varrho^e(\mathfrak{C}_\delta)) \,/\, \left(\Delta - \sum_{\delta=1}^{\Delta} \varrho^e(\mathfrak{C}_\delta)\right), \delta = 1, 2, \ldots, \Delta$, and $w_\delta = (\varpi_\kappa(\mathfrak{C}_\delta)) \,/\, \left(\sum_{\delta=1}^{\Delta} \varpi_\kappa(\mathfrak{C}_\delta)\right), \delta = 1, 2, \ldots, \Delta$ are used for entropy measure ϱ^e and knowledge measure ϖ_κ respectively.

Now, a decision matrix is taken and criteria weights are computed for this decision matrix by using the above formulas. Have a look at the following example given below

Example 4.3. Take a $\mathcal{C}^V_{\mathcal{F}}$-decision matrix $\mathcal{D}$ having alternative as its rows and criteria as its columns given by

$$\mathcal{D} = \begin{bmatrix} 0.44 + 0.56\,j & 0.74 + 0.50\,j & 0.86 + 0.51\,j & 0.54 + 0.30\,j & 0.65 + 0.20\,j \\ 0.81 + 0.49\,j & 0.38 + 0.84\,j & 0.54 + 0.69\,j & 0.92 + 0.05\,j & 0.87 + 0.23\,j \\ 0.29 + 0.40\,j & 0.61 + 0.15\,j & 0.52 + 0.37\,j & 0.04 + 0.18\,j & 0.80 + 0.59\,j \\ 0.65 + 0.31\,j & 0.89 + 0.17\,j & 0.36 + 0.85\,j & 0.42 + 0.15\,j & 0.42 + 0.49\,j \\ 0.23 + 0.78\,j & 0.10 + 0.41\,j & 0.54 + 0.42\,j & 0.12 + 0.14\,j & 0.02 + 0.78\,j \\ 0.82 + 0.43\,j & 0.54 + 0.16\,j & 0.54 + 0.29\,j & 0.31 + 0.25\,j & 0.42 + 0.36\,j \\ 0.29 + 0.78\,j & 0.68 + 0.61\,j & 0.21 + 0.24\,j & 0.16 + 0.15\,j & 0.35 + 0.49\,j \\ 0.70 + 0.50\,j & 0.36 + 0.75\,j & 0.26 + 0.84\,j & 0.02 + 0.01\,j & 0.56 + 0.78\,j \end{bmatrix}$$

The criteria weights are estimated by the previously known entropy measures given in Eqs. (4.1) - (4.4) and by the measure given in Eq. (3.3). By using the formulas given above, the result is computed in Table 3. It is easy to see from Table 3 that the measures given in Eqs. (4.1) - (4.4)

Table 3: Criteria weights.

Measures ↓	← Criteria Weights →				
	w_1	w_2	w_3	w_4	w_5
ϱ^e_1	**0.199**	**0.199**	0.192	0.212	**0.199**
ϱ^e_2	**0.193**	0.195	**0.193**	0.214	0.204
ϱ^e_3	0.207	**0.193**	0.204	**0.193**	0.202
ϱ^e_4	0.204	0.190	**0.206**	0.194	**0.206**
ϖ_κ	0.118	0.186	0.142	0.398	0.156

are not able to estimate the criteria weights in the given $\mathfrak{DM}$ issue. But the measure given in Eq. (3.3) did its job well. Therefore, there is a need for a measure that can estimate the criteria weights correctly.

In the upcoming section, some other measures are derived from the proposed measure ϖ_κ.

5 Derivations of some measures

5.1 Standard Complex-valued Fuzzy Accuracy measure with Properties

In any context, there is a strong connection between knowledge measure and accuracy measure. The concept of accuracy measure comes into existence when the accuracy between any two data sets is required. In literature, there does not exist any accuracy measure in $\mathcal{C}_{\mathcal{F}}^{V}$-context. Thus, an accuracy measure is deducted from the measure ϖ_κ as follows

Take a nonempty universal set $\mathfrak{R}$ containing more than one element. For $\mathfrak{C}, \mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, an Accuracy measure in $\mathcal{C}_{\mathcal{F}}^{V}$-context is defined as follows

$$\xi^A(\mathfrak{C}, \mathfrak{D}) = \frac{1}{4\Gamma(\sqrt[3]{4}-1)} \sum_{\gamma=1}^{\Gamma} \left[\begin{array}{c} \left(\begin{array}{c} \sqrt[3]{4\left[(m(\mathfrak{C}))^{\frac{3}{2}} + (m^c(\mathfrak{C}))^{\frac{3}{2}}\right]} \\ +\sqrt[3]{4\left[(m(\mathfrak{D}))^{\frac{3}{2}} + (m^c(\mathfrak{D}))^{\frac{3}{2}}\right]} - 2 \end{array} \right) \\ \left(\begin{array}{c} \sqrt[3]{4\left[\sqrt{m(\mathfrak{C})}m(\mathfrak{D}) + \sqrt{m^c(\mathfrak{C})}m^c(\mathfrak{D})\right]} \\ +\sqrt[3]{4\left[\sqrt{m(\mathfrak{D})}m(\mathfrak{C}) + \sqrt{m^c(\mathfrak{D})}m^c(\mathfrak{C})\right]} - 2 \end{array} \right) \end{array} \right], \quad (5.1)$$

where $m(\mathfrak{C}) = (\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2 + (\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2$, $m(\mathfrak{D}) = (\alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma))^2 + (\beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma))^2$, $m^c(C) = (1-\alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2 + (1-\beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma))^2$, $m^c(\mathfrak{D}) = (1-\alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma))^2 + (1-\beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma))^2$.

Now, the credibility of the measure $\xi^A(\mathfrak{C}, \mathfrak{D})$ is checked.

Theorem 5.1. *For $\mathcal{C}_{\mathcal{F}}^{V}$-sets $\mathfrak{C} = \{\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{C}}^{\mu}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$, and $\mathfrak{D} = \{\langle \mathfrak{r}_\gamma, \alpha_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma) + j \cdot \beta_{\mathfrak{D}}^{\mu}(\mathfrak{r}_\gamma)\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$, a credible $\mathcal{C}_{\mathcal{F}}^{V}$-Accuracy measure $\xi^A : \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \times \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \to [0,1]$ defined in Eq. (5.1) should meet the following requirements:*

(A1) *For all $\mathfrak{C}, \mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, the value of $\xi^A(\mathfrak{C}, \mathfrak{D})$ lies in between 0 & 1;*

(A2) *The value of $\xi^A(\mathfrak{C}, \mathfrak{D})$ is equals to 0 if both $\mathcal{C}_{\mathcal{F}}^{V}$-sets $\mathfrak{C}$ and $\mathfrak{D}$ are equals to a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\{\langle \mathfrak{r}_\gamma, \frac{1}{2} + j \cdot \frac{1}{2}\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$;*

(A3) *The value of $\xi^A(\mathfrak{C}, \mathfrak{D})$ is equals to 1 if both $\mathcal{C}_{\mathcal{F}}^{V}$-sets $\mathfrak{C}$ and $\mathfrak{D}$ are either equals to a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\{\langle \mathfrak{r}_\gamma, 0 + j \cdot 0\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$, or equals to a $\mathcal{C}_{\mathcal{F}}^{V}$-set $\{\langle \mathfrak{r}_\gamma, 1 + j \cdot 1\rangle | \mathfrak{r}_\gamma \in \mathfrak{R}\}$;*

(A4) *$\xi^A(\mathfrak{C}, \mathfrak{D}) = \varpi_\kappa(\mathfrak{C})$ if $\mathfrak{C} = \mathfrak{D}$, where ϖ_κ is the knowledge measure.*

Proof. This theorem can easily prove from Theorem 3.1 and by using Eq. (5.1). □

The measure ξ^A fulfil some properties. They are-

(A). $\xi^A(\mathfrak{C}_1, \mathfrak{C}_3 \cap \mathfrak{C}_2) + \xi^A(\mathfrak{C}_1, \mathfrak{C}_3 \cup \mathfrak{C}_2) = \xi^A(\mathfrak{C}_1, \mathfrak{C}_2) + \xi^A(\mathfrak{C}_1, \mathfrak{C}_3)$.

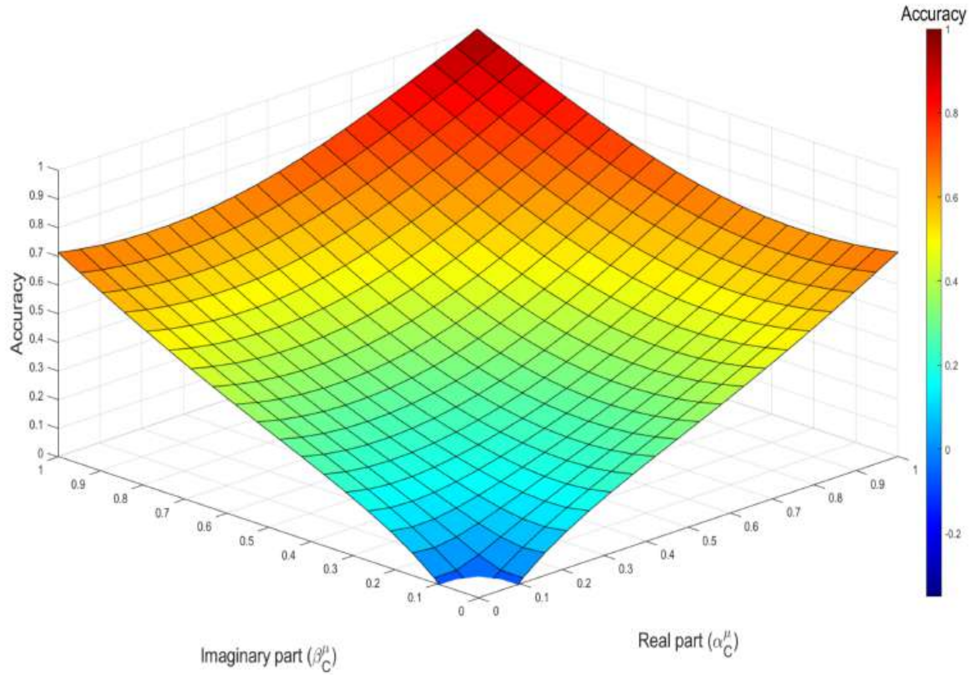
(B). $\xi^A(\mathfrak{C}_2 \cap \mathfrak{C}_1, \mathfrak{C}_3) + \xi^A(\mathfrak{C}_2 \cup \mathfrak{C}_1, \mathfrak{C}_3) = \xi^A(\mathfrak{C}_1, \mathfrak{C}_3) + \xi^A(\mathfrak{C}_2, \mathfrak{C}_3)$.

(C). $\xi^A(\mathfrak{C}_3, \mathfrak{C}_3^*) = \xi^A(\mathfrak{C}_3^*, \mathfrak{C}_3)$.

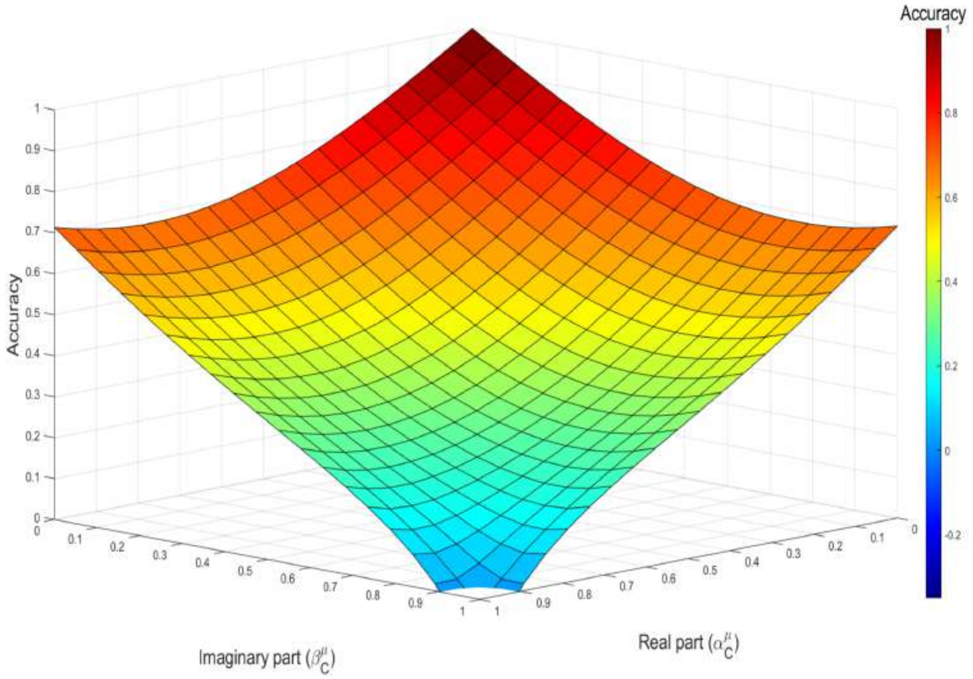
(D). $\xi^A(\mathfrak{C}_2, \mathfrak{C}_3) + \xi^A(\mathfrak{C}_2, \mathfrak{C}_3^*) = \xi^A({\mathfrak{C}_2}^*, \mathfrak{C}_3^*) + \xi^A({\mathfrak{C}_2}^*, \mathfrak{C}_3)$.

where $\mathfrak{C}_3, \mathfrak{C}_2, \mathfrak{C}_1 \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, and the notation * represents the complement.

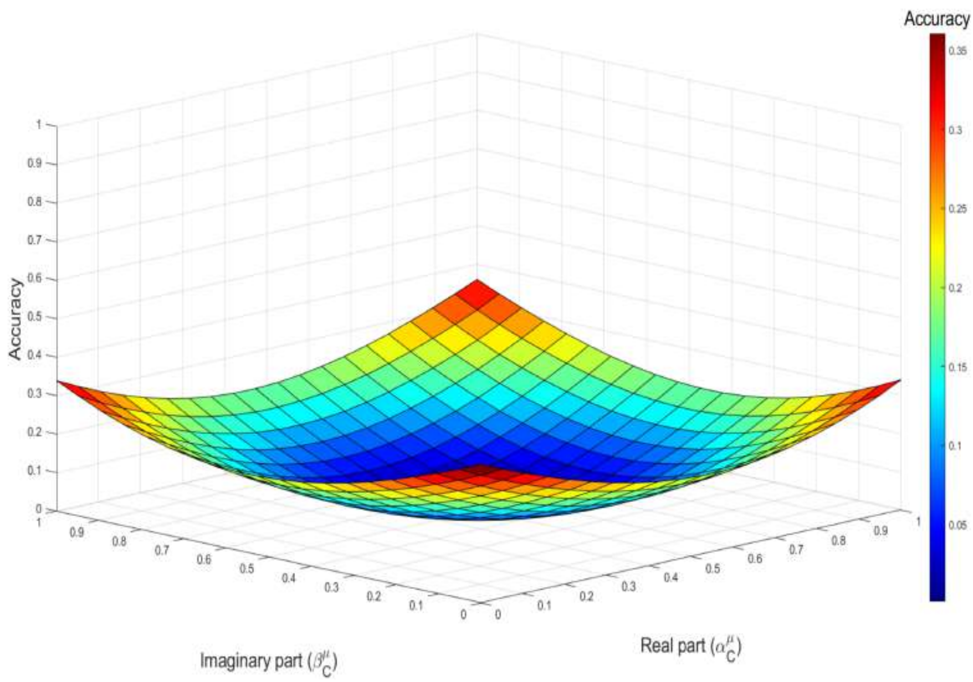
Example 5.2. Let us take an arbitrary $\mathcal{C}_{\mathcal{F}}^{V}$-number $\mathfrak{C} = \langle \alpha_{\mathfrak{C}}^{\mu}, \beta_{\mathfrak{C}}^{\mu}\rangle$. Take three $\mathcal{C}_{\mathcal{F}}^{V}$-numbers $\mathfrak{C}_1 = \langle 1, 1\rangle$, $\mathfrak{C}_2 = \langle 0, 0\rangle$, & $\mathfrak{C}_3 = \langle 0.5, 0.5\rangle$. The accuracy between the arbitrary $\mathcal{C}_{\mathcal{F}}^{V}$-number $\mathfrak{C}$ and the fixed $\mathcal{C}_{\mathcal{F}}^{V}$-numbers $\mathfrak{C}_1$, $\mathfrak{C}_2$, & $\mathfrak{C}_3$ is shown in Figure 3. From this figure, it is seen that the accuracy measure ξ^A satisfies all the properties of a credible measure.



(a) $\xi^{A}(\mathfrak{C}, \mathfrak{C}_1)$



(b) $\xi^{A}(\mathfrak{C}, \mathfrak{C}_2)$



(c) $\xi^{A}(\mathfrak{C}, \mathfrak{C}_3)$

Figure 3: Accuracy between $\mathcal{C}_{\mathcal{F}}^{V}$-number $\mathfrak{C}$ and $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$.

5.2 Standard Complex-valued Fuzzy Similarity measure with Properties

Take a nonempty universal set $\mathfrak{R}$ containing more than one element. For $\mathfrak{C}, \mathfrak{D} \in \mathcal{C}_{\mathcal{F}}^{V}$, a Similarity measure in $\mathcal{C}_{\mathcal{F}}^{V}$-context is defined as follows

$$\xi^{S}(\mathfrak{C}, \mathfrak{D}) = 1 - \big|\varpi_{\kappa}(\mathfrak{C}) - \varpi_{\kappa}(\mathfrak{D})\big|. \tag{5.2}$$

Now, the credibility of the measure ξ^{S} is checked.

Theorem 5.3. *For $\mathcal{C}_{\mathcal{F}}^{V}$-sets $\mathfrak{C}_1$ & $\mathfrak{C}_2$, a credible $\mathcal{C}_{\mathcal{F}}^{V}$-Similarity measure $\xi^{S} : \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \times \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R}) \to [0, 1]$ defined in Eq. (5.2) should meet the following requirements:*

(S1) For all $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, the value of $\xi^{S}(\mathfrak{C}_1, \mathfrak{C}_2)$ lies in between 0 & 1;

(S2) The values of $\xi^{S}(\mathfrak{C}_1, \mathfrak{C}_2)$ and $\xi^{S}(\mathfrak{C}_2, \mathfrak{C}_1)$ are equals for all $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$;

(S3) The value of $\xi^{S}(\mathfrak{C}_1, \mathfrak{C}_2)$ is equals to 1 if both $\mathcal{C}_{\mathcal{F}}^{V}$-sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equal;

(S4) If $\mathfrak{C}_3 \subseteq \mathfrak{C}_2$, & $\mathfrak{C}_2 \subseteq \mathfrak{C}_1$ then $\xi^{S}(\mathfrak{C}_2, \mathfrak{C}_3) \geq \xi^{S}(\mathfrak{C}_1, \mathfrak{C}_3)$, & $\xi^{S}(\mathfrak{C}_1, \mathfrak{C}_2) \geq \xi^{S}(\mathfrak{C}_1, \mathfrak{C}_3)$.

Proof. This theorem can easily be proved from Theorem 3.1 and by using Eq. (5.2). □

The measure ξ^{S} fulfills some properties. They are-

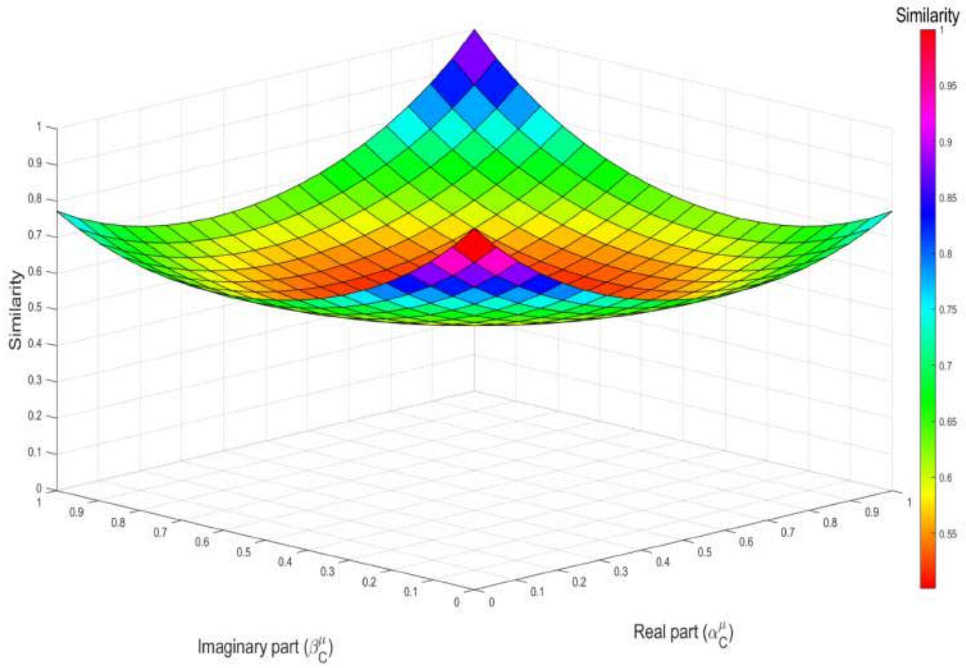
(A). $\xi^{S}({\mathfrak{C}_2}^{*}, {\mathfrak{C}_1}^{*}) = \xi^{S}(\mathfrak{C}_2, \mathfrak{C}_1)$.

(B). $\xi^{S}(\mathfrak{C}_2, {\mathfrak{C}_1}^{*}) = \xi^{S}({\mathfrak{C}_2}^{*}, \mathfrak{C}_1)$

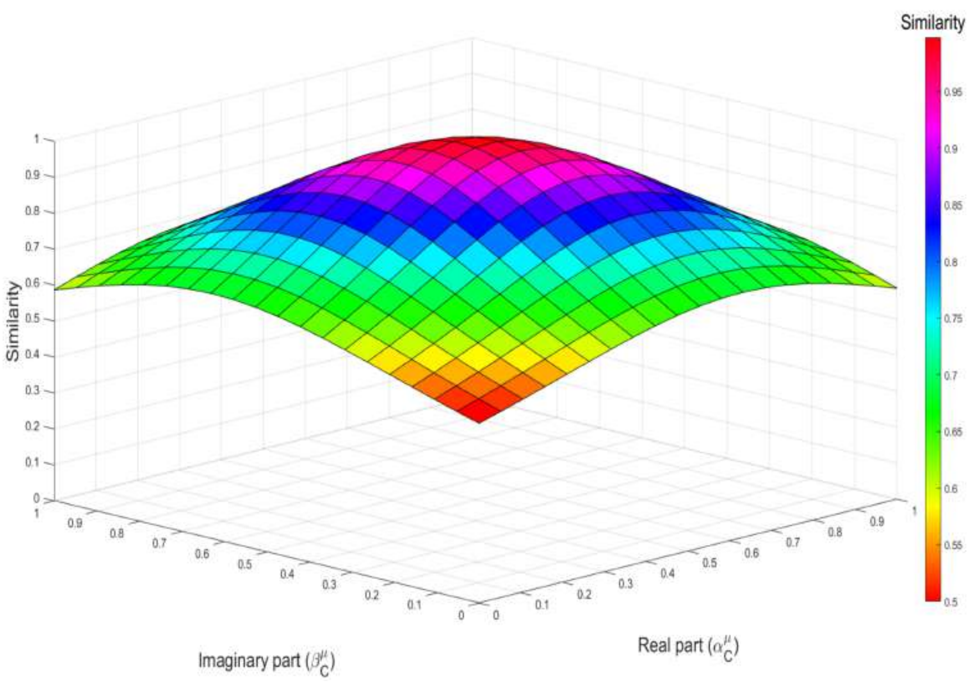
(C). For given Similarity measures ξ^{S}_1 & ξ^{S}_2, $\zeta\xi^{S}_1 + (1-\zeta)\xi^{S}_2$, $(0 \leq \zeta \leq 1)$ is again a Similarity measure.

where $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}_{\mathcal{F}}^{V}(\mathfrak{R})$, and the notation * represents the complement.

Example 5.4. Let us take an arbitrary $\mathcal{C}_{\mathcal{F}}^{V}$-number $\mathfrak{C} = \langle \alpha^{\mu}_{\mathfrak{C}}, \beta^{\mu}_{\mathfrak{C}} \rangle$. Take three $\mathcal{C}_{\mathcal{F}}^{V}$-numbers $\mathfrak{C}_1 = \langle 1, 1 \rangle$, $\mathfrak{C}_2 = \langle 0, 0 \rangle$, & $\mathfrak{C}_3 = \langle 0.5, 0.5 \rangle$. The similarity between the arbitrary $\mathcal{C}_{\mathcal{F}}^{V}$-number $\mathfrak{C}$ and the fixed $\mathcal{C}_{\mathcal{F}}^{V}$-numbers $\mathfrak{C}_1$, $\mathfrak{C}_2$, & $\mathfrak{C}_3$ is shown in Figure 4. From this figure, it is seen that the similarity measure ξ^{S} satisfies all the properties of a credible measure.



(a) $\xi^S(\mathfrak{C},\mathfrak{C}_1)$ or $\xi^S(\mathfrak{C},\mathfrak{C}_2)$



(b) $\xi^S(\mathfrak{C},\mathfrak{C}_3)$

Figure 4: Similarity between $\mathcal{C}^V_{\mathcal{F}}$-number $\mathfrak{C}$ and $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$.

5.3 Standard Complex-valued Fuzzy Dissimilarity measure with Properties

Take a nonempty universal set $\mathfrak{R}$ containing more than one element. For $\mathfrak{C},\mathfrak{D} \in \mathcal{C}^V_{\mathcal{F}}(\mathfrak{R})$, a Dissimilarity measure in $\mathcal{C}^V_{\mathcal{F}}$-context is defined as follows

$$\xi^D(\mathfrak{C},\mathfrak{D}) = \frac{\left|\varpi_\kappa(\mathfrak{C}) - \varpi_\kappa(\mathfrak{D})\right|}{1+\left|\varpi_\kappa(\mathfrak{C}) - \varpi_\kappa(\mathfrak{D})\right|}. \tag{5.3}$$

Now, the credibility of the measure ξ^D is checked.

Theorem 5.5. *For $\mathcal{C}^{V}_{\mathcal{F}}$-sets $\mathfrak{C}_1$ & $\mathfrak{C}_2$, a credible $\mathcal{C}^{V}_{\mathcal{F}}$-Dissimilarity measure $\xi^D : \mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R}) \times \mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R}) \to [0,1]$ defined in Eq. (5.3) should meet the following requirements:*

(S1) For all $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$, the value of $\xi^D(\mathfrak{C}_1, \mathfrak{C}_2)$ lies in between 0 & 1;

(S2) The values of $\xi^D(\mathfrak{C}_1, \mathfrak{C}_2)$ and $\xi^D(\mathfrak{C}_2, \mathfrak{C}_1)$ are equals for all $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$;

(S3) The value of $\xi^D(\mathfrak{C}_1, \mathfrak{C}_2)$ is equals to 0 if both $\mathcal{C}^{V}_{\mathcal{F}}$-sets $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are equal;

(S4) If $\mathfrak{C}_3 \subseteq \mathfrak{C}_2$, & $\mathfrak{C}_2 \subseteq \mathfrak{C}_1$ then $\xi^D(\mathfrak{C}_2, \mathfrak{C}_3) \le \xi^D(\mathfrak{C}_1, \mathfrak{C}_3)$, & $\xi^D(\mathfrak{C}_1, \mathfrak{C}_2) \le \xi^D(\mathfrak{C}_1, \mathfrak{C}_3)$.

Proof. This theorem can easily be proved from Theorem 3.1 and by using Eq. (5.3). □

The measure ξ^D fulfills some properties. They are-

(A). $\xi^D(\mathfrak{C}_2{}^*, \mathfrak{C}_1{}^*) = \xi^D(\mathfrak{C}_2, \mathfrak{C}_1)$.

(B). $\xi^D(\mathfrak{C}_2, \mathfrak{C}_1{}^*) = \xi^D(\mathfrak{C}_2{}^*, \mathfrak{C}_1)$

(C). For given Dissimilarity measures $\xi_1^{\mathfrak{D}}$ & $\xi_2^{\mathfrak{D}}$, $\zeta\xi_1^{\mathfrak{D}} + (1-\zeta)\xi_2^{\mathfrak{D}}$, $(0 \le \zeta \le 1)$ is again a Dissimilarity measure.

where $\mathfrak{C}_1, \mathfrak{C}_2 \in \mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$, and the notation * represents the complement.

Example 5.6. Let us take an arbitrary $\mathcal{C}^{V}_{\mathcal{F}}$-number $\mathfrak{C} = \langle \alpha^{\mu}_{\mathfrak{C}}, \beta^{\mu}_{\mathfrak{C}} \rangle$. Take three $\mathcal{C}^{V}_{\mathcal{F}}$-numbers $\mathfrak{C}_1 = \langle 1, 1 \rangle$, $\mathfrak{C}_2 = \langle 0, 0 \rangle$, & $\mathfrak{C}_3 = \langle 0.5, 0.5 \rangle$. The dissimilarity between the arbitrary $\mathcal{C}^{V}_{\mathcal{F}}$-number $\mathfrak{C}$ and the fixed $\mathcal{C}^{V}_{\mathcal{F}}$-numbers $\mathfrak{C}_1$, $\mathfrak{C}_2$, & $\mathfrak{C}_3$ is shown in Figure 5. From this figure, it is seen that the dissimilarity measure ξ^D satisfies all the properties of a credible measure [79].

Now, the implementation of the deducted measures is discussed in Pattern Recognition and Clustering issues.

6 Applications of the proposed measures

6.1 Pattern Recognition

The process of classifying an unknown pattern into certain known patterns is known as Pattern recognition. Many academics have studied pattern recognition techniques and have found success with the use of similarity, accuracy, and dissimilarity measures. Using examples, we will demonstrate how to apply the deducted measures to the Pattern recognition problem. Before starting, the methodology used for Pattern recognition is given below.

Problem: Let $\mathfrak{R} = \{\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_p\}$ is a set. It is taken that 'Γ' pattern are shown by $\mathcal{C}^{V}_{\mathcal{F}}$-sets $\mathfrak{C}_\gamma, (\gamma = 1, 2, \ldots, \Gamma)$. Let a pattern $\mathfrak{D}$ is to be fit in these patterns. We try to find the pattern that best matches with the given pattern $\mathfrak{D}$.

The methods used for pattern recognition are given below:

- **Accuracy measure method:** For a $\mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$-accuracy measure ξ^A, the given pattern $\mathfrak{D}$ is recognized in pattern $\mathfrak{C}_{\overline{\gamma}}$, where

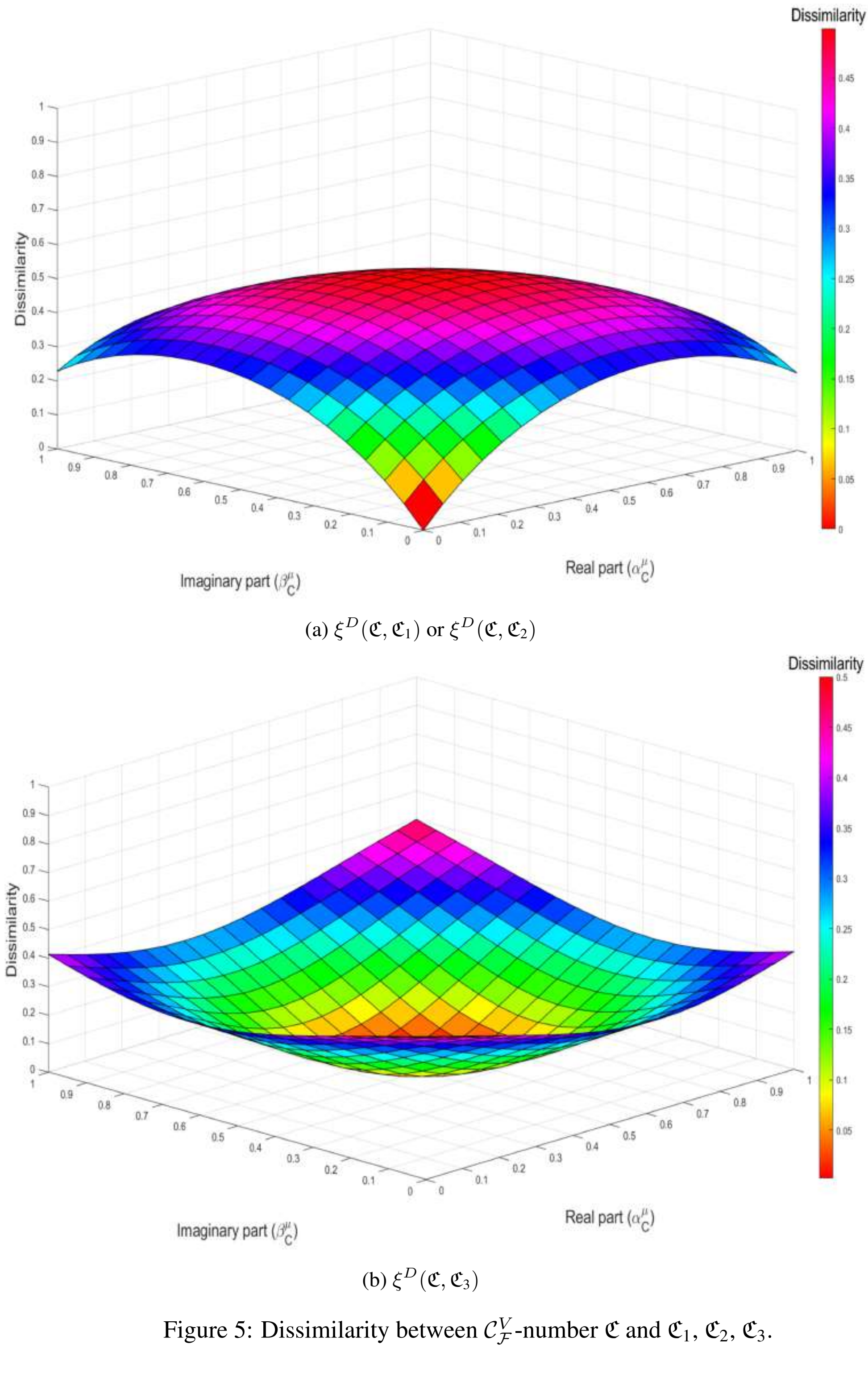
$$\xi^A(\mathfrak{D}, \mathfrak{C}_{\overline{\gamma}}) = \max_{\gamma=1,2,\ldots,\Gamma} (\xi^A(\mathfrak{D}, \mathfrak{C}_\gamma)).$$

- **Similarity measure method:** For a $\mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$-similarity measure ξ^S, the given pattern $\mathfrak{D}$ is recognized in pattern $\mathfrak{C}_{\overline{\gamma}}$, where

$$\xi^S(\mathfrak{D}, \mathfrak{C}_{\overline{\gamma}}) = \max_{\gamma=1,2,\ldots,\Gamma} (\xi^S(\mathfrak{D}, \mathfrak{C}_\gamma)).$$

- **Dissimilarity measure method:** For a $\mathcal{C}^{V}_{\mathcal{F}}(\mathfrak{R})$-dissimilarity measure ξ^D, the given pattern $\mathfrak{D}$ is recognized in pattern $\mathfrak{C}_{\overline{\gamma}}$, where

$$\xi^D(\mathfrak{D}, \mathfrak{C}_{\overline{\gamma}}) = \max_{\gamma=1,2,\ldots,\Gamma} (\xi^D(\mathfrak{D}, \mathfrak{C}_\gamma)).$$



(a) $\xi^D(\mathfrak{C}, \mathfrak{C}_1)$ or $\xi^D(\mathfrak{C}, \mathfrak{C}_2)$

(b) $\xi^D(\mathfrak{C}, \mathfrak{C}_3)$

Figure 5: Dissimilarity between $\mathcal{C}^V_{\mathcal{F}}$-number $\mathfrak{C}$ and $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$.

Similarity and dissimilarity measures are frequently used by academics to address problems with pattern recognition. Comparing accuracy measures to similarity and dissimilarity measures, however, may be a better way to address pattern recognition issues. Therefore, we address pattern identification problems by applying the deducted measures and comparing the result with some other known measures in the literature. Some similarity as well as dissimilarity measures in $\mathcal{C}^V_{\mathcal{F}}$-context are given in Table 4 and Table 5 respectively.

Table 4: Some Similarity measures in $\mathcal{C}_{\mathcal{F}}^{V}$-context.

Sr. No.	Similarity measures	Authors
1	$s_1^A(\mathfrak{C},\mathfrak{D}) = 1 - \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \sin\left\{\frac{\pi}{2}\left[\max\left(\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)\right]\right\}$	Ali [67]
2	$s_2^A(\mathfrak{C},\mathfrak{D}) = 1 - \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \sin\left\{\frac{\pi}{4}\left[\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert + \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right]\right\}$	Ali [67]
3	$s_3^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \cos\left\{\frac{\pi}{2}\left[\max\left(\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)\right]\right\}$	Ali [67]
4	$s_4^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \cos\left\{\frac{\pi}{4}\left[\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert + \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right]\right\}$	Ali [67]
5	$s_5^A(\mathfrak{C},\mathfrak{D}) = 1 - \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \tan\left\{\frac{\pi}{4}\left[\max\left(\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)\right]\right\}$	Ali [67]
6	$s_6^A(\mathfrak{C},\mathfrak{D}) = 1 - \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \tan\left\{\frac{\pi}{8}\left[\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert + \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right]\right\}$	Ali [67]
7	$s_7^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma} \cot\left\{\frac{\pi}{4} + \frac{\pi}{4}\left[\max\left(\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma) - P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)\right]\right\}$	Ali [67]

Table 5: Some Distance measures in $\mathcal{C}_{\mathcal{F}}^{V}$-context.

Sr. No.	Distance measures	Authors
1	$d^{Zh}(\mathfrak{C},\mathfrak{D}) = \max\left(\sup\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert, \sup\frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)$	Zhang et al. [80]
2	$d_1^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{2}\sum_{\gamma=1}^{\Gamma}\left(\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert + \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)$	Alkouri & Salleh [81]
3	$d_2^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{2\Gamma}\sum_{\gamma=1}^{\Gamma}\left(\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert + \frac{\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert}{2\pi}\right)$	Alkouri & Salleh [81]
4	$d_3^A(\mathfrak{C},\mathfrak{D}) = \left[\frac{1}{2}\sum_{\gamma=1}^{\Gamma}\left(\left(M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right)^2 + \frac{(P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma))^2}{4\pi^2}\right)\right]^{\frac{1}{2}}$	Alkouri & Salleh [81]
5	$d_4^A(\mathfrak{C},\mathfrak{D}) = \left[\frac{1}{2\Gamma}\sum_{\gamma=1}^{\Gamma}\left(\left(M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right)^2 + \frac{(P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma))^2}{4\pi^2}\right)\right]^{\frac{1}{2}}$	Alkouri & Salleh [81]
6	$d_5^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{2}\sum_{\gamma=1}^{\Gamma}\left(\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert^p + \frac{\min(\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, 2\pi-\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert)^p}{\pi^p}\right)^{\frac{1}{p}}$	Alkouri & Salleh [81]
7	$d_6^A(\mathfrak{C},\mathfrak{D}) = \frac{1}{2\Gamma}\sum_{\gamma=1}^{\Gamma}\left(\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma) - M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert^p + \frac{\min(\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert, 2\pi-\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\rvert)^p}{\pi^p}\right)^{\frac{1}{p}}$	Alkouri & Salleh [81]
8	$d^{Ze}(\mathfrak{C},\mathfrak{D}) = \frac{1}{\Gamma}\sum_{\gamma=1}^{\Gamma}\left(\frac{\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma)-M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert}{1+\left\lvert M_{\mathfrak{C}}(\mathfrak{r}_\gamma)-M_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert} + \frac{\left\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert}{2\pi+\left\lvert P_{\mathfrak{C}}(\mathfrak{r}_\gamma)-P_{\mathfrak{D}}(\mathfrak{r}_\gamma)\right\rvert}\right)$	Zeeshan et al. [64]

An example of pattern recognition is given below to check the reliability of the deducted measures and other measures.

Example 6.1. Take a universal set $\mathfrak{R} = \{10, 8, 6, 4, 2\}$. For this set $\mathfrak{R}$, define five patterns given below

$$\mathfrak{C}_1 = \{\langle 10, 0.6 + 0.6\,j\rangle, \langle 8, 0.5 + 0.4\,j\rangle, \langle 6, 0.6 + 0.5\,j\rangle, \langle 4, 0.6 + 0.7\,j\rangle, \langle 2, 0.5 + 0.8\,j\rangle\}\,;$$
$$\mathfrak{C}_2 = \{\langle 10, 0.8 + 0.4\,j\rangle, \langle 8, 0.5 + 0.3\,j\rangle, \langle 6, 0.7 + 0.4\,j\rangle, \langle 4, 0.6 + 0.4\,j\rangle, \langle 2, 0.4 + 0.6\,j\rangle\}\,;$$
$$\mathfrak{C}_3 = \{\langle 10, 0.6 + 0.5\,j\rangle, \langle 8, 0.5 + 0.3\,j\rangle, \langle 6, 0.7 + 0.2\,j\rangle, \langle 4, 0.8 + 0.4\,j\rangle, \langle 2, 0.6 + 0.5\,j\rangle\}\,;$$
$$\mathfrak{C}_4 = \{\langle 10, 0.4 + 0.3\,j\rangle, \langle 8, 0.5 + 0.4\,j\rangle, \langle 6, 0.6 + 0.5\,j\rangle, \langle 4, 0.2 + 0.5\,j\rangle, \langle 2, 0.3 + 0.4\,j\rangle\}\,;$$
$$\mathfrak{C}_5 = \{\langle 10, 0.5 + 0.4\,j\rangle, \langle 8, 0.2 + 0.8\,j\rangle, \langle 6, 0.4 + 0.2\,j\rangle, \langle 4, 0.6 + 0.5\,j\rangle, \langle 2, 0.5 + 0.6\,j\rangle\}\,.$$

Consider the fixed pattern is

$$\mathfrak{C} = \{\langle 10, 0.5 + 0.3\,j\rangle, \langle 8, 0.4 + 0.2\,j\rangle, \langle 6, 0.0 + 0.5\,j\rangle, \langle 4, 0.3 + 0.7\,j\rangle, \langle 2, 0.5 + 0.5\,j\rangle\}\,.$$

Now, this example is solved by methods given above for the best matching of pattern $\mathfrak{C}$ with patterns $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$. Some Similarity and Distance measures are given in Table 4 & Table 5 and the deducted measures are used to solve this example. The output is shown in Table 6 theoretically and in Figure 6 & Figure 7 graphically.

Table 6: Pattern recognition by using previously known measures and the proposed measures.

Measures	$(\mathfrak{C}, \mathfrak{C}_1)$	$(\mathfrak{C}, \mathfrak{C}_2)$	$(\mathfrak{C}, \mathfrak{C}_3)$	$(\mathfrak{C}, \mathfrak{C}_4)$	$(\mathfrak{C}, \mathfrak{C}_5)$	Recognized Pattern
s_1^A (Ali [67])	0.651	0.733	0.762	0.696	0.769	$\mathfrak{C}_5$
s_2^A (Ali [67])	0.777	0.826	0.823	0.813	0.845	$\mathfrak{C}_5$
s_3^A (Ali [67])	0.935	0.947	0.968	0.947	0.955	$\mathfrak{C}_3$
s_4^A (Ali [67])	0.973	0.977	0.980	0.979	0.979	$\mathfrak{C}_5$
s_5^A (Ali [67])	0.819	0.860	0.879	0.843	0.879	Not Recognized
s_6^A (Ali [67])	0.887	0.911	0.910	0.905	0.921	$\mathfrak{C}_5$
s_7^A (Ali [67])	0.696	0.766	0.786	0.733	0.796	$\mathfrak{C}_5$
d^{Zh} (Zhang et al. [80])	0.281	0.311	0.228	0.281	0.377	$\mathfrak{C}_3$
d_1^A (Alkouri & Salleh [81])	0.716	0.562	0.569	0.602	0.503	$\mathfrak{C}_5$
d_2^A (Alkouri & Salleh [81])	0.143	0.112	0.114	0.120	0.101	$\mathfrak{C}_5$
d_3^A (Alkouri & Salleh [81])	0.383	0.353	0.307	0.344	0.324	$\mathfrak{C}_3$
d_4^A (Alkouri & Salleh [81])	0.172	0.158	0.137	0.154	0.145	$\mathfrak{C}_3$
d_5^A (Alkouri & Salleh [81])	0.579	0.521	0.522	0.504	0.507	$\mathfrak{C}_4$
d_6^A (Alkouri & Salleh [81])	0.116	0.104	0.104	0.101	0.101	Not Recognized
d^{Ze} (Zeeshan et al. [64])	0.239	0.185	0.196	0.202	0.168	$\mathfrak{C}_5$
ξ^A (Proposed)	0.111	0.121	0.140	0.128	0.152	$\mathfrak{C}_5$
ξ^S (Proposed)	0.906	0.902	0.920	0.920	0.932	$\mathfrak{C}_5$
ξ^D (Proposed)	0.094	0.098	0.081	0.080	0.068	$\mathfrak{C}_5$

p=9 is used in the distance measures d_5^A & d_6^A.

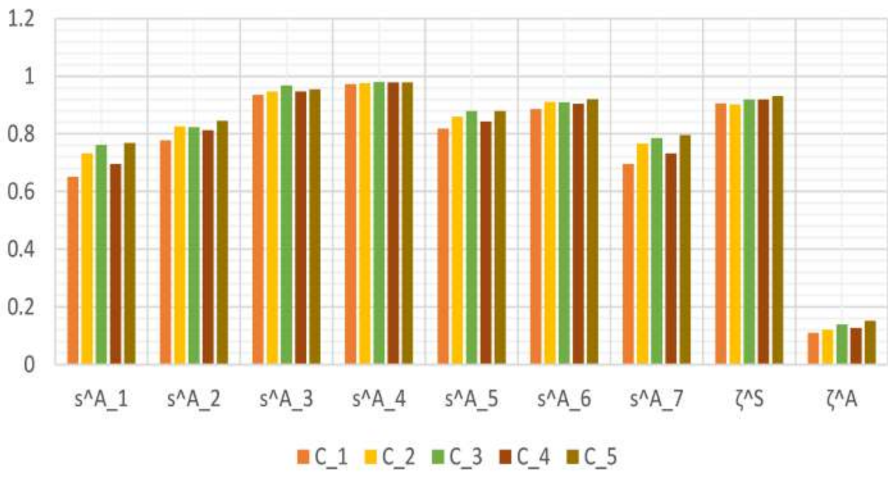


Figure 6: Pattern recognition by using previously known Similarity measures and the proposed measures.

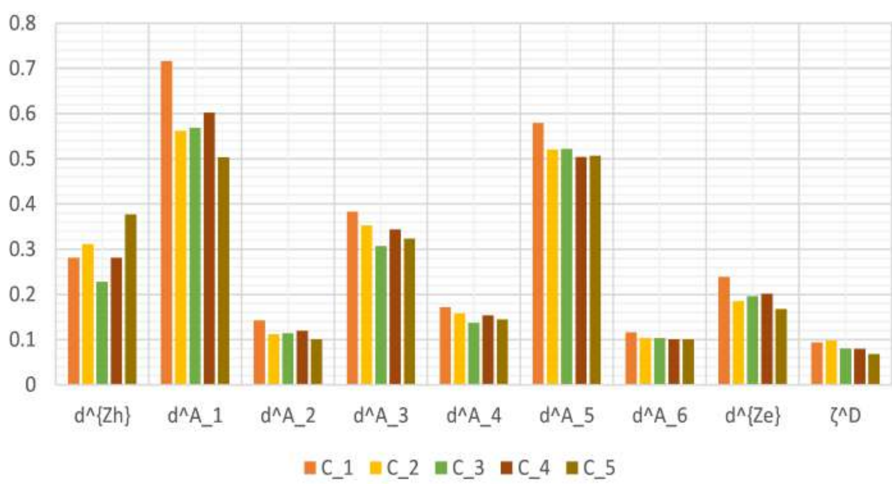


Figure 7: Pattern recognition by using previously known Dissimilarity measures and the proposed measures.

From Table 6, we can identify that the fixed pattern $\mathfrak{C}$ is recognized in pattern $\mathfrak{C}_5$ by proposed measures. Also, the fixed pattern $\mathfrak{C}$ is recognized in pattern $\mathfrak{C}_5$ by most of the previously known measures. Thus, we may conclude that the proposed measures can successfully address pattern recognition concerns.

One more example is given below.

Example 6.2. Take a universal set $\mathfrak{R} = \{9,7,6,3,1\}$. For this set $\mathfrak{R}$, define five patterns given below

$$\mathfrak{C}_1 = \{\langle 9, 0.5+0.5\,j\rangle, \langle 7, 0.7+0.1\,j\rangle, \langle 5, 0.1+0.2\,j\rangle, \langle 3, 0.2+0.4\,j\rangle, \langle 1, 0.4+0.4\,j\rangle\}\,;$$
$$\mathfrak{C}_2 = \{\langle 9, 0.4+0.2\,j\rangle, \langle 7, 0.2+0.3\,j\rangle, \langle 5, 0.3+0.5\,j\rangle, \langle 3, 0.5+0.3\,j\rangle, \langle 1, 0.4+0.4\,j\rangle\}\,;$$
$$\mathfrak{C}_3 = \{\langle 9, 0.5+0.1\,j\rangle, \langle 7, 0.4+0.5\,j\rangle, \langle 5, 0.2+0.4\,j\rangle, \langle 3, 0.1+0.8\,j\rangle, \langle 1, 0.3+0.5\,j\rangle\}\,;$$
$$\mathfrak{C}_4 = \{\langle 9, 0.5+0.6\,j\rangle, \langle 7, 0.2+0.2\,j\rangle, \langle 5, 0.1+0.3\,j\rangle, \langle 3, 0.2+0.3\,j\rangle, \langle 1, 0.5+0.2\,j\rangle\}\,;$$
$$\mathfrak{C}_5 = \{\langle 9, 0.5+0.5\,j\rangle, \langle 7, 0.6+0.8\,j\rangle, \langle 5, 0.1+0.2\,j\rangle, \langle 3, 0.3+0.4\,j\rangle, \langle 1, 0.5+0.5\,j\rangle\}\,.$$

Consider the fixed pattern is

$$\mathfrak{C} = \{\langle 9, 0.8+0.5\,j\rangle, \langle 7, 0.3+0.0\,j\rangle, \langle 5, 0.6+0.2\,j\rangle, \langle 3, 0.1+0.6\,j\rangle, \langle 1, 0.0+0.3\,j\rangle\}\,.$$

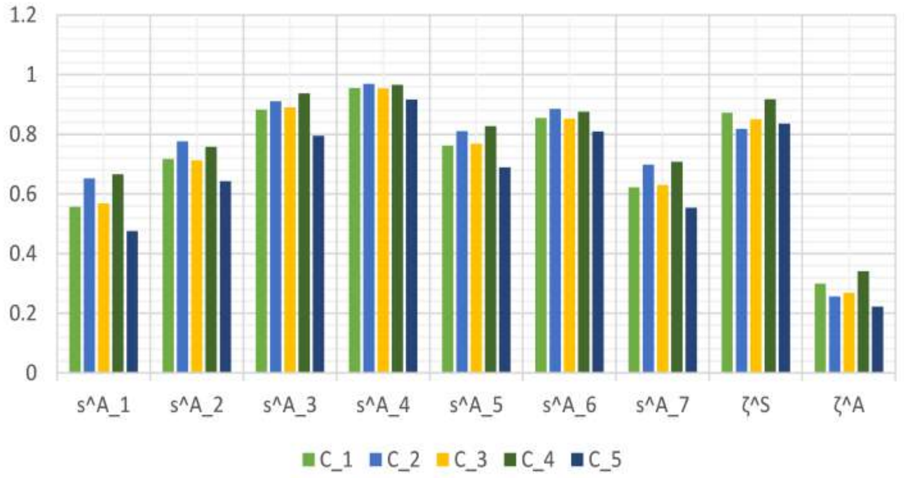


Figure 8: Pattern recognition by using previously known Similarity measures and the proposed measures.

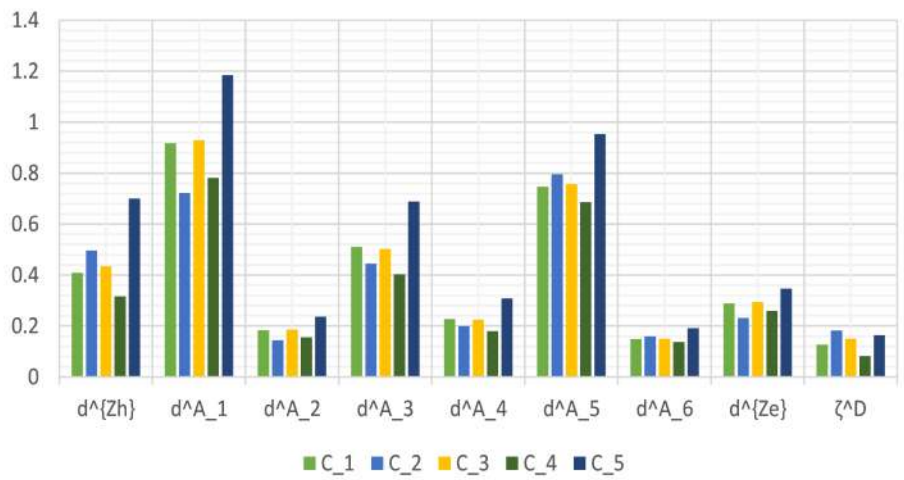


Figure 9: Pattern recognition by using previously known Dissimilarity measures and the proposed measures.

Now, this example is solved by methods given above for the best matching of pattern $\mathfrak{C}$ with patterns $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5$. Some Similarity and Distance measures are given in Table 4 & Table 5 and the deducted measures are used to solve this example. The output is shown in Table 7 theoretically and in Figure 8 & Figure 9 graphically.

Table 7: Pattern recognition by using previously known measures and the proposed measures.

Measures	$(\mathfrak{C}, \mathfrak{C}_1)$	$(\mathfrak{C}, \mathfrak{C}_2)$	$(\mathfrak{C}, \mathfrak{C}_3)$	$(\mathfrak{C}, \mathfrak{C}_4)$	$(\mathfrak{C}, \mathfrak{C}_5)$	Recognized Pattern
s_1^A (Ali [67])	0.557	0.652	0.568	0.666	0.476	$\mathfrak{C}_4$
s_2^A (Ali [67])	0.717	0.777	0.713	0.758	0.643	$\mathfrak{C}_2$
s_3^A (Ali [67])	0.884	0.911	0.890	0.937	0.795	$\mathfrak{C}_4$
s_4^A (Ali [67])	0.955	0.969	0.954	0.966	0.916	$\mathfrak{C}_2$
s_5^A (Ali [67])	0.762	0.811	0.769	0.827	0.690	$\mathfrak{C}_4$
s_6^A (Ali [67])	0.855	0.886	0.853	0.876	0.810	$\mathfrak{C}_2$
s_7^A (Ali [67])	0.622	0.698	0.630	0.708	0.554	$\mathfrak{C}_4$
d^{Zh} (Zhang et al. [80])	0.409	0.496	0.434	0.316	0.700	$\mathfrak{C}_4$
d_1^A (Alkouri & Salleh [81])	0.918	0.722	0.929	0.781	1.185	$\mathfrak{C}_4$
d_2^A (Alkouri & Salleh [81])	0.184	0.144	0.186	0.156	0.237	$\mathfrak{C}_2$
d_3^A (Alkouri & Salleh [81])	0.510	0.445	0.502	0.403	0.689	$\mathfrak{C}_4$
d_4^A (Alkouri & Salleh [81])	0.228	0.199	0.225	0.180	0.308	$\mathfrak{C}_4$
d_5^A (Alkouri & Salleh [81])	0.747	0.795	0.757	0.686	0.953	$\mathfrak{C}_4$
d_6^A (Alkouri & Salleh [81])	0.149	0.159	0.151	0.137	0.191	$\mathfrak{C}_4$
d^{Ze} (Zeeshan et al. [64])	0.289	0.231	0.295	0.260	0.347	$\mathfrak{C}_2$
ξ^A (Proposed)	0.300	0.257	0.269	0.342	0.223	$\mathfrak{C}_4$
ξ^S (Proposed)	0.873	0.818	0.850	0.918	0.836	$\mathfrak{C}_4$
ξ^D (Proposed)	0.127	0.183	0.150	0.082	0.164	$\mathfrak{C}_4$

p=9 is used in the distance measures d_5^A & d_6^A.

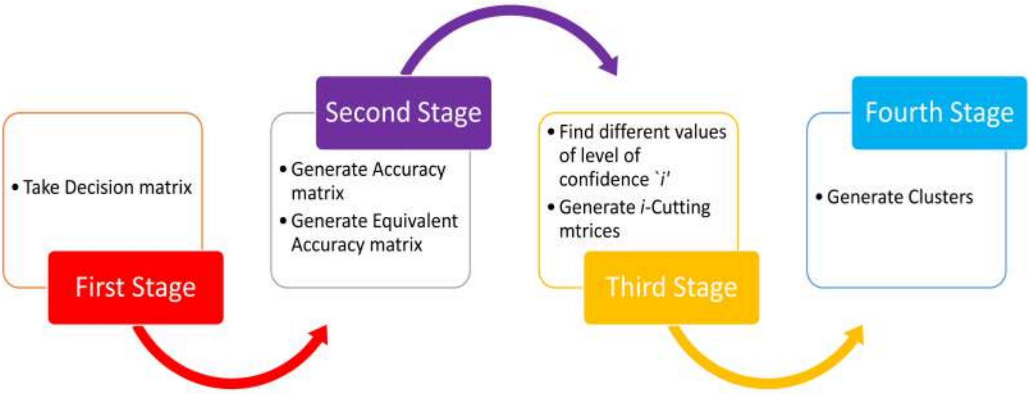


Figure 10: Stages of the proposed method.

From Table 7, we can identify that the fixed pattern $\mathfrak{C}$ is recognized in pattern $\mathfrak{C}_5$ by proposed measures. Also, the fixed pattern $\mathfrak{C}$ is recognized in pattern $\mathfrak{C}_5$ by most of the previously known measures. Thus, we may conclude that the deducted measures are reliable for pattern recognition concerns.

6.2 Cluster Analysis

Clustering classifies data by splitting a large number of data items into groups based on their similarities. In clustering issues, we strive to match other possibilities in a group such that they are more similar to the other alternatives in that cluster but more distinct from the alternates in different clusters. As a result, an effort is taken to demonstrate the human propensity to organize related experiences into consistent groupings. There may be many reasons for selecting and developing lessons in this manner. Clustering is an approach for detecting patterns in a collection of unsorted items that did not exist. Rico et al. [82] propose a clustering approach based on similarity measures. However, in this publication, the proposed accuracy measure is used. Take some definitions given below.

Definition 6.3. Take a collection of $\mathcal{C}^V_{\mathcal{F}}$-sets represented by $\{\mathfrak{C}_\gamma\}_{\gamma=1}^{\Gamma}$. For define knowledge measure ϖ_κ and accuracy measure ξ^A, a Symmetrical Accuracy matrix is represented by $\mathfrak{X} = \{\mathfrak{x}_{\gamma\delta}\}_{\Gamma\times\Gamma}$, where $\mathfrak{x}_{\gamma\delta} = \frac{\xi^A(\mathfrak{C}_\gamma,\mathfrak{C}_\delta)}{\varpi_\kappa(\mathfrak{C}_\gamma)}$ for $\gamma = \delta$ and $\mathfrak{x}_{\gamma\delta} = \xi^A(\mathfrak{C}_\gamma, \mathfrak{C}_\delta)$ for $\gamma \neq \delta$. Also, all the entries in this matrix lie in the interval $[0, 1]$ with unity at its diagonal.

Definition 6.4. The Composition matrix is defined by $\mathfrak{X}^2 = \mathfrak{X} \otimes \mathfrak{X} = \{\mathfrak{e}_{\gamma\delta}\}_{\gamma\times\gamma}$, where $\mathfrak{e}_{\gamma\delta} = \max_\beta \{\min\{\mathfrak{x}_{\gamma\beta}, \mathfrak{x}_{\beta\delta}\}\}$; $\gamma, \delta = 1, 2, \ldots, \Gamma$ for given Symmetrical accuracy matrix $\mathfrak{X}$. Also, $\mathfrak{X}^2 \subseteq \mathfrak{X}$, i.e., $\mathfrak{e}_{\gamma\delta} \leq \mathfrak{x}_{\gamma\delta}$; $\forall\, \gamma, \delta = 1, 2, \ldots, \Gamma$.

Definition 6.5. After taking compositions again and again, a stage $(\mathfrak{X}^{2^n})$ is arrived at which $\mathfrak{X}^{2^n} = \mathfrak{X}^{2^{n+1}}$. Also, this matrix $(\mathfrak{X}^{2^n})$ is again an Equivalent accuracy matrix.

Definition 6.6. (Rico et al. [82]) For any Equivalent accuracy matrix $\mathfrak{X} = \{\mathfrak{x}_{\gamma\delta}\}_{\Gamma\times\Gamma}$, the different values of the level of confidence 'i' are selected from the interval $[0, 1]$. For these values of level of confidence (i), different i-cutting matrices $\mathfrak{U}_i = \{\mathfrak{u}_{\gamma\delta}\}_{\Gamma\times\Gamma}$ are generated, where $\mathfrak{u}_{\gamma\delta} = 1$ for $\mathfrak{x}_{\gamma\delta} \geq i$, and $\mathfrak{u}_{\gamma\delta} = 0$ for $\mathfrak{x}_{\gamma\delta} < i$.

The proposed Clustering method

The proposed Clustering method contains four stages which are represented by Figure 10. Let us consider a clustering problem of Γ alternatives represented by $\{\mathfrak{C}_\gamma\}_{\gamma=1}^{\Gamma}$. Let this problem contains Δ criteria represented by $\{\mathfrak{D}_\delta\}_{\delta=1}^{\Delta}$ There are the following stages in the proposed method:

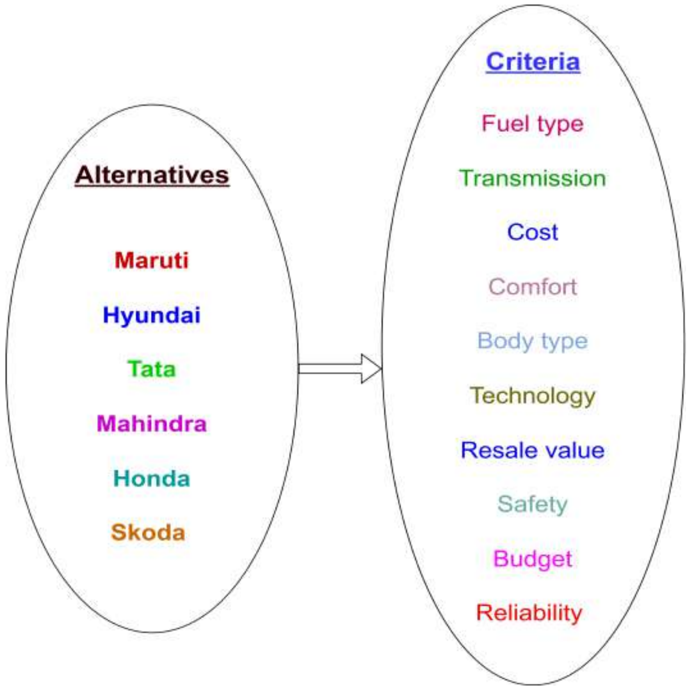


Figure 11: Alternatives & Criteria.

Stage 1. Take a decision matrix and generate Accuracy matrix: Take a decision matrix $\mathfrak{Y} = \{\mathfrak{y}_{\gamma\delta}\}_{\gamma\times\delta}$ in $\mathcal{C}^{V}_{\mathcal{F}}$-context. Generate an Accuracy matrix $\mathfrak{X} = \{\mathfrak{x}_{\gamma\delta}\}_{\gamma\times\gamma}$ by using Definition 6.3. The resultant matrix $\mathfrak{X}$ is symmetrical.

Stage 2. Generate an Equivalent matrix: After applying composition on the accuracy matrix $\mathfrak{X}$ again and again, a stage $(\mathfrak{X}^{2^n})$ is arrived at which $\mathfrak{X}^{2^n} = \mathfrak{X}^{2^{n+1}}$ for some $n \in \mathbb{N}$. This matrix $\mathfrak{X}^{2^n}$ also behaves like an accuracy matrix and is called an Equivalent matrix.

Stage 3. Find different values of levels of Confidence & generate $\imath$-cutting matrices: Select different values of level of confidence '$\imath$' from the Equivalent matrix $\mathfrak{X}^{2^n}$ that all lies in the interval $[0, 1]$. For these values of level of confidence, generate $\imath$-cutting matrix $\mathfrak{U}_{\imath} = \{\mathfrak{u}_{\gamma\delta}\}_{\Gamma\times\Gamma}$ by using Definition 6.6.

Stage 4. Generate Clusters: The alternatives $\mathfrak{C}_\gamma$ and $\mathfrak{C}_\delta$ lies a cluster if the columns γ^{th} & δ^{th} are equal in a $\imath$-cutting matrix for a particular value of levels of Confidence. The same result can be carried out for rows. In the same way, other clusters are generated.

An example of Clustering is taken which is solved by using the proposed method.

Example 6.7. Take a problem of clustering in which there are six cars companies taken as alternatives, represented by Maruti ($\mathfrak{C}_1$), Hyundai ($\mathfrak{C}_2$), Tata ($\mathfrak{C}_3$), Mahindra ($\mathfrak{C}_4$), Honda ($\mathfrak{C}_5$), and Skoda ($\mathfrak{C}_6$). Let their criteria or feathers are represented by- Fuel type ($\mathfrak{D}_1$), Transmission ($\mathfrak{D}_2$), Cost ($\mathfrak{D}_3$), Comfort ($\mathfrak{D}_4$), Body type ($\mathfrak{D}_5$), Technology ($\mathfrak{D}_6$), Resale value ($\mathfrak{D}_7$), Safety ($\mathfrak{D}_8$), Budget ($\mathfrak{D}_9$), and Reliability ($\mathfrak{D}_{10}$). These car companies & their feathers are represented by Figure 11. The data for this problem is given in Table 8 & Table 9.

The given clustering problem is solved by the suggested method. There are the following stages in the proposed method:

Stage 1. Take a decision matrix and generate Accuracy matrix: Let us take decision matrix $\mathfrak{Y}$ is represented by Table 8 & Table 9 in $\mathcal{C}^{V}_{\mathcal{F}}$-context. By using Definition 6.3, the Accuracy

Table 8: Dataset taken (For criteria $\mathfrak{D}_1$, $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, and $\mathfrak{D}_5$).

Car company	$\mathfrak{D}_1$	$\mathfrak{D}_2$	$\mathfrak{D}_3$	$\mathfrak{D}_4$	$\mathfrak{D}_5$
Maruti ($\mathfrak{C}_1$)	$0.2+j\ 0.6$	$0.5+j\ 0.7$	$0.7+j\ 0.2$	$0.8+j\ 0.5$	$0.6+j\ 0.2$
Hyundai ($\mathfrak{C}_2$)	$0.3+j\ 0.6$	$0.5+j\ 0.7$	$0.4+j\ 0.0$	$0.5+j\ 0.2$	$0.6+j\ 0.4$
Tata ($\mathfrak{C}_3$)	$0.6+j\ 0.3$	$0.8+j\ 0.4$	$0.7+j\ 0.5$	$0.0+j\ 0.6$	$0.1+j\ 0.9$
Mahindra ($\mathfrak{C}_4$)	$0.7+j\ 0.3$	$0.5+j\ 0.0$	$0.4+j\ 0.8$	$0.6+j\ 0.1$	$0.5+j\ 0.4$
Honda ($\mathfrak{C}_5$)	$0.6+j\ 0.6$	$0.5+j\ 0.4$	$0.7+j\ 0.5$	$0.1+j\ 0.7$	$0.1+j\ 0.8$
Skoda ($\mathfrak{C}_6$)	$0.5+j\ 0.6$	$0.3+j\ 0.5$	$0.0+j\ 0.7$	$0.4+j\ 0.8$	$0.5+j\ 0.4$

Table 9: Dataset taken (For criteria $\mathfrak{D}_6$, $\mathfrak{D}_7$, $\mathfrak{D}_8$, $\mathfrak{D}_9$, and $\mathfrak{D}_{10}$).

Car company	$\mathfrak{D}_6$	$\mathfrak{D}_7$	$\mathfrak{D}_8$	$\mathfrak{D}_9$	$\mathfrak{D}_{10}$
Maruti ($\mathfrak{C}_1$)	$0.3+j\ 0.1$	$0.1+j\ 0.0$	$0.4+j\ 0.3$	$0.5+j\ 0.2$	$0.2+j\ 0.3$
Hyundai ($\mathfrak{C}_2$)	$0.2+j\ 0.6$	$0.4+j\ 0.8$	$0.2+j\ 0.9$	$0.3+j\ 0.1$	$0.2+j\ 0.2$
Tata ($\mathfrak{C}_3$)	$0.6+j\ 0.8$	$0.5+j\ 0.7$	$0.4+j\ 0.5$	$0.7+j\ 0.2$	$0.8+j\ 0.0$
Mahindra ($\mathfrak{C}_4$)	$0.4+j\ 0.5$	$0.5+j\ 0.3$	$0.6+j\ 0.7$	$0.2+j\ 0.8$	$0.3+j\ 0.9$
Honda ($\mathfrak{C}_5$)	$0.0+j\ 0.9$	$0.3+j\ 0.5$	$0.5+j\ 0.3$	$0.4+j\ 0.2$	$0.6+j\ 0.1$
Skoda ($\mathfrak{C}_6$)	$0.7+j\ 0.3$	$0.5+j\ 0.0$	$0.6+j\ 0.4$	$0.5+j\ 0.5$	$0.4+j\ 0.6$

matrix $\mathfrak{X}$ is given in Eq. (6.1).

$$\mathfrak{X} = \begin{pmatrix} 1 & 0.2415 & 0.2076 & 0.1980 & 0.2353 & 0.1980 \\ 0.2415 & 1 & 0.2262 & 0.1901 & 0.2219 & 0.1736 \\ 0.2076 & 0.2262 & 1 & 0.1968 & 0.2181 & 0.1649 \\ 0.1980 & 0.1901 & 0.1968 & 1 & 0.1962 & 0.1614 \\ 0.2353 & 0.2219 & 0.2181 & 0.1962 & 1 & 0.1718 \\ 0.1980 & 0.1736 & 0.1649 & 0.1614 & 0.1718 & 1 \end{pmatrix}. \tag{6.1}$$

The resultant matrix $\mathfrak{X}$ is symmetrical.

Stage 2. Generate an Equivalent matrix: After applying composition on the accuracy matrix $\mathfrak{X}$ again and again, a stage $(\mathfrak{X}^{2^n})$ is arrived at which $\mathfrak{X}^{2^n} = \mathfrak{X}^{2^{n+1}}$ for some $n \in \mathbb{N}$. This matrix $\mathfrak{X}^{2^n}$ also behaves like an accuracy matrix and is called an Equivalent matrix.

$$\mathfrak{X}^2 = \mathfrak{X} \; o \; \mathfrak{X} = \begin{pmatrix} 1 & 0.2415 & 0.2262 & 0.1980 & 0.2353 & 0.1980 \\ 0.2415 & 1 & 0.2262 & 0.1980 & 0.2353 & 0.1980 \\ 0.2262 & 0.2262 & 1 & 0.1980 & 0.2219 & 0.1980 \\ 0.1980 & 0.1980 & 0.1980 & 1 & 0.1980 & 0.1980 \\ 0.2353 & 0.2353 & 0.2219 & 0.1980 & 1 & 0.1980 \\ 0.1980 & 0.1980 & 0.1980 & 0.1980 & 0.1980 & 1 \end{pmatrix},$$

$$\mathfrak{X}^4 = \mathfrak{X}^2 \; o \; \mathfrak{X}^2 = \begin{pmatrix} 1 & 0.2415 & 0.2262 & 0.1980 & 0.2353 & 0.1979 \\ 0.2415 & 1 & 0.2262 & 0.1980 & 0.2353 & 0.1979 \\ 0.2262 & 0.2262 & 1 & 0.1980 & 0.2262 & 0.1979 \\ 0.1980 & 0.1980 & 0.1980 & 1 & 0.1980 & 0.1979 \\ 0.2353 & 0.2353 & 0.2262 & 0.1980 & 1 & 0.1979 \\ 0.1979 & 0.1979 & 0.1979 & 0.1979 & 0.1979 & 1 \end{pmatrix},$$

$$\mathfrak{X}^8 = \mathfrak{X}^4 \, o \, \mathfrak{X}^4 = \begin{pmatrix} 1 & 0.2415 & 0.2262 & 0.1980 & 0.2353 & 0.1979 \\ 0.2415 & 1 & 0.2262 & 0.1980 & 0.2353 & 0.1979 \\ 0.2262 & 0.2262 & 1 & 0.1980 & 0.2262 & 0.1979 \\ 0.1980 & 0.1980 & 0.1980 & 1 & 0.1980 & 0.1979 \\ 0.2353 & 0.2353 & 0.2262 & 0.1980 & 1 & 0.1979 \\ 0.1979 & 0.1979 & 0.1979 & 0.1979 & 0.1979 & 1 \end{pmatrix}.$$

At last, we get $\mathfrak{X}^4 = \mathfrak{X}^8$. Thus, $\mathfrak{X}^8$ is an equivalent matrix.

Stage 3. Find different values of levels of Confidence & generate i-cutting matrices: Select different values of level of confidence 'i' from the Equivalent matrix $\mathfrak{X}^{2^n}$ that all lies in the interval $[0, 1]$. Different values of 'i' taken from matrix $\mathfrak{X}^8$ are-$\{0.1979, 0.1980, 0.2262, 0.2353, 0.2415, 1\}$. For these values of level of confidence, generate i-cutting matrix $\mathfrak{U}_i = \{\mathfrak{u}_{\gamma\delta}\}_{\Gamma\times\Gamma}$ by using Definition 6.6. The i-cutting matrices are-

For $0 < i \leq 0.1979$,

$$\mathfrak{U}_{0.1979} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

For $0.1979 < i \leq 0.1980$,

$$\mathfrak{U}_{0.1980} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $0.1980 < i \leq 0.2262$,

$$\mathfrak{U}_{0.2262} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $0.2262 < i \leq 0.2353$,

$$\mathfrak{U}_{0.2353} = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Table 10: Clustering by proposed accuracy method.

Sr. No.	Level of Confidence	Clusters
1	$0 < i \leq 0.1979$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6\}$
2	$0.1979 < i \leq 0.1980$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$
3	$0.1980 < i \leq 0.2262$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_5\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_6\}$
4	$0.2262 < i \leq 0.2353$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_5\}$, $\{\mathfrak{C}_3\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_6\}$
5	$0.2353 < i \leq 0.2415$	$\{\mathfrak{C}_1, \mathfrak{C}_2\}$, $\{\mathfrak{C}_3\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$
6	$0.2415 < i \leq 1$	$\{\mathfrak{C}_1\}$, $\{\mathfrak{C}_2\}$, $\{\mathfrak{C}_3\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$

Table 11: Clustering by proposed similarity method.

Sr. No.	Level of Confidence	Clusters
1	$0 < i \leq 0.9490$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5, \mathfrak{C}_6\}$
2	$0.9490 < i \leq 0.9740$	$\{\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$
3	$0.9740 < i \leq 0.9751$	$\{\mathfrak{C}_1, \mathfrak{C}_2\}$, $\{\mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$
4	$0.9751 < i \leq 0.9769$	$\{\mathfrak{C}_1\}$, $\{\mathfrak{C}_2\}$, $\{\mathfrak{C}_3, \mathfrak{C}_4, \mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$
5	$0.9769 < i \leq 0.9841$	$\{\mathfrak{C}_1\}$, $\{\mathfrak{C}_2\}$, $\{\mathfrak{C}_3, \mathfrak{C}_5\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_6\}$
6	$0.9841 < i \leq 1$	$\{\mathfrak{C}_1\}$, $\{\mathfrak{C}_2\}$, $\{\mathfrak{C}_3\}$, $\{\mathfrak{C}_4\}$, $\{\mathfrak{C}_5\}$, $\{\mathfrak{C}_6\}$

For $0.2353 < i \leq 0.2415$,

$$\mathfrak{U}_{0.2415} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $0.2415 < i \leq 1$,

$$\mathfrak{U}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Stage 4. Generate Clusters: The alternatives $\mathfrak{C}_\gamma$ and $\mathfrak{C}_\delta$ lies a cluster if γ^{th} & δ^{th} columns are equal in a i-cutting matrix for a particular value of levels of Confidence. The same result can be carried out for rows. In the same way, other clusters are generated. The cluster of the given problem is given in Table 11

Thus, there is an efficacy in the proposed accuracy measure to handle the clustering problems as obtained from Table 10. However, if the proposed similarity measure is used above, then the result obtained is shown in Table 11. Hence, there is a potential in the proposed similarity measure to handle clustering problems as obtained from Table 10. Hence, the new approach offers high accuracy.

7 Advantages of the proposed study

Numerous real-life problems that arise in practical settings may be addressed with the proposed measures and the proposed method, such as the following:

(1). The proposed clustering method is used to solve the problems of detecting patterns, simplifying data, improving visual representation, making informed decisions, image segmentation, etc.

(2). The proposed pattern recognition method is used in solving issues involving speech recognition, image recognition, handwriting recognition, medical diagnosis, financial fraud detection, biometric identification, etc.

(3). The proposed measures can be used to solve the real-life $\mathfrak{DM}$ problems such as mineral processing, sustainable energy planning, technology investment, water and agriculture management, mobile crowd computing, check quality products, etc.

(4). The proposed measures can be used to propose a method for solving supplier selection issues, machine learning issues, signal, and image processing problems, etc.

(5). We can also use the proposed measure in solving the problems of source coding, data compression, channel encoding, error detection, statistical inferences, cryptography, etc.

(6). The proposed measures can be used to handle the problems of medical diagnosis.

From the above points, it can be seen that the proposed measure has a wide range of applications. Besides these applications, many real-life issues may be handled by the proposed measures and the proposed methods.

8 Conclusion

Since the phase function in the traditional $\mathcal{C}_{\mathcal{F}}$-set is not a fuzzy function, and the complement of the phase function does not affect the phase function [70], therefore, a $\mathcal{C}_{\mathcal{F}}^{V}$-set is proposed. Based on the notion of this $\mathcal{C}_{\mathcal{F}}^{V}$-set, a knowledge measure is provided in $\mathcal{C}_{\mathcal{F}}^{V}$-context. Its credibility is checked along with some basic properties. The proposed measure is compared with some previously known measures to prove its efficacy. The comparison is taken in the form of criteria weight computation and linguistic comparison. From the proposed measure, some new measures such as accuracy, similarity, and distance measures are deducted. Their credibility is checked with the help of suitable axioms. In the end, the deducted measures are successfully applied to the problems involving Pattern recognition and Clustering. To check the efficacy of the deducted measures, some examples of Clustering and Pattern recognition are taken. The suggested method also handles some real-world issues. Some applications of the proposed measures are discussed. The result of any method depends on the idea that is in the mind of its creator when designing a method. For a particular situation, a specific method may give accurate results. Under the same conditions, it is not possible for different methods to give the same results. So, the idea in the mind of the creator of a specific method at the time of designing is important. The proposed method also has some limitations. They are-

- The proposed method contains a lengthy computation. They may consume the time of the users. Errors at earlier stages can not be corrected in the last stages without precomputation. For the correct result, we have to do all our computations again from the very first step.
- The results obtained from the proposed method for the inconsistent and uncertain data are less accurate.

Taking these points into account, it can be seen that the proposed method has some limitations. Every method has some pros and cons. No method is completely accurate in this world. From the examples, it can be seen that the proposed method is reliable.

However, in the future, the proposed study may be extended to Complex-valued Bipolar fuzzy sets, Complex-valued picture fuzzy sets, Complex-valued intuitionistic fuzzy sets, etc. We may also use the proposed methods to solve several real-world problems of Marketing, Healthcare, Human resources, Credit risk analysis, Image processing, Data mining, Social network analysis, etc. Also, a concerted effort will be made in the proposed manuscript to remove any deficiencies in the upcoming years.

Declarations

Authorship Contribution statement

Amandeep Singh: Conceptualization, Writing - original draft, Formal analysis, Writing - review & editing, Methodology, Resources, Investigation, Validation, Software, Visualization.
Satish Kumar: Supervision, Project administration, Conceptualization, Data curation, Investigation.

Competing interest & Ethical approval

None.

Funding

None.

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Received: 2024-10-14
Accepted: 2025-05-08