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Localization of the weak solution to the non-local parabolic problem with the $p(z)$-Kirchhoff term

Mouad Allalou, Soukaina Yacini and Abderrahmane Raji

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Corresponding Author. S. YACINI

Abstract *In this paper, we investigate the existence of a weak solution for a nonlinear parabolic initial boundary value problem related to the* $p(z)$*-Kirchhoff-type equation. The equation is given by:*

$$\frac{\partial\vartheta}{\partial t} - g\Big(\int_\Omega \big(\mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)}\big)dz\Big)\Big(diva(z,t,\nabla\vartheta) - |\vartheta|^{p(z)-2}\vartheta\Big) = f(z,t)$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ *is a bounded domain with Lipschitz boundary* $\partial\Omega$*,* $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *is the* $p(z)$*-Kirchhoff-type function, and* $a : Q \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is a Carathéodory function. We show the existence of a weak solution for this problem by applying Berkovits and Mustonen topological degree theory in the space* $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$*, provided that certain assumptions hold.*

1 Introduction

In this paper, we study a specific class of parabolic problems, namely the $p(z)$*-Kirchhoff parabolic problem, which is defined as follows:*

$$(\mathcal{P})\begin{cases} \dfrac{\partial\vartheta}{\partial t} - g\big(\mathcal{L}(\vartheta)\big)\Big(div\ a(z,t,\nabla\vartheta) - |\vartheta|^{p(z)-2}\vartheta\Big) = f(z,t) & in \quad Q := \Omega\times(0,T) \\ \vartheta(z,0) = \vartheta_0(z) & in \quad \Omega \\ \vartheta(z,t) = 0 & on \quad \Gamma = \partial\Omega\times(0,T). \end{cases}$$

Where

$$\mathcal{L}(\vartheta) = \int_\Omega \left(\mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)}\right)dz.$$

The motivation for studying problem $(\mathcal{P})$ *lies in its relevance to nonlinear diffusion processes that combine nonlocal effects with spatially varying anisotropy. The presence of the variable exponent* $p(z)$ *allows the medium's properties to vary across space, reflecting real-world materials whose resistance to diffusion is non-uniform. Moreover, the Kirchhoff-type nonlocal term involving the functional* $\mathcal{L}(\vartheta)$ *models situations where the diffusion rate depends on the total energy of the system, which is typical in stretched membranes, thermo-mechanical systems, and even in certain biological and image-processing models. These features make equation* $(\mathcal{P})$ *an important and realistic subject of modern nonlinear analysis.*

Here, $\Omega \subset \mathbb{R}^N$ *is a bounded smooth domain,* $a(z,t,\varsigma) : Q\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ *is a Carathéodory function with a continuous derivative with respect to* ς*, and* $\mathcal{A}(z,t,\varsigma) : Q\times\mathbb{R}^N \longrightarrow \mathbb{R}$ *is a continuous mapping. For all* $(z,t) \in Q$ *and all* $\varsigma,\varsigma' \in \mathbb{R}^N$*,* $\varsigma \neq \varsigma'$*, we assume the following* *hypotheses on $\mathcal{M}$, $\mathcal{A}$, a, and f: there exist positive constants α, β, and a positive function $k(z,t) \in L^{q(z)}(Q)$ such that*

(h_1) $\mathcal{A}(z,t,0) = 0$ *and* $a(z,t,\varsigma) = \nabla_\varsigma \mathcal{A}(z,t,\varsigma)$,

(h_2) $\alpha|\xi|^{p(z)} \leq a(z,t,\varsigma)\cdot\varsigma \leq p(z)\mathcal{A}(z,t,\varsigma)$,

(h_3) $|a(z,t,\varsigma)| \leq k(z,t) + \beta\,|\varsigma|^{p(z)-1}$,

(h_4) $[a(z,t,\varsigma) - a(z,t,\varsigma')]\cdot(\varsigma - \varsigma') > 0$.

To ensure the existence of weak solutions, we assume that the Kirchhoff function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, non-decreasing, and satisfies the following growth condition:

(M_0) *There exist two positive constants $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1 t^{r(z)-1} \leq g(t) \leq \mathcal{B}_2 t^{r(z)-1}$.*

Here $r \in C_+(\overline{\Omega})$ with $1 \leq r^- \leq r(z) \leq r^+ \leq p^- \leq p(z) \leq p^+$ for all $t \in [0,+\infty)$.

Within this framework, Ω denotes a bounded open domain in $\mathbb{R}^N$, $N \geq 2$, with smooth boundary $\partial\Omega$. We define $Q := \Omega \times (0,T)$, where $\varGamma = \partial\Omega \times (0,T)$ represents its lateral boundary, and $T > 0$ is fixed. The term $-\mathrm{div}\, a(z,t,\nabla\vartheta)$ acts as a Leray–Lions operator from $\mathcal{V}$ to its dual $\mathcal{V}^*$, with

$$\mathcal{V} = L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) \quad \textit{and} \quad \mathcal{V}^* = L^{(p^-)'}(0,T;W^{-1,p(z)}(\Omega)),$$

where $\frac{1}{p(z)} + \frac{1}{p'(z)} = 1$. The right-hand side f is assumed to belong to $\mathcal{V}^*$, and the Kirchhoff-type function $g : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and satisfies the above assumptions.

Problem $(\mathcal{P})$ constitutes a new variant of the class of $p(z)$-Kirchhoff parabolic equations. Such models were originally introduced by Kirchhoff [22] as a generalization of the classical D'Alembert wave equation in the study of vibrating elastic strings.

The $p(z)$-Kirchhoff parabolic problem studied here extends classical Kirchhoff-type models, where the diffusion operator depends not only on the local behavior of the gradient but also on a nonlocal integral term involving the solution. This dual dependence makes the problem both anisotropic and nonlocal, reflecting many complex physical phenomena such as non-Newtonian fluids, thermorheological materials, or image processing problems with spatial adaptivity. The presence of the variable exponent $p(z)$ adds further challenges, as it allows the growth conditions to vary with the spatial position, making standard tools such as monotonicity methods or variational approaches less straightforward.

In recent years, Kirchhoff-type problems have attracted considerable attention (see e.g., [38, 6, 16, 30, 33, 19, 1, 17, 32, 36]), with a wide range of analytical approaches including variational methods, Galerkin approximations, topological techniques, and the sub- and super-solution method.

To establish the existence of weak solutions, we rely on the topological degree theory developed by Berkovits and Mustonen. This framework is well suited to problems where the operator is the sum of a maximal monotone linear operator and a bounded demicontinuous perturbation of type (S_+), which is precisely the case for Kirchhoff-type structures. Unlike classical degree theories, this approach can handle both the lack of compactness and the variable growth imposed by the $p(z)$-structure.

The concept of weak solutions is fundamental in the analysis of nonlinear partial differential equations, particularly when the equations involve irregular data or degenerate/singular behavior. Weak solutions provide a rigorous framework in which existence (and sometimes uniqueness) can be proved without requiring strong differentiability of the solution. They also form the foundation for further studies on stability, regularity, and numerical approximation schemes for complex physical models governed by such PDEs.

The remainder of this paper is organized as follows: In Section 2, we present the functional framework required for our analysis. Section 3 introduces several classes of operators and the *corresponding topological degree. Finally, in Section 4, we provide the proof of the main result of the paper.*

2 Notations and preliminaries

2.1 Sobolev space with variable exponent

In this subsection, we will examine a selection of fundamental characteristics and definitions concerning Lebesgue-Sobolev spaces with variable exponents $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$, for more insights into the properties of Lebesgue-Sobolev spaces with variable exponents, we recommend consulting the work of Fan and Zhao [15].

Let Ω be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$), we say that a real-valued continuous function $p(\cdot)$ is log-Höder continuous in Ω if

$$|p(x) - p(y)| \leq \frac{C}{|\log|x-y||} \quad \forall x, y \in \overline{\Omega}$$

such that $|x-y| < \frac{1}{2}$, with possible different constant C. We denote

$$C_+(\Omega) = \{\text{log-Höder continuous function } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N\},$$

where $p^- = \min\{p(z) : z \in \overline{\Omega}\}$ and $p^+ = \max\{p(z) : z \in \overline{\Omega}\}$.

We denote by $\mathcal{P}(\Omega)$ the set of Lebesgue measurable functions: $\mathcal{P}(\Omega) = \{w : \Omega \to \mathbb{R} \text{ measurable}\}$ and $\mathcal{P}^+(\Omega) = \{w : \Omega \to [1, \infty) \text{ measurable}\}$.

We define the variable exponent Lebesgue space for $p \in C_+(\overline{\Omega})$ by

$$L^{p(\cdot)}(\Omega) = \{w \in \mathcal{P}(\Omega) : \int_\Omega |w(z)|^{p(z)}\, dz < \infty\},$$

this space is endowed with the (Luxembourg) norm defined by the formula

$$\|w\|_{L^{p(\cdot)}(\Omega)} = \|w\|_{p(\cdot)} = \inf\{\lambda > 0 : \int_\Omega \left|\frac{w(z)}{\lambda}\right|^{p(z)} dz \leq 1\}.$$

If $1 < p^- \leq p^+ < \infty$ then $L^{p(\cdot)}(\Omega)$ is a uniformly convex Banach space and therefore reflexive, and if $p \in \mathcal{P}^+(\Omega) \cap L^\infty(\Omega)$, then $L^{p(\cdot)}(\Omega)$ is a separable space. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$, see [14].

Proposition 2.1. *(Young's Inequality): Let $p, p' \in C_+(\overline{\Omega})$, where p' the conjugate of p, $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. For all $a, b > 0$, we have*

$$ab \leq \frac{a^{p(z)}}{p(z)} + \frac{b^{p'(z)}}{p'(z)}.$$

Proposition 2.2. *(Generalised Hölder Inequality): see [15, 21]*

i_0) *For any functions $w \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega |wv|\, dz \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right) \|w\|_{p(\cdot)} \|v\|_{p'(\cdot)} \leq 2\|w\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

i_1) *For all $p, q \in C_+(\overline{\Omega})$ such that $p(z) \leq q(z)$ a.e. in Ω, we have $L^{q(\cdot)} \hookrightarrow L^{p(\cdot)}$ and the embedding is continuous.*

Lemma 2.3. *(See [15]) If we denote $\sigma(w) = \int_\Omega |w(z)|^{p(z)}\, dx$ for all $w \in L^{p(\cdot)}(\Omega)$, then*

$$\min\left\{\|w\|_{p(.)}^{p^-}, \|w\|_{p(.)}^{p^+}\right\} \leq \sigma(w) \leq \max\left\{\|w\|_{p(.)}^{p^-}, \|w\|_{p(.)}^{p^+}\right\}.$$

Proposition 2.4. *(See [14]) For $w \in L^{p(\cdot)}(\Omega)$ and $\{w_k\}_{k\in\mathbb{N}} \subset L^{p(\cdot)}(\Omega)$, the following assertions hold:*

t_1) $w \neq 0 \Rightarrow \left(\|w\|_{p(\cdot)} = \lambda \Leftrightarrow \sigma\left(\frac{w}{\lambda}\right) = 1\right)$,

t_2) $\|w\|_{p(\cdot)} > 1 \Rightarrow \|w\|_{p(.)}^{p^-} \leq \sigma(w) \leq \|w\|_{p(.)}^{p^+}$,

t_3) $\|w\|_{p(\cdot)} < 1 \Rightarrow \|w\|_{p(.)}^{p^+} \leq \sigma(w) \leq \|w\|_{p(.)}^{p^-}$,

t_4) $\lim_{k\to\infty} \|w_k\|_{p(\cdot)} = 0 \Leftrightarrow \lim_{k\to\infty} \sigma(w_k) = 0$,

t_5) $\lim_{k\to\infty} \|w_k\|_{L^{p(\cdot)}(\Omega)} = \infty \Leftrightarrow \lim_{k\to\infty} \sigma(w_k) = \infty$.

Lemma 2.5. *Let $h_n \to h$ a.e. and $h_n \rightharpoonup h$ in $L^{p(\cdot)}(\Omega)$. Then,*

$$\lim_{n\to\infty} \int_\Omega |h_n|^{p(z)}\, dz - \int_\Omega |h - h_n|^{p(z)}\, dz = \int_\Omega |h|^{p(z)}\, dz.$$

Theorem 2.6. *For any function $w \in L^{p(\cdot)}(\Omega)$ and $w_n \in L^{p(\cdot)}(\Omega)$, the following statements are equivalent :*

a_1) $\lim_{n\to\infty} \|w_n - w\|_{p(\cdot)} = 0$,

a_2) $\lim_{k\to\infty} \sigma(w_n - w) = 0$,

a_3) *w_n converges to w in measure and* $\lim_{n\to\infty} \sigma(w_n) = \sigma(w)$.

Which share the same type of properties as $L^{p(\cdot)}(\Omega)$, we define also the variable Sobolev space by

$$W^{1,p(\cdot)}(\Omega) = \{w \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla w| \in L^{p(\cdot)}(\Omega)\},$$

where the norm is defined by

$$\|w\|_{1,p(\cdot)} = \|w\|_{p(\cdot)} + \|\nabla w\|_{p(\cdot)} \quad \text{for all } w \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, i.e.,

$$W_0^{1,p(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)},$$

and we define the Sobolev exponent by

$$p^*(z) = \frac{Np(z)}{N - p(z)} \quad \text{for } p(z) < N.$$

Proposition 2.7. *(See [14])*

r_1) *Assuming $1 < p^- \leq p^+ < \infty$, the spaces $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

r_2) *If $q \in C_+(\overline{\Omega})$ and $q(z) < p^*(z)$ for any $z \in \Omega$, then the embedding $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is continuous and compact.*

r_3) *Poincaré inequality: there exists a constant $C > 0$, such that*

$$\|w\|_{p(\cdot)} \leq C\|\nabla w\|_{p(\cdot)} \quad \text{for all } w \in W_0^{1,p(\cdot)}(\Omega).$$

Remark 2.8. . By (t_3) of Proposition 2.4, we deduce that $\|\nabla w\|_{p(\cdot)}$ and $\|w\|_{1,p(\cdot)}$ are equivalent norms in $W_0^{1,p(\cdot)}(\Omega)$.

2.2 Spaces $L^{p^-}(0,T;W_0^{1,p(z)}(\Omega))$

In this subsection, we revisit the fundamental definitions and properties of Bochner spaces, as discussed in various references such as [10, 13, 25, 26, 28, 31]. We will also use the standard notation for Bochner spaces. Specifically, if $q \geq 1$ and X is a Banach space then $L^q(0,T;X)$ denotes the space of strongly measurable functions $w:(0,T)\to X$ for which $t\mapsto \|w(t)\|_X \in L^q((0,T))$. Additionally, $C([0,T];X)$ denotes the space of continuous functions $w:[0,T]\to X$, equipped with the norm $\|w\|_{C([0,T];X)} = \max_{t\in[0,T]}\|w(t)\|_X$.

$$L^{p^-}(0,T;W_0^{1,p(z)}(\Omega)) = \Big\{ w:(0,T)\to W_0^{1,p(z)}(\Omega) \text{ measurable};\ \Big(\int_0^T \|w(t)\|^{p^-}_{W_0^{1,p(z)}(\Omega)}\,dt\Big)^{1/p^-} < \infty \Big\}.$$

and we define the space

$$L^\infty(0,T;X) = \Big\{ w:(0,T)\to X \text{ measurable}, \exists\, C>0 : \|w(t)\|_X \leq C \text{ a.e.} \Big\},$$

where the norm is defined by

$$\|w\|_{L^\infty(0,T;X)} = \inf\Big\{ C>0 : \|w(t)\|_X \leq C \text{ a.e.} \Big\}.$$

We introduce the functional space (see [9])

$$V = \Big\{ f \in L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega));\ |\nabla f| \in L^{p(\cdot)}(\mathcal{Q}) \Big\}, \tag{2.1}$$

which is endowed with the norm

$$\|f\|_V = \|\nabla f\|_{L^{p(\cdot)}(\mathcal{Q})},$$

or, the equivalent norm

$$|||f|||_V = \|f\|_{L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega))} + \|\nabla f\|_{L^{p(\cdot)}(\mathcal{Q})}.$$

Space V is a separable and reflexive Banach space. The equivalence of the two norms is an easy consequence of the continuous embedding $L^{p(\cdot)}(Q) \hookrightarrow L^{p^-}(0,T;L^{p(\cdot)}(\Omega))$ and the Poincaré inequality. We state some further properties of V in the following lemma.

Lemma 2.9. *Let V be defined as in (2.1) and its dual space be denoted by V^*. Then,*

h_1**)** *we have the following continuous dense embeddings:*

$$L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T;W_0^{1,p(\cdot)}(\Omega)).$$

In particular, since $D(Q)$ is dense in $L^{p^+}(0,T;W_0^{1,p(\cdot)}(\Omega))$, it is dense in V and for the corresponding dual spaces, we have

$$L^{(p^-)'}(0,T;(W_0^{1,p(\cdot)}(\Omega))^*) \hookrightarrow V^* \hookrightarrow L^{(p^+)'}(0,T;(W_0^{1,p(\cdot)}(\Omega))^*).$$

h_2**)** *One can represent the elements of V^* as follows: if $T\in V^*$, then there exists $F = (f_1,\dots,f_N) \in (L^{P'}(\cdot)(Q))^N$ such that $T = divF$ and*

$$\langle T,\xi\rangle_{V^*,V} = \int_0^T\int_\Omega F\cdot\nabla\xi\,dz\,dt,$$

for any $\xi \in V$. Moreover, we have

$$\|T\|_{V^*} = \max_n \Big\{ \|f_i\|_{L^{p(\cdot)}(Q)}, \quad i=1,\dots,n. \Big\}$$

Remark 2.10. The space $V\cap L^\infty(Q)$ endowed with the norm defined by the formula

$$\|v\|_{V\cap L^\infty(Q)} = \max_n\{\|v\|_V, \|v\|_{L^\infty(Q)}\}, \quad v\in V\cap L^\infty(Q),$$

is a Banach space. In fact, it is the dual space of the Banach space $V^* + L^1(Q)$ endowed with the norm

$$\|v\|_{V^*+L^1(Q)} = \inf\{\|v_1\|_{V^*} + \|v_2\|_{L^1(Q)} : v = v_1+v_2, v_1\in V^*, v_2\in L^1(Q)\}.$$

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3 Classes of mappings and Topological degree

In this section, we present certain outcomes and characteristics derived from Berkovits and Mustonen's degree theory for demicontinuous operators of generalized (S_+) type in real reflexive Banach spaces. Throughout the discussion, let $\mathcal{X}$ denote a real separable reflexive Banach space with its dual space $\mathcal{X}^*$ and continuous dual pairing denoted by $\langle \cdot , \cdot \rangle$. Consider a nonempty subset Ω of $\mathcal{X}$, and let $\rightharpoonup$ represent weak convergence.
Let $\mathcal{T} : \mathcal{X} \longrightarrow 2^{\mathcal{X}^*}$ be a multi-valued mapping. The graph of $\mathcal{T}$, denoted by $G(\mathcal{T})$, is defined as:

$$G(\mathcal{T}) = \big\{(w, v) \in \mathcal{X} \times \mathcal{X}^* \ : \ v \in \mathcal{T}(w)\big\}.$$

Definition 3.1. The multi-values mapping $\mathcal{T}$ is termed

(i) Monotone if, for each pair of elements $(w_1, w_1), (v_2, v_2)$ in $G(\mathcal{T})$, the inequality

$$\langle w_1 - w_2, v_1 - v_2\rangle \geq 0$$

holds.

(ii) Maximal monotone, if it is monotone and maximal in the sense of graph inclusion among monotone multi-values mappings from $\mathcal{X}$ to $2^{\mathcal{X}^*}$. Equivalently, for any $(w_0, v_0) \in \mathcal{X} \times \mathcal{X}^*$ for which $\langle w_0 - w, v_0 - v\rangle \geq 0$, for all $(w, v) \in G(T)$, we have $(w_0, v_0) \in G(T)$.

Definition 3.2. Let $\mathcal{Y}$ be another real Banach space. A mapping F: $D(\mathcal{J}) \subset \mathcal{X} \to \mathcal{Y}$ is said to be

(i) bounded, if it takes any bounded set into a bounded set.

(ii) demicontinuous, if for each sequence $(w_n) \subset \Omega,\ w_n \to u$ implies $\mathcal{J}(w_n) \rightharpoonup \mathcal{J}(w)$.

(iii) of type (S_+), if for any sequence $(w_n) \subset D(\mathcal{J})$ with $w_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle \mathcal{J} w_n, w_n - w\rangle \leq 0$, we have $w_n \to w$.

In the sequel, let $\hbar : D(\hbar) \subset \mathcal{X} \to \mathcal{X}^*$ be a linear maximal monotone map such that $D(\hbar)$ is dense in $\mathcal{X}$.
For each open and bounded subset G on $\mathcal{X}$, we consider the following classes of operators:

$$\mathcal{F}_G(\Omega) := \big\{\hbar + S : \overline{G} \cap D(\hbar) \to \mathcal{X}^* \mid S \quad \textit{is bounded, demicontinuous} \\ \textit{and of type } (S_+) \textit{ with respect to } D(\hbar) \textit{ from } G \textit{ to } \mathcal{X}^*\big\},$$

$$\mathcal{H}_G := \big\{\hbar + S(\tau) \ : \ \overline{G} \cap D(\hbar) \to \mathcal{X}^* \mid S(\tau) \ \textit{ is a bounded homotopy} \\ \textit{of type } (S_+) \ \textit{ with respect to } D(\hbar) \ \textit{ from } \overline{G} \ \textit{ to } X^*\big\}.$$

Remark 3.3. ([7]). Remark that the class $\mathcal{H}_G$ encompasses all affine homotopy

$$\hbar + (1-\tau)S_1 + \tau S_2 \quad \text{with} \quad (\hbar + S_i) \in \mathcal{F}_G \quad \text{and } i = 1, 2.$$

We present the Berkovits and Mustonen topological degree for a category of demicontinuous operators satisfying condition $(S_+)_T$. For a more detailed discussion, see [7].

Theorem 3.4. *Let $\hbar$ be a linear maximal monotone densely defined map from $D(\hbar) \subset \mathcal{X}$ to $\mathcal{X}^*$. There exists a unique degree function*

$$d : \big\{(\mathcal{J}, G, h) \ : \ \mathcal{J} \in \mathcal{F}_G,\ G \textit{ an open bounded subset in } \mathcal{X},\ h \notin \mathcal{J}\big(\partial G \cap D(\hbar)\big)\big\} \longrightarrow \mathbb{Z},$$

which satisfies the following properties :

(i) (Normalization) $\hbar + F$ is a normalising map, where F is the duality mapping of $\mathcal{X}$ into $\mathcal{X}^*$, that is, $d(\hbar + F, G, h) = 1$, when $h \in (\hbar + F)(G \cap D(\hbar))$.

(ii) (Additivity) *Let $F \in \mathcal{F}_G$. If G_1 and G_2 are two disjoint open subsets of G such that $h \notin F\big((\overline{G}\backslash(G_1 \cup G_2)) \cap D(\hbar)\big)$, then we have*

$$d(F,G,h) = d(F,G_1,h) + d(F,G_2,h)\ .$$

(iii) (Homotopy invariance) *If $\mathcal{J}(z) \in \mathcal{H}_G$ and $h(\tau) \notin F(\tau)(\partial G \cap D(\hbar))$ for every $\tau \in [0,1]$, where $h(\tau)$ is a continuous curve in $\mathcal{X}^*$, then*

$$d(\mathcal{J}(z),\ G,\ h(z)) = \ constant, \qquad \forall z \in [0,1].$$

(iv) (Existence) *if $d(\mathcal{J},G,h) \neq 0$, then the equation $\mathcal{J}w = h$ has a solution in $G \cap D(L)$.*

Lemma 3.5. *Let $\hbar + S \in \mathcal{F}_{\mathcal{X}}$ and $h \in \mathcal{X}^*$. Suppose that there exists $R > 0$ such that*

$$\langle \hbar w + Sw - h, w\rangle > 0, \tag{3.1}$$

for any $w \in \partial B_R(0) \cap D(\hbar)$. Then

$$(\hbar + S)(D(\hbar)) = \mathcal{X}^*. \tag{3.2}$$

Proof. Let $\varepsilon > 0,\ \tau \in [0,1]$ and

$$\mathcal{J}_\varepsilon(z,w) = \hbar w + (1-\tau)Fw + \tau(Sw + \varepsilon Fw - h).$$

As $0 \in \hbar(0)$ and applying the boundary condition (3.1), we have

$$\begin{aligned}\langle \mathcal{J}_\varepsilon(\tau,w),w\rangle &= \langle \tau(\hbar w + Sw - h, w\rangle + \langle (1-\tau)\hbar w + (1-\tau+\varepsilon)Fw, w\rangle\\ &\geq \langle (1-\tau)\hbar w + (1-z+\varepsilon)Fw, w\rangle\\ &= (1-\tau)\langle \hbar w, w\rangle + (1-\tau+\varepsilon)\langle Fw, w\rangle\\ &\geq (1-\tau+\varepsilon)\|w\|^2 = (1-\tau+\varepsilon)R^2 > 0.\end{aligned}$$

Which means that $0 \notin \mathcal{J}_\varepsilon(z,w)$. Since F and $S+\varepsilon F$ are bounded, continuous and of type (S_+), $\{\mathcal{J}_\varepsilon(\tau,\cdot)\}_{\tau\in[0,1]}$ is an admissible homotopy. Hence, by using the normalisation and invariance under homotopy, we get

$$d(\mathcal{J}_\varepsilon(\tau,\cdot), B_R(0), 0) = d(\hbar + F, B_R(0), 0) = 1.$$

As a result, there exists $w_\varepsilon \in D(\hbar)$ such that $0 \in \mathcal{J}_\varepsilon(\tau,\cdot)$.
If we take $\tau = 1$ and when $\varepsilon \to 0^+$, then we have $h \in \hbar w + Sw$ for some $w \in D(\hbar)$. Since $h \in \mathcal{X}^*$ is arbitrary, we deduce that $(\hbar + S)(D(\hbar)) = \mathcal{X}^*$. □

4 Main results

Lemma 4.1. *([4]). Assume that (h_2)-(h_4) hold, and let $(\vartheta_n)_n$ be a sequence in $L^{p^-}\big(0,T;W_0^{1,p(z)}(\Omega)\big)$ such that $\vartheta_n \rightharpoonup \vartheta$ weakly in $L^{p^-}\big(0,T;W_0^{1,p(z)}(\Omega)\big)$ and*

$$\int_Q \big[a(z,t,\nabla\vartheta_n) - a(z,t,\nabla\vartheta)\big]\nabla(\vartheta_n - \vartheta)dz \longrightarrow 0. \tag{4.1}$$

Then $\vartheta_n \longrightarrow \vartheta$ strongly in $L^{p^-}\big(0,T;W_0^{1,p(z)}(\Omega)\big)$.

Let us consider the following functional:

$$\mathcal{G}(\vartheta) = \int_0^T \widehat{g}\Big(\int_\Omega \big(\mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)}\big)dz\Big)dt \qquad \forall \vartheta \in \mathcal{V},$$

where $\widehat{g} : [0,+\infty[\longrightarrow [0,+\infty[$ is the primitive of g, defined by

$$\widehat{g}(n) = \int_0^n g(\xi)d\xi.$$

It is well known that $\mathcal{G}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $\vartheta \in \mathcal{V}$ is the functional $\mathcal{F}(\vartheta) := \mathcal{G}'(\vartheta) \in \mathcal{V}^*$ given by

$$\langle \mathcal{F}(\vartheta)\,,\,\Phi\rangle =$$

$$\int_0^T \Big\{ g\Big(\int_\Omega (\mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)})\,dz\Big)\Big[\int_\Omega a(z,t,\nabla\vartheta)\nabla\Phi\,dz + \int_\Omega |\vartheta|^{p(z)-2}\vartheta\Phi\,dz\Big]\Big\}dt$$

for all $\Phi \in \mathcal{V}$.

Lemma 4.2. *Suppose that the assumption (h_1)-(h_4) and (M_0) hold, then*

- *$\mathcal{F}$ is continuous and bounded mapping.*
- *the mapping $\mathcal{F}$ is of class (S_+).*

Proof. • The continuity of $\mathcal{F}$ is immediate, since it is the Fréchet derivative of $\mathcal{G}$. Now, we prove that the operator $\mathcal{F}$ is bounded.

$$\left\{\begin{aligned}
|\langle \mathcal{F}\vartheta, \Phi\rangle| &= \Big|\int_0^T \Big\{g\Big(\int_\Omega \mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)})dz\Big)\\
&\qquad \times\Big(\int_\Omega a(z,t,\nabla\vartheta)\nabla\Phi\,dz + \int_\Omega |\vartheta|^{p(z)-2}\vartheta\Phi dz\Big\}dt\Big|\\
&\le \mathcal{B}_2\int_0^T \Big(\int_\Omega \mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)})dz\Big)^{r(z)-1}\\
&\qquad \times\Big(\int_\Omega |a(z,t,\nabla\vartheta)|\cdot|\nabla\Phi|\,dz + \int_\Omega |\vartheta|^{p(z)-1}\cdot|\Phi|dz\Big)dt\\
&\le \mathcal{C}onst\int_0^T \left(\int_\Omega \Big(\mathcal{A}(z,t,\nabla\vartheta)dz\Big)^{r(z)-1} + \Big(\int_\Omega |\vartheta|^{p(z)}dz\Big)^{r(z)-1}\right)\\
&\qquad \times\left(\int_\Omega |a(z,t,\nabla\vartheta)|\cdot|\nabla\Phi|\,dz + \int_\Omega |\vartheta|^{p(z)-1}\cdot|\Phi|dz\right)dt\\
&\le \mathcal{C}onst\int_0^T \left(\int_\Omega \Big(\mathcal{A}(z,t,\nabla\vartheta)dz\Big)^{r(z)-1} + \|\vartheta\|^{\alpha(r(z)-1)}_{W_0^{1,p(z)}(\Omega)}\right)\\
&\qquad \times\left[\Big(\|a(z,t,\nabla\vartheta)\|_{p'(z)}\cdot\|\nabla\Phi\|_{p(z)}\Big) + \Big(\int_\Omega |\vartheta|^{(p(z)-1)p'(z)}dz\Big)^{\frac{1}{\eta}}\cdot\|\Phi\|_{p(z)}\right]dt\\
&\le \mathcal{C}onst\int_0^T \left(\int_\Omega \Big(\mathcal{A}(z,t,\nabla\vartheta)dz\Big)^{r(z)-1} + \|\vartheta\|^{\alpha(r(z)-1)}_{W_0^{1,p(z)}(\Omega)}\right)\\
&\qquad \times\left[\Big(\|a(z,t,\nabla\vartheta)\|_{p'(z)}\cdot\|\Phi\|_{W_0^{1,p(z)}(\Omega)}\Big) + \|\vartheta\|^{\frac{\alpha}{\eta}}_{W_0^{1,p(z)}(\Omega)}\cdot\|\Phi\|_{W_0^{1,p(z)}(\Omega)}\right]dt\\
&\le \mathcal{C}onst\int_0^T \left(\int_\Omega \Big(\mathcal{A}(z,t,\nabla\vartheta)dz\Big)^{r(z)-1} + \|\vartheta\|^{\alpha(r(z)-1)}_{W_0^{1,p(z)}(\Omega)}\right)\\
&\qquad \times\left[\Big(\|a(z,t,\nabla\vartheta)\|_{p'(z)}\cdot\|\Phi\|^{p-}_{W_0^{1,p(z)}(\Omega)}\Big) + \|\vartheta\|^{\frac{\alpha}{\eta}}_{W_0^{1,p(z)}(\Omega)}\cdot\|\Phi\|^{p-}_{W_0^{1,p(z)}(\Omega)}\right]dt.
\end{aligned}\right.$$

Where

$$\alpha = \begin{cases} p^- & \text{if } \|\vartheta\|_{p(z)} \le 1\\ p^+ & \text{if } \|\vartheta\|_{p(z)} \ge 1\end{cases}$$

and

$$\eta = \begin{cases} p'^- & \text{if } |\vartheta^{p(z)-1}|_{p'(z)} \le 1\\ p'^+ & \text{if } |\vartheta^{p(z)-1}|_{p'(z)} \ge 1.\end{cases}$$

Or by (h_1) we have for any $z \in \Omega$ and $\xi \in \mathbb{R}^n$

$$\mathcal{A}(z,t,\xi) = \int_0^1 \frac{d}{ds}\mathcal{A}(z,t,s\xi)ds = \int_0^1 a(z,t,s\xi)\xi ds,$$

by combining (h_3), Fubini's theorem and Young's inequality we have

$$(I_1)\begin{cases} \displaystyle\int_\Omega \mathcal{A}(z,t,\nabla\vartheta)dz &= \displaystyle\int_\Omega\int_0^1 a(z,t,s\nabla\vartheta)\nabla\vartheta ds\,dz \\ &= \displaystyle\int_0^1 \Big[\int_\Omega a(z,t,s\nabla\vartheta)\nabla\vartheta dz\Big] ds \\ &\le \displaystyle\int_0^1 \Big[C_{p'}\int_\Omega \big|a(z,t,s\nabla\vartheta)\big|^{p'(z)}dz + C_p\int_\Omega |\nabla\vartheta|^{p(z)}dz\Big] ds \\ &\le C_1 + C'\displaystyle\int_0^1\int_\Omega |s\,\nabla\vartheta|^{p(z)}\,dz\,ds + C_p\|\vartheta\|^\alpha_{W_0^{1,p(z)}(\Omega)} \\ &\le C_1 + C_2\displaystyle\int_\Omega |\nabla\vartheta|^{p(z)}\,dz + C_p\|\vartheta\|^\alpha_{W_0^{1,p(z)}(\Omega)} \\ &\le C\|\vartheta\|^\alpha_{W_0^{1,p(z)}(\Omega)}. \end{cases}$$

From the growth condition (h_2), we can easily show that $\|a(z,t,\nabla\vartheta)\|_{L^{p'(z)}(\Omega)}$ is bounded for all $\vartheta \in W_0^{1,p(z)}(\Omega)$. Then,

$$\begin{aligned}|\langle \mathcal{F},\Phi\rangle| &\le \mathcal{C}onst\int_0^T \|\Phi\|^{p-}_{W_0^{1,p(z)}(\Omega)}\,dt \\ &\le \mathcal{C}onst\|\Phi\|^{p-}_{\mathcal{V}}.\end{aligned}$$

Which means that the operator $\mathcal{F}$ is bounded.

- Next, we verify that the operator $\mathcal{F}$ is of type (S_+).

Assume that $(\vartheta_n)_n \subset \mathcal{V}$ and

$$\begin{cases}\vartheta_n \rightharpoonup \vartheta \quad \text{in} \quad \mathcal{V} \\ \limsup\limits_{n\to\infty}\langle \mathcal{F}\vartheta_n, \vartheta_n - \vartheta\rangle \le 0.\end{cases} \tag{4.2}$$

We will show that $\vartheta_n \to \vartheta$ in $\mathcal{V}$.

On one hand, it is indeed true that $\vartheta_n \rightharpoonup \vartheta$ in $\mathcal{V}$. Consequently, the sequence $(\vartheta_n)_n$ is bounded in $\mathcal{V}$. Given that $\mathcal{V}$ compactly embeds in $L^{p(z)}(Q)$, there exists a subsequence, still denoted by $(\vartheta_n)_n$, such that $\vartheta n \to \vartheta$ in $L^{p(z)}(Q)$, we obtain

$$\limsup_{n\to\infty}\langle \mathcal{F}\vartheta_n,\vartheta_n-\vartheta\rangle = \limsup_{n\to\infty}\langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta,\ \vartheta_n-\vartheta\rangle = \lim_{n\to\infty}\langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta,\ \vartheta_n-\vartheta\rangle \le 0. \tag{4.3}$$

Which means

$$\begin{aligned}\lim_{n\to\infty}\langle \mathcal{F}\vartheta_n - \mathcal{F}\vartheta,\ \vartheta_n-\vartheta\rangle = \lim_{n\to\infty}\Bigg(\int_0^T g\big(\mathcal{L}(\vartheta_n)\big)\Big[\int_\Omega a(z,t,\nabla\vartheta_n)\nabla(\vartheta_n-\vartheta)dz \\ + \int_\Omega |\vartheta_n|^{p(z)-2}\vartheta_n\cdot(\vartheta_n-\vartheta)dz\Big]dt - \int_0^T g\big(\mathcal{L}(\vartheta)\big)\Big[\int_\Omega a(z,t,\nabla\vartheta)\nabla(\vartheta_n-\vartheta)dz \\ + \int_\Omega |\vartheta|^{p(z)-2}\vartheta\cdot(\vartheta_n-\vartheta)dz\Big]dt\Bigg) \le 0.\end{aligned} \tag{4.4}$$

Since $\displaystyle\int_\Omega \frac{1}{p(z)}|\vartheta|^{p(z)})dz$ is bounded and by (I_1), we infer that $g\big(\mathcal{L}(\vartheta_n)\big)$ is bounded.

As g is continuous, up to a subsequence there is $k \geq 0$ such that

$$g\Big(\mathcal{L}(\vartheta_n)\Big) \longrightarrow g(k) \geq \mathcal{B}_1 k^{r(z)-1} \qquad \text{as} \quad n \to \infty. \tag{4.5}$$

Using the compact embedding $\mathcal{V} \hookrightarrow\hookrightarrow L^p(Q)$, we have

$$\lim_{n\to\infty} \int_Q |\vartheta_n|^{p(z)-2}\vartheta_n \cdot (\vartheta_n - \vartheta)\, \mathrm{d}z\, \mathrm{d}t = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_Q |\vartheta|^{p(z)-2}\vartheta \cdot (\vartheta_n - \vartheta)\, \mathrm{d}z\, \mathrm{d}t = 0. \tag{4.6}$$

From (4.5), (4.4), (4.6) and since $\mathcal{B}_1 \geq 0$, we have

$$\lim_{n\to\infty} \int_Q \Big(a(z,t,\nabla\vartheta_n) - a(z,t,\nabla\vartheta)\Big)\nabla(\vartheta_n - \vartheta)\, \mathrm{d}z\, \mathrm{d}t \leq 0. \tag{4.7}$$

By combining (4.7) and (h_4), we have

$$\lim_{n\to\infty} \int_Q \Big(a(z,t,\nabla\vartheta_n) - a(z,t,\nabla\vartheta)\Big)\nabla(\vartheta_n - \vartheta)\, \mathrm{d}z\, \mathrm{d}t = 0,$$

and therefore, in light of Lemma 4.1, we obtain

$$\vartheta_n \longrightarrow \vartheta \qquad \text{in } \mathcal{V},$$

which implies that $\mathcal{F}$ is of type (S_+). □

Let us consider the following operator $\hbar$ defined from the subset $D(\hbar)$ of $\mathcal{V}$ into its dual $\mathcal{V}^*$, where

$$D(\hbar) = \big\{\Phi \in \mathcal{V} \;:\; \Phi' \in \mathcal{V}^*,\ \Phi(0) = 0\big\},$$

such that

$$\langle \hbar\vartheta, \Phi\rangle = -\int_Q \vartheta\Phi_t\, \mathrm{d}z\, \mathrm{d}t \qquad \textit{for all } \vartheta \in D(\hbar),\quad \Phi \in \mathcal{V}.$$

Consequently, the operator $\hbar$ is generated by $\partial/\partial t$ by means of the relation

$$\langle \hbar\vartheta, \Phi\rangle = \int_0^T \langle \vartheta'(t), \Phi(t)\rangle\, \mathrm{d}t, \ \ \textit{for all } \vartheta \in D(\hbar),\ \Phi \in \mathcal{V}.$$

Lemma 4.3. *([38]). $\hbar$ is a linear maximal monotone densely defined map.*

Our main result is the following existence theorem:

Theorem 4.4. *Let $f \in \mathcal{V}^*$ and $\vartheta_0 \in L^2(\Omega)$, suppose that the assumptions (h_1)-(h_4) and (M_0) are satisfied. Then there exists at least one weak solution $\vartheta \in D(\hbar)$ of problem $(\mathcal{P})$ in the following sense*

$$-\int_Q \vartheta\Phi_t\, \mathrm{d}z\, \mathrm{d}t + \int_0^T \Big\{g(\mathcal{L}(\vartheta))\Big[\int_\Omega a(z,t,\nabla\vartheta)\nabla\Phi\, \mathrm{d}z + \int_\Omega |\vartheta|^{p(z)-2}\vartheta\Phi\, dz\Big]\Big\}\, \mathrm{d}t = \int_0^T \langle f, \Phi\rangle\, \mathrm{d}t$$

Proof. On the one hand, from the lemma 4.3, the operator

$$\hbar : D(\hbar) \subset \mathcal{V} \longrightarrow \mathcal{V},$$

$$\langle \hbar\vartheta, \Phi\rangle_{\mathcal{V}} = \int_0^T \langle \vartheta'(t), \Phi(t)\rangle\, \mathrm{d}t, \ \ \text{for all } \vartheta \in D(\hbar),\ \Phi \in \mathcal{V}.$$

is a densely defined maximal monotone operator.
By the monotonicity of $\hbar$, we have

$$\langle \hbar\vartheta, \vartheta\rangle \geq 0 \ \ \text{for all } \vartheta \in D(\hbar)\,,$$

then we obtain

$$\begin{aligned}\langle \hbar\vartheta + \mathcal{F}\vartheta, \vartheta\rangle &\geq \langle \mathcal{F}\vartheta, \vartheta\rangle \\ &= \int_0^T \Big\{ g\Big(\int_\Omega \mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)}\Big)dz\Big) \times \Big(\int_\Omega a(z,t,\nabla\vartheta)\nabla\vartheta dz + \int_\Omega |\vartheta|^{p(z)}\, dz\Big)\Big\}\, \mathrm{d}t,\end{aligned}$$

by using the assumptions (h_2) and (M_0), we get

$$\begin{aligned}\langle \hbar\vartheta + \mathcal{F}\vartheta, \vartheta\rangle &\geq \mathcal{B}_1 \int_0^T \Big[\Big(\int_\Omega \mathcal{A}(z,t,\nabla\vartheta) + \frac{1}{p(z)}|\vartheta|^{p(z)} dz\Big)^{r(z)-1} \times \Big(\int_\Omega a(z,t,\nabla\vartheta)\nabla\vartheta dz + \int_\Omega |\vartheta|^{p(z)}\, dz\Big)\Big] dt \\ &\geq \mathcal{B}_1 \int_0^T \Big[\Big(\frac{\alpha}{p^+} \int_\Omega |\nabla\vartheta|^{p(z)}\, dz + \frac{1}{p^+}\int_\Omega |\vartheta|^{p(z)}\, dz\Big)^{r(z)-1} \times \Big(\int_\Omega \alpha|\nabla\vartheta|^{p(z)}\, dz + \int_\Omega |\vartheta|^{p(z)}\, dz\Big)\Big] dt \\ &\geq \mathcal{B}_1 \int_0^T \Big[C_1\Big(\int_\Omega |\nabla\vartheta|^{p(z)} + |\vartheta|^{p(z)}\, dz\Big)^{r(z)-1} \times C_2\Big(\int_\Omega |\nabla\vartheta|^{p(z)}\, dz + \int_\Omega |\vartheta|^{p(z)}\, dz\Big)\Big] dt \\ &\geq \mathcal{C}st \int_0^T \Big[\Big(\int_\Omega |\nabla\vartheta|^{p(z)} + |\vartheta|^{p(z)}\, dz\Big)^{r(z)}\Big] dt \\ &\geq \mathcal{C}st \int_0^T \|\vartheta\|^{\gamma r(z)}_{W_0^{1,p(z)}(\Omega)}\, dt \\ &\geq \mathcal{C}st\|\vartheta\|^{\gamma r(z)}_{\mathcal{V}} \quad \text{for all } \vartheta \in \mathcal{V}. \end{aligned} \tag{4.8}$$

Where

$$\gamma = \begin{cases} p^- & \text{if } \|\vartheta\|_{1,p(z)} \leq 1 \\ p^+ & \text{if } \|\vartheta\|_{1,p(z)} \geq 1 \end{cases}$$

Since the right side of the above inequality (4.8) tends to ∞ as $\|\vartheta\|_{\mathcal{V}} \to \infty$, then for each $f \in \mathcal{V}^*$ there exists $R = R(f)$ such that

$$\langle \hbar\vartheta + \mathcal{F}\vartheta - f, \vartheta\rangle > 0, \quad \text{for all } \vartheta \in B_R(0) \cap D(\hbar). \tag{4.9}$$

From Lemma 4.3, $\hbar : D(\hbar) \subset \mathcal{V} \to \mathcal{V}^*$ is a maximal monotone operator. By combining the monotonicity of $\hbar$, assumptions (h_2)–(M_0), and coercivity estimates, one proves that for each $f \in \mathcal{V}^*$, there exists $\vartheta \in D(\hbar)$ satisfying

$$\hbar\vartheta + \mathcal{F}(\vartheta) = f.$$

Hence, problem $(\mathcal{P})$ admits at least one weak solution. □

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Author information

Mouad Allalou, Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal,, Morocco.
E-mail: mouadallalou@gmail.com

Soukaina Yacini, Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal,, Morocco.
E-mail: yacinisoukaina@gmail.com

Abderrahmane Raji, Applied Mathematics and Scientific Computing Laboratory, Faculty of Science and Technics, Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal,, Morocco.
E-mail: rajiabd2@gmail.com

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