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On the dual notion of annihilator conditions in modules over commutative rings

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Abstract Let R be a commutative ring with $1 \neq 0$, and M be unital R-module. We say that M satisfies the dual of property (A) if, for each finitely generated ideal I of R with $I \subseteq W_R(M)$, we have $IM \neq M$. Similarly, M is said to satisfy the dual of strong Property (A) if, for any $r_1, \cdots, r_n \in W_R(M)$, there exists a completely irreducible submodule N of M such that $r_i M \subseteq N \neq M$ [13]. In this paper, we establish several results concerning these classes of modules. Additionally, we investigate the relationships between Property (A), strong Property (A), the dual of Property (A) and the dual of strong Property (A). Moreover, we highlight a notable phenomenon in the theory: the absence of a complementary notion to "Property (A)". Our aim is to bridge this gap and explore key aspects of this dualization.

1 Introduction

In this paper, all rings are assumed to be commutative with a unit element, and all modules are unital. Let R be a commutative ring and N an R-module. The subset $Z_R(N)$ of R,known as the set of zero-divisors of N, is defined as

$$Z_R(N) = \{a \in R : ax = 0 \text{ for some nonzero element } x \in N\}.$$

Similarly, the set of zero-divisors of R is denoted by $Z(R) = Z_R(R)$. Also, $E(N)$ denotes the injective envelope of N. A ring R is said to satisfy Property (A) (or to be an (A)-ring) if, for each finitely generated ideal I of R with $I \subseteq Z(R)$, there exists a nonzero element $b \in R$ such that $Ib = (0)$ [16]. Note that the class of (A)-rings includes Noetherian rings and $\mathbb{Z}$-graded rings (see [16, Theorem 2.5] and [19, p 63]). In [9], Darani initiated the study of Property (A) for modules and used the term F-McCoy (following Faith McCoy terminology) instead of Property (A). He also introduced strong Property (A) under the name super coprimal and defined a module N coprimal if $\mathrm{Z}_R(N)$ is an ideal. In [23], Mahdou and Hassani independently introduced the concept of a strong (A)-ring. A ring R is said to satisfy strong Property (A) (or to be a strong (A)-ring) if, for each $a_1, a_2, \cdots, a_n \in \mathrm{Z}_R(R)$, there exists a nonzero element $b \in R$ such that $a_1 b = a_2 b = \cdots = a_n b = 0$.

In [3], Anderson and Chun extended Property (A) from rings to modules. According to their definition, an R-module N is said to satisfy Property (A) if, for each finitely generated ideal I of R with $I \subseteq Z_R(N)$, there exists a nonzero element $x \in N$ such that $Ix = (0)$, or equivalently $\mathrm{ann}_R(N) \neq (0)$. An R-module N is said to satisfy strong Property (A) if, for each $a_1, a_2, \cdots, a_n \in \mathrm{Z}_R(N)$, there exists a nonzero element $x \in N$ such that $a_1 x = a_2 x = \cdots = a_n x = 0$. The dual notion of $\mathrm{Z}_R(N)$ is defined as

$$\mathrm{W}_R(N) = \{a \in R : aN \neq N\}.$$

Similarly, the dual notion of $Z(R)$ is given by $W(R) = W_R(R)$. Then, $\mathrm{W}(R)$ is given by

$$\mathrm{W}(R) = \bigcup \{m : m \in \max(R)\},$$

where max(R) denoted the set of maximal ideals of R. A non-zero submodule L of N is said to be secondal if $\mathrm{W}_R(L)$ is an ideal of R. In this case, $\mathrm{W}_R(N)$ is a prime ideal of R (see [4]).
Recall that a proper submodule C of R-module M is said to be completely irreducible if, whenever $C = \cap_{i\in I} C_i$, where $\{C_i\}_{i\in I}$ is a family of submodules of M, it follows that $C = C_i$ for some $i \in I$. Equivalently, this means that M/C is a cocyclic R-module. It is easy to see that every submodule of M can be expressed as the intersection of completely irreducible submodules of M (see [21]).

In [13], F. Farshadifar initiated the study of the dual of Property (A) and, respectively, the dual of strong Property (A) in modules. In fact, an R-module M satisfies the dual of property (A) if, for each finitely generated ideal I of R with $I \subseteq W_R(M)$, we have $IM \neq M$, or equivalently, $\frac{R}{I} \otimes_R M \neq 0$. An R-module M satisfies the dual of strong Property (A) if, for any $r_1, \cdots, r_n \in W_R(M)$, there exists a completely irreducible submodule N of M such that $r_i M \subseteq N \neq M$. Since every submodule of M is an intersection of completely irreducible submodules of M, it follows that M satisfies the dual of strong Property (A) if and only if, for any $r_1, \cdots, r_n \in W_R(M)$, there exists a submodule N of M such that $r_i M \subseteq N \neq M$. Clearly, if an R-module M satisfies the dual of strong Property (A), then M satisfies the dual of Property (A). In general, the converse does not hold. For example, the $\mathbb{Z}$-module $\mathbb{Z}$ satisfies the dual of Property (A), but does not satisfy the dual of strong Property (A), see for instance [13, Example 2.3]. Now, let S be a ring such that R is an S-algebra, E an injective cogenerator for S-modules, and M an R-module. Then M is an (R, S)-bimodule. We define $M^* = \mathrm{Hom}_S(M, E)$. If $N \cong M^*$ as an (R, S)-bimodule, then N is called a dual module of M. Similarly, we define the (R, S)-bimodule M^{**}. In particular case, when $S = \mathbb{Z}$, $E = \mathbb{Q}/\mathbb{Z}$, the dual module $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called the character module of M and denoted by M^+. If $S = R$ and $E = E(\oplus_{m\in \max(R)} R/m)$, then E is an injective cogenerator for R-modules, providing another example of a dual module.

In this paper, we first present some fundamental results related to the notion of the dual set of zero divisors. Among these results, Theorem 2.9 plays a crucial role in the remainder of our paper: Let S be a ring such that R is an S-algebra. If E is an injective cogenerator for S-modules and M is an R-module, then

(1) $W_R(M) = Z_R(M^*)$,

(2) $W_R(M^*) = Z_R(M)$.

We then utilize this result to characterize the dual of (strong) Property (A) for modules in terms of (strong) Property (A). This provides a duality between the dual of (strong) Property (A) and (strong) Property (A). For example, in Theorem 3.5 and Theorem 4.1, we prove if R is an algebra commutative over a commutative ring S and E is an injective cogenerator for S-modules, then, for an R-module M, the following holds:

(1) M satisfies the dual of (strong) Property $(A) \Leftrightarrow M^*$ satisfies (strong) Property (A).

(2) M^* satisfies the dual of (strong) Property $(A) \Leftrightarrow M$ satisfies (strong) Property (A).

Moreover, we present several examples, many of which are pathological, for modules satisfying the dual of Property (A). For instance, the R-module $B_1 \oplus B_2$ satisfies the dual of Property (A), whereas the R-modules B_1 and B_2 do not (Example 3.8). Conversely, the R-modules B_1 and B_2 satisfy the dual of Property (A), but $B_1 \oplus B_2$ does not (Example 3.9). This demonstrates that the dual of Property (A) is not preserved under submodules, homomorphic images, direct sums, or direct summands. In fact, as shown in [13, Example 2.14], every module is a direct summand (and thus a submodule and homomorphic image) of a module satisfying the dual of Property (A).
Finally, if an R-module M is an Artinian (resp., a Noetherian) module, then M satisfies both a Property (A) and the dual of Property (A) (see Corollary 3.2). Hence, while most notions related to the Artinian (resp., a Noetherian) modules can easily be dualized, we demonstrate in this paper a notable phenomenon: the absence of a complementary notion to "Property (A)". Our goal is to bridge this gap and explore key results of this dualization.

2 On the set $\mathrm{W}_R(M)$

Let's start with the following proposition.

Proposition 2.1. *Let R be a commutative ring, and let $\Sigma: \ 0 \longrightarrow N \longrightarrow M \xrightarrow{\pi} \frac{M}{N} \longrightarrow 0$ be an exact sequence of R-modules. Then we have the following.*

(1) $\mathrm{W}_R(\frac{M}{N}) \subseteq \mathrm{W}_R(M) \subseteq \mathrm{W}_R(N) \cup \mathrm{W}_R(\frac{M}{N})$.

(2) *If Σ is* RD-exact, *then* $\mathrm{W}_R(M) = \mathrm{W}_R(N) \cup \mathrm{W}_R(\frac{M}{N})$.

(3) *If R is an integral domain and $\frac{M}{N}$ is torsion-free, then* $\mathrm{W}_R(M) = \mathrm{W}_R(N) \cup \mathrm{W}_R(\frac{M}{N})$.

(4) *If $(M_i)_{i\in I}$ is a family of R-modules, then we have* $\mathrm{W}_R(\oplus_{i\in I} M_i) = \cup_{i\in I} \mathrm{W}_R(M_i)$.

(5) *If N is a small submodule of M, then* $\mathrm{W}_R(\frac{M}{N}) = \mathrm{W}_R(M)$.

Proof. (1) Suppose $a \in \mathrm{W}_R\left(\frac{M}{N}\right)$, which means that $\frac{M}{N} \otimes_R \frac{R}{Ra} \neq 0$. Consider the exact sequence:

$$(*) \qquad N \otimes_R \frac{R}{Ra} \longrightarrow M \otimes_R \frac{R}{Ra} \longrightarrow \frac{M}{N} \otimes_R \frac{R}{Ra} \longrightarrow 0.$$

Since $\frac{M}{N} \otimes_R \frac{R}{Ra} \neq 0$, it follows that $M \otimes_R \frac{R}{Ra} \neq 0$, and hence $a \in \mathrm{W}_R(M)$.

Now, suppose $a \in \mathrm{W}_R(M)$. Then, $M \otimes_R \frac{R}{Ra} \neq 0$. Consider again the sequence $(*)$. It follows that either $\frac{M}{N} \otimes_R \frac{R}{Ra} \neq 0$ or $N \otimes_R \frac{R}{Ra} \neq 0$. Thus, $a \in \mathrm{W}_R(N) \cup \mathrm{W}_R\left(\frac{M}{N}\right)$.

(2) Suppose $a \in \mathrm{W}_R(N) \cup \mathrm{W}_R\left(\frac{M}{N}\right)$. Then either $a \in \mathrm{W}_R\left(\frac{M}{N}\right)$ or $a \in \mathrm{W}_R(N)$. If $a \in \mathrm{W}_R\left(\frac{M}{N}\right)$, then by (1), we obtain $a \in \mathrm{W}_R(M)$. If $a \in \mathrm{W}_R(N)$, then $N \otimes_R \frac{R}{Ra} \neq 0$. Since Σ is RD-exact, we have the exact sequence:

$$0 \longrightarrow N \otimes_R \frac{R}{Ra} \longrightarrow M \otimes_R \frac{R}{Ra} \longrightarrow \frac{M}{N} \otimes_R \frac{R}{Ra} \longrightarrow 0.$$

Thus, $M \otimes_R \frac{R}{Ra} \neq 0$, which implies $a \in \mathrm{W}_R(M)$.

(3) This follows from (2).

(4) Let $a \in R$. Then:

$$\frac{R}{Ra} \otimes_R \left(\bigoplus_{i\in I} M_i \right) \cong \bigoplus_{i\in I} \left(\frac{R}{Ra} \otimes_R M_i \right).$$

Hence, $\frac{R}{Ra} \otimes_R \left(\bigoplus_{i\in I} M_i\right) \neq 0$ if and only if there exists some $i_0 \in I$ such that $\frac{R}{Ra} \otimes_R M_{i_0} \neq 0$.

(5) It suffices to prove that $\mathrm{W}_R(M) \subseteq \mathrm{W}_R\left(\frac{M}{N}\right)$. Indeed, if $a \in \mathrm{W}_R(M)$, then $aM \neq M$. Suppose, for contradiction, that $a\left(\frac{M}{N}\right) = \frac{M}{N}$. Then for every $x \in M$, there exists $m \in M$ such that $x - am \in N$, implying that $x \in (aM + N)$. This leads to $M = aM + N$. Since N is a small submodule of M, it follows that $M = aM$, contradicting our assumption. Therefore, $a \in \mathrm{W}_R\left(\frac{M}{N}\right)$.

□

It is well known that, given a ring R, an R-module N is called cocyclic if N is isomorphic to a submodule of $E(S)$ for some simple R-module S in (see [17], [26]). A prime ideal P of R is said to be a coassociated prime of an R-module M if there exists a cocyclic homomorphic image N of M such that $P = \mathrm{ann}_R(N)$. The set of coassociated prime ideals of M is denoted by $\mathrm{Coass}(M)$ [26]. Also, a prime ideal P of R is said to be a weakly coassociated prime of M if there exists a cocyclic homomorphic image N of M such that P is a minimal element in $V(\mathrm{ann}_R(N))$. The set of weakly coassociated prime ideals of M is denoted by $\widetilde{\mathrm{Coass}}(M)$ [26]. By [26, Theorem 2.11], we have $\widetilde{\mathrm{Coass}}(M) \neq \emptyset$ whenever $M \neq 0$. Moreover, by [26, Theorem 2.15],

$$\mathrm{W}_R(M) = \bigcup_{P \in \widetilde{\mathrm{Coass}}(M)} P.$$

One can easily verify that

$$\mathrm{W}_R(R) = \bigcup\{P : P \in \mathrm{Spec}(R)\} = \bigcup\{m : m \in \max(R)\}.$$

Proposition 2.2. *Let R be a ring, and let M be a nonzero finitely generated R-module. Then,*

$$\mathrm{W}_R(M) = \bigcup_{P \in \mathrm{Supp}(M)} P.$$

To prove this proposition, we require the following two lemmas:

Lemma 2.3. *Let R be a ring, and let $0 \neq M$ be a finitely generated R-module. Then $\mathrm{J}(R) \subseteq \mathrm{W}_R(M)$, where $\mathrm{J}(R)$ denotes the Jacobson radical of R. In particular, if R is a local ring with a maximal ideal m, then $\mathrm{W}_R(M) = m$.*

Proof. Assume the contrary, that $\mathrm{J}(R) \nsubseteq \mathrm{W}_R(M)$. Then there exists $a \in \mathrm{J}(R)$ such that $aM = M$. Since M is a finitely generated R-module, and by Nakayama's Lemma, we have $M = 0$, which is a contradiction. □

Lemma 2.4. *Let R be a ring, and $0 \neq M$ be a finitely generated R-module. Then the following are equivalent:*

(1) $I \subseteq \mathrm{W}_R(M)$;

(2) $IM \neq M$.

Proof. (1) ⇒ (2) Suppose that $I \subseteq \mathrm{W}_R(M)$. Assume, for the sake of contradiction, that $IM = M$. Since M is a finitely generated R-module, and by Nakayama's Lemma, there exists $a \in I$ such that $(1-a)M = 0$. As $a \in I \subseteq \mathrm{W}_R(M)$, there exists $x \in M \backslash aM$. Now, $(1-a)x = 0$ implies that $x \in aM$, which leads to a contradiction.
(2) ⇒ (1) This is straightforward. □

Proof of Proposition 2.2. Let $a \in \mathrm{W}_R(M)$. By [26, Theorem 2.15], there exists $P \in \widetilde{Coass}(M)$ such that $a \in P$, and P is a minimal element in $\mathrm{V}(\mathrm{ann}_R(L))$, where L is a cocyclic homomorphic image of M. Thus, $\mathrm{ann}_R(M) \subseteq \mathrm{ann}_R(L)$, so $P \in \mathrm{V}(\mathrm{ann}_R(M))$. Since M is finitely generated, $P \in \mathrm{Supp}(M)$. This implies that $\mathrm{W}_R(M) \subseteq \bigcup_{P \in \mathrm{Supp}(M)} P$.
On the other hand, let $P \in \mathrm{Supp}(M)$. Then M_P is non-zero finitely generated R_P-module. Since R_P is local ring with maximal ideal PR_P of R_P, by Lemma 2.3, we have $\mathrm{W}_{R_P}(M_P) = PR_P$. Now, by Lemma 2.4, $(PR_P)M_P \neq M_P$. Thus, $PM \neq M$, and therefore $P \subseteq \mathrm{W}_R(M)$ by Lemme 2.4. Consequently, $\mathrm{W}_R(M) = \bigcup_{P \in \mathrm{Supp}(M)} P$. □

Remark 2.5. Let R be a ring and M an R-module. Recall that M is called faithfully flat if M is a flat R-module and $IM \neq M$ for all ideal proper I of R, or equivalently, M is flat R-module and $mM \neq M$ for every maximal ideal m of R, as stated in [6]. Now, if M is faithfully, then $\mathrm{W}_R(M) = \mathrm{W}_R(R)$. By [6, Exercie 1, § 3], consider a family of R-modules $\{M_i\}_{i \in I}$, where I is an arbitrary set. Then the module $\oplus_{i \in I} M_i$ is faithfully flat if all the M_i are flat and at least one of them is faithfully flat. However, the converse is false. For example, let $R = \mathbb{Z}/15\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, which shows that the $\mathbb{Z}$-module $\mathbb{Z}/5\mathbb{Z}$ of R is projective. However, it is not faithfully flat because $\mathbb{Z}/5\mathbb{Z} \otimes_R \mathbb{Z}/3\mathbb{Z} = 0$. Now, if all the M_i are flat and at least one of them is faithfully flat, then $\mathrm{W}_R(\oplus_{i \in I} M_i) = \cup_{i \in I} \mathrm{W}_R(M_i) = \mathrm{W}_R(R)$. In particular, if F is a free R-module, then $\mathrm{W}_R(F) = \mathrm{W}_R(R)$.

The following proposition characterizes local rings R for which the set $\mathrm{W}_R(M)$, for any finitely generated R-module M, is equal to $\mathrm{W}_R(R)$.

Proposition 2.6. *Let R be a ring. Then the following assertions are equivalent:*

(1) *R is a semi-perfect and any projective R-module is faithfully flat;*

(2) *Any non-zero finitely generated R-module M, $\mathrm{W}_R(M) = \mathrm{W}_R(R)$;*

(3) *Any non-zero cyclic R-module N, $\mathrm{W}_R(N) = \mathrm{W}_R(R)$;*

(4) *R is a local ring.*

Proof. (1) $\Rightarrow$ (2) Suppose that R is a semi-perfect and any projective R-module is faithfully flat. Let M be any finitely generated R-module. Since R is semi-perfect, M has a projective cover. Now let $q : P \longrightarrow M$ be a small epimorphism with P projective. This implies $\mathrm{W}_R(M) = \mathrm{W}_R(P)$, by Proposition 2.1 (5). Since P is faithfully flat, by Remark 2.5, we have $\mathrm{W}_R(M) = \mathrm{W}_R(R)$.
(2) $\Rightarrow$ (3) This is straightforward.
(3) $\Rightarrow$ (4) Assume that for each cyclic R-module N, $\mathrm{W}_R(N) = \mathrm{W}_R(R)$. Suppose that R has at least two maximal ideals, say Q_1, Q_2. As R/Q_1 is cyclic R-module, by hypothesis $Q_2 \subseteq \mathrm{W}_R(R/Q_1)$. Thus, $Q_1 + Q_2 \neq R$, which is a contradiction since $Q_1 + Q_2 = R$. Hence R must be a local ring.
(4) $\Rightarrow$ (1) Suppose that R is a local ring. Then, by [15, Corollary 26.7], every projective R-module P is free. As R is a local ring, it follows that R is semi-perfect.

□

Recall that a faithfully flat module M over a ring R is always flat and faithful ($\mathrm{ann}_R(M) = 0$). However, the converse does not hold in general. For example, $\mathbb{Q}$ is a faithful and flat $\mathbb{Z}$-module, but it is not faithfully flat. In fact, $\mathbb{Q} \otimes_{\mathbb{Z}} -$, which reduces all the quotient modules $\mathbb{Z}/a\mathbb{Z}$ to zero ($\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/a\mathbb{Z} = 0$). In this context, the following theorem characterizes when the set $\mathrm{W}_R(M)$ of finitely generated flat modules M over a ring R is equal to $\mathrm{W}_R(R)$.

Proposition 2.7. *Let R be a ring, and let M be a non-zero finitely generated flat R-module. Then the following assertions are equivalent:*

(1) *M is faithfully flat;*

(2) *M is faithful;*

(3) $\max(R) \subseteq$ *Coass*(M)*;*

(4) $\mathrm{W}_R(M) = \mathrm{W}_R(R)$*;*

(5) *ann*$_R(M) \subseteq \mathrm{J}(R)$*;*

(6) $\mathrm{W}_{R_P}(M_P) = PR_P$ *for each prime ideal P of R;*

(7) $\mathrm{W}_{R_P}(M_P) = PR_P$ *for each maximal ideal P of R.*

Proof. The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (5), and (6) $\Rightarrow$ (7) are clear.

(1) $\Rightarrow$ (3) Suppose that M is faithfully flat. For any $m \in \max(R)$, we have $M \otimes_R \frac{R}{m} \neq 0$, which implies that $M \neq mM$. Thus, there exists a cocyclic homomorphic image $\frac{M}{D}$ of $\frac{M}{mM}$ with $mM \subseteq D$, and hence $m = \mathrm{ann}_R\left(\frac{M}{D}\right)$. Therefore, $m \in \mathrm{Coass}(M)$.

(3) $\Rightarrow$ (4) Suppose that $\max(R) \subseteq \mathrm{Coass}(M)$. By [26, Theorem 2.15], we have $\mathrm{W}_R(M) = \bigcup_{P\in\mathrm{Coass}(M)} P$. Since $\max(R) \subseteq \mathrm{Coass}(M)$, it follows that $\max(R) \subseteq \mathrm{Coass}(M)$. This implies that $\mathrm{W}_R(M) = \mathrm{W}_R(R)$.

(4) $\Rightarrow$ (1) Suppose that $\mathrm{W}_R(M) = \mathrm{W}_R(R)$. Since M is a flat R-module, we can show that $mM \neq M$ for each $m \in \max(R)$. Moreover, as M is a finitely generated R-module and $\mathrm{W}_R(M) = \mathrm{W}_R(R)$, Lemma 2.4 implies that $mM \neq M$ for each $m \in \max(R)$.

(5) $\Rightarrow$ (7) Assume that $\mathrm{ann}_R(M) \subseteq \mathrm{J}(R)$. Then $\max(R) \subseteq \mathrm{Supp}(M)$. By the proof of Proposition 2.2, we obtain $\mathrm{W}_{R_P}(M_P) = PR_P$ for each maximal ideal P of R.

(7) $\Rightarrow$ (4) This follows from the proof of Proposition 2.2.

(1) $\Rightarrow$ (6) This follows from the proofs of the implication (1) $\Rightarrow$ (5) and Proposition 2.2. □

The next theorem will play an important role in the rest of the paper.

Theorem 2.8. *Let S be a ring such that R is an S-algebra. Let E be an injective cogenerator for S-modules and M be an R-module. Then the following holds:*

(1) $W_R(M) = Z_R(M^*)$.

(2) $W_R(M^*) = Z_R(M)$.

Proof. We can assume that $M \neq 0$.

1) Note that $((R/aR) \otimes_R M)^* \cong \mathrm{Hom}_R(R/aR, \mathrm{Hom}_S(M, E)) \cong \mathrm{ann}_{M^*}(a)$ for any element $a \in R$. Since E is a cogenerator for S-modules, it is easy to check that $a \in \mathrm{W}_R(M)$ if and only if $a \in \mathrm{Z}_R(M^*)$.

2) Let $a \in \mathrm{W}_R(M^*)$. Then we have $(R/aR) \otimes_R M^* \neq 0$. Since E is an injective S-module, we also have

$$0 \neq (R/aR) \otimes_R \mathrm{Hom}_S(M, E) \cong \mathrm{Hom}_S(\mathrm{Hom}_R(R/aR, M), E) \cong \mathrm{Hom}_S(\mathrm{ann}_M(a), E).$$

By hypothesis, since E is a cogenerator for S-modules, we conclude that $\mathrm{ann}_M(a) \neq 0$, and thus $a \in \mathrm{Z}_R(M)$. Conversely, let $a \in Z_R(M)$. Then $\mathrm{ann}_M(a) \neq 0$. Since E is an injective cogenerator for S-modules, we have:

$$(R/aR) \otimes_R \mathrm{Hom}_S(M, E) \cong \mathrm{Hom}_S(\mathrm{Hom}_R(R/aR, M), E) \cong \mathrm{Hom}_S(\mathrm{ann}_M(a), E) \neq 0.$$

This implies that $(R/aR) \otimes_R \mathrm{Hom}_S(M, E) \neq 0$, hence $a \in \mathrm{W}_R(M^*)$.

□

Corollary 2.9. *Let R be a ring and E an injective cogenerator for R-modules. The following hold:*

(1) $\mathrm{W}_R(E) = \mathrm{Z}_R(R)$.

(2) $\mathrm{Z}_R(E) = \mathrm{W}_R(R)$.

Recall that if $\varphi : R \longrightarrow T$ is a ring map where R and T are commutative rings, and E is an injective cogenerator for R-modules, then $\mathrm{Hom}_R(T, E)$ is an injective cogenerator for T-modules. In the particular case when $T = R/\mathrm{ann}_R(M)$, where M is an R-module, then $\mathrm{ann}_E(\mathrm{ann}_R(M))$ is an injective cogenerator for $(R/\mathrm{ann}_R(M))$-modules. Moreover, by Corollary 2.9, we have.

Proposition 2.10. *Let R be a ring and E an injective cogenerator for R-modules. If M is an R-module, then*

(1) $\mathrm{W}_{(R/\,\mathrm{ann}_R(M))}(\mathrm{ann}_E(\mathrm{ann}_R(M))) = \mathrm{Z}(R/\,\mathrm{ann}_R(M))$.

(2) $\mathrm{W}(R/\,\mathrm{ann}_R(M)) = \mathrm{Z}_{(R/\,\mathrm{ann}_R(M))}(\mathrm{ann}_E(\mathrm{ann}_R(M)))$.

Recall that given a ring R with a total quotient ring $Q(R)$, any element of $Q(R)$ is either a zero-divisor or invertible in $Q(R)$. The following proposition addresses the case when a ring R is equal to its total quotient ring in terms of W_R and Z_R.

Proposition 2.11. *Let R be a ring and E an injective cogenerator for R-modules. Then the following assertions are equivalent:*

(1) *R is equal to its total quotient ring.*

(2) $\mathrm{W}_R(R) = \mathrm{Z}_R(R)$*;*

(3) $\mathrm{W}_R(E) = \mathrm{Z}_R(E)$.

Proof. (1) $\Rightarrow$ (2) Assume that R is equal to its total quotient ring. If $a \in \mathrm{W}_R(R)$, then a is not invertible in R, and thus a is a zero divisor in R. Hence, $\mathrm{W}_R(R) \subseteq \mathrm{Z}_R(R)$. Conversely, let $a \in \mathrm{Z}_R(R)$. Then a is not invertible in R, and thus $aR \neq R$. Hence, $\mathrm{Z}_R(R) \subseteq \mathrm{W}_R(R)$.
(2) $\Rightarrow$ (1) Assume that $\mathrm{W}_R(R) = \mathrm{Z}_R(R)$ and suppose that an element a of R is not invertible in R. Then $a \in \mathrm{W}_R(R) = \mathrm{Z}_R(R)$, and thus a is a zero divisor.
(2) $\Leftrightarrow$ (3) Apply Theorem 2.8. □

For any commutative ring R and M an R-module, if $a, b \in R \setminus \mathrm{W}_R(M)$, then $aM = M$ and $bM = M$, implying $abM = a(bM) = aM = M$. Hence, $R \setminus \mathrm{W}_R(M)$ is a multiplicatively closed subset of R. Now, let $S_M = R \setminus \mathrm{W}_R(M)$, $\mathrm{q}_M(R) = S_M^{-1}R$, and $\mathrm{q}_M(M) = S_M^{-1}M$. Then, $\mathrm{q}_M(M)$ is an $\mathrm{q}_M(R)$-module.

Proposition 2.12. *Let R be a ring and M an R-module. Then*

(1) $\mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M)) \subseteq S_M^{-1}\,\mathrm{W}_R(M)$.

(2) *If* $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$, *then*

  (a) $\mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M)) = S_M^{-1}\,\mathrm{W}_R(M)$.

  (b) *Any element of* $\mathrm{q}_M(R)$ *is either an element of* $\mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M))$ *or invertible in* $\mathrm{q}_M(R)$.

Proof. (1) Let $\frac{r}{s} \in \mathrm{q}_M(R)$ such that $\frac{r}{s} \in \mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M))$. Then $\frac{r}{s}\,\mathrm{q}_M(M) \neq \mathrm{q}_M(M)$. Assume, for contradiction, that $r \notin \mathrm{W}_R(M)$. Then $rM = M$, and since $\frac{r}{s}\,\mathrm{q}_M(M) \neq \mathrm{q}_M(M)$, there exists an element $\frac{m}{t} \in \mathrm{q}_M(M) \setminus \frac{r}{s}\,\mathrm{q}_M(M)$. Now, $rM = M$ implies that $\frac{m}{t} \in \frac{r}{s}\,\mathrm{q}_M(M)$, which is a contradiction.
(2) Now, assume that $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$.
(a) Let $\frac{r}{s} \in S_M^{-1}\,\mathrm{W}_R(M)$ with $r \in \mathrm{W}_R(M)$ and $s \in S_M$. Assume, for contradiction, that $\frac{r}{s} \notin \mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M))$. Then $\frac{r}{s}\,\mathrm{q}_M(M) = \mathrm{q}_M(M)$, so for every $m \in M$, there exists $m' \in M$ and $t \in S_M$ such that $\frac{m}{1} = \frac{r}{s}\frac{m'}{t}$. Since $st \in S_M$, we have $stM = M$, and there exists $m'' \in M$ such that $m' = stm''$. Therefore, $\frac{m}{1} = \frac{rm''}{1}$. Thus, $rM = M$, which is a contradiction. By (1), we conclude that $\mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M)) = S_M^{-1}\,\mathrm{W}_R(M)$.
(b) Let $\frac{a}{s} \in \mathrm{q}_M(R)$ with $a \in R$ and $s \in S_M$. Suppose that $\frac{a}{s} \notin \mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M))$. Then, by (a), $a \notin \mathrm{W}_R(M)$, so $a \in S_M$. Hence, $\frac{a}{s}$ is invertible in $\mathrm{q}_M(R)$, as desired. □

3 On the duals of Property (A)

We begin this section with some examples of modules that satisfy the dual of Property (A), as demonstrated by the following lemma.

Lemma 3.1 ([13], Theorem 2.12). (1) *The trivial module vacuously satisfies the dual of Property (A).*

(2) *Every module over a Bezout ring R, satisfies the dual of Property (A).*

(3) *Let M be a finitely generated module over a commutative ring R. Then M satisfies the dual of Property (A). In fact, for any ideal I of R with $I \subseteq W_R(M)$, $IM \neq M$.*

(4) *Let M be an artinian module over a commutative ring R. Then M satisfies the dual of Property (A). In fact, for any ideal I of R with $I \subseteq W_R(M)$, $IM \neq M$.*

(5) *Let R be a zero-dimensional commutative ring (e.g. R is artinian). Then every R-module satisfies the dual of Property (A).*

(6) *Let M and N be R-modules with $W_R(N) \subseteq W_R(M)$. If M satisfies the dual of Property (A), then $M \oplus N$ does.*

(7) *Let R be a commutative ring and N be a small submodule in M. Then M satisfies the dual of Property (A) if and only if M/N does.*

Corollary 3.2. *Let R be a ring and M an R-module. Then the following hold:*

(1) *R satisfies the dual of Property (A);*

(2) *If M is an Artinian (resp., a Noetherian) module, then M satisfies both a Property (A) and the dual of Property (A).*

Proof. (1) Apply Lemma 3.1.

(2) Apply [3, Theorem 2.2] and Lemma 3.1.

Example 3.3. Let R be a ring, and M be a faithfully flat R-module. Then, if for each finitely generated ideal I of R with $I \subseteq W_R(M)$, we have $IM \neq M$, it follows that M satisfies the dual of Property (A). In particular, If M is a free R-module, then M is a faithfully flat R-module, and thus M satisfies the dual of property (A).

□

Corollary 3.4. *Let R be a ring. Then, every R-module is a submodule, homomorphic image, or direct factor of a module satisfying the dual of Property (A).*

Proof. Let M be any R-module, then $\mathrm{W}_R(M) \subseteq \mathrm{W}_R(R)$. By Corollary 3.2, R satisfies the dual of Property (A). Then, by Lemma 3.1(6), $M \oplus R$ also satisfies the dual of Property (A). □

The following theorem establishes a relationship between Property (A) and its dual.

Theorem 3.5. *Let R be an algebra commutative over a commutative ring S, and let E be an injective cogenerator for S-modules. For an R-module M, the following holds:*

(1) *M satisfies the dual of Property (A) if and only if M^* satisfies Property (A).*

(2) *M^* satisfies the dual of Property (A) if and only if M satisfies Property (A).*

Proof. (1) Note that $((R/I) \otimes_R M)^* \cong \mathrm{Hom}_R(R/I, \mathrm{Hom}_S(M, E)) \cong \mathrm{ann}_{M^*}(I)$ for any ideal I of R. As E is a cogenerator for S-modules, by Theorem 2.8(1), $\mathrm{W}_R(M) = \mathrm{Z}_R(M^*)$. It is easy to check that M satisfies the dual of Property (A) if and only if M^* satisfies Property (A).

2) Let I be any finitely generated ideal I of R. Since E is an injective S-module, we have $(R/I) \otimes_R \mathrm{Hom}_S(M, E) \cong \mathrm{Hom}_S(\mathrm{Hom}_R(R/I, M), E) \cong \mathrm{Hom}_S(\mathrm{ann}_M(I), E)$. Since E is an injective cogenerator for S-modules, and by Theorem 2.8(2), $\mathrm{Z}_R(M) = \mathrm{W}_R(M^*)$. It is easy to check that M^* satisfies the dual of Property (A) if and only if M satisfies Property (A).

□

According to this theorem, the following corollary can be easily derived.

Corollary 3.6. *Let R be a ring and E an injective cogenerator for R-modules. Then,*

(1) *E satisfies Property (A).*

(2) *R is an (A)-ring if and only if E satisfies the dual of Property (A).*

Proposition 3.7. *Let M be an R-module over a ring R, and let E be an injective cogenerator for R-modules. The following statements are equivalent:*

(1) *$R/\mathrm{ann}_R(M)$ is an (A)-ring;*

(2) *$\mathrm{ann}_E(\mathrm{ann}_R(M))$ satisfies de dual of Property (A) as $(R/\mathrm{ann}_R(M))$-module;*

(3) *$\mathrm{ann}_E(\mathrm{ann}_R(M))$ satisfies de dual of Property (A) as R-module.*

Proof. (1) ⇔ (2) Apply Corollary 3.6 since $\mathrm{ann}_E(\mathrm{ann}_R(M))$ is an injective cogenerator for $(R/\mathrm{ann}(M))$-modules
(2) ⇔ (3) Apply [13, Corollary 2.7]. □

Next, we show that the dual of Property (A) is not preserved under submodules, homomorphic images, or direct factors.

Example 3.8 ($B_1 \oplus B_2$ satisfies the dual of Property (A), but B_1 and B_2 do not). Let R be a commutative ring in which each maximal ideal is finitely generated, and let $\max(R)$ denote the set of maximal ideals of R. Suppose that $\max(R)$ has a decomposition $\max(R) = \Lambda_1 \cup \Lambda_2$, where

$$\cup_{m \in \Lambda_1} m = \cup_{m \in \max(R)} m = \cup_{m \in \Lambda_2} m.$$

Put $B_i = A_i^+$, where $A_i = \oplus_{m \in \Lambda_i} R/m$, $i = 1, 2$, and $B = B_1 \oplus B_2$. Then, by [3, Example 2.6], the module $A_1 \oplus A_2$ satisfies Property (A), but neither A_1 nor A_2 satisfies Property (A). By Theorem 3.5 (2), B satisfies the dual of Property (A), but neither B_1 nor B_2 satisfies the dual of Property (A).

Example 3.9 (B_1 and B_2 satisfies the dual of Property (A), but $B_1 \oplus B_2$ does not). Take D to be a two-dimensional regular local ring with maximal ideal $m = (x_1, x_2)$, as in [3, Example 2.9]. Define $A_1 = D/(x_1)$ and

$$A_2 = \oplus_{P \in X^1(D) \setminus \{(x_1)\}} D/P,$$

where $X^1(D)$ denotes the set of height-one prime ideals of D. Then, $A_1 \oplus A_2$ does not satisfy Property (A), but A_1 and A_2 satisfies Property (A) (see [3, Example 2.11]). Now, define $B_i = A_i^+$ for $i = 1, 2$. By Theorem 3.5 (2), both B_1 and B_2 satisfy the dual of Property (A), but $B_1 \oplus B_2 =$ does not satisfy the dual of Property (A).

Recall that an R-module M is called a multiplication module if, for every submodule N of M, there exists an ideal I of R such that $N = IM$. The following theorem characterizes multiplication modules that satisfy Property (A).

Theorem 3.10. *let R be a ring and M be a multiplication R-module. If E is an injective cogenerator for R-modules, then the following statements are equivalent:*

(1) *M satisfies Property (A);*

(2) *$R/\operatorname{ann}_R(M)$ is an (A)-ring;*

(3) *$\operatorname{ann}_E(\operatorname{ann}_R(M))$ satisfies the dual of Property (A) as R-module.*

Proof. (1) ⇔ (2) It follows from [22, Theorem 1].
(2) ⇔ (3) Apply Proposition 3.7.

□

Recall that an R-module M is called a comultiplication module if, for every submodule N of M, there exists an ideal I of R such that $N = \operatorname{ann}_M(I)$. The following theorem characterizes comultiplication modules that satisfy the dual of Property (A).

Theorem 3.11. *let R be a ring and M be a comultiplication R-module. If E is an injective cogenerator for R-modules, then the following statements are equivalent:*

(1) *M satisfies the dual of Property (A);*

(2) *$R/\operatorname{ann}_R(M)$ is an (A)-ring;*

(3) *$\operatorname{ann}_E(\operatorname{ann}_R(M))$ satisfies the dual of Property (A) as R-module.*

Proof. (2) ⇔ (3) Apply Proposition 3.7.
(1) ⇒ (2) Suppose that M satisfies the dual of Property (A). Let $\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n} \in R/\operatorname{ann}_R(M)$ such that $a_i \in R$ for $i = 1, 2, \ldots, n$, and assume that $G = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}) \subseteq \mathrm{Z}(R/\operatorname{ann}_R(M))$. Let $I = (a_1, a_2, \ldots, a_n)$ be an ideal of R. Take any $x \in I$. Then $\overline{x} \in G$, and so there exists some $\overline{y} \in R/\operatorname{ann}_R(M)$ such that $\overline{xy} = \overline{0}$, with $\overline{y} \neq \overline{0}$. This implies that $xyM = 0$, so we conclude $y \notin \operatorname{ann}_R(M)$, and hence there exists $m \in M$ such that $ym \neq 0$. Since $xyM = 0$, we have $x(ym) = 0$, which implies that $x \in \mathrm{Z}_R(M)$. Therefore, $I \subseteq \mathrm{Z}_R(M)$. Since M satisfies the dual of Property (A), it follows that $IM \neq M$. Since M is a comultiplication R-module, we have $IM = \operatorname{ann}_M(L)$ for some ideal L of R. Thus, we conclude that $LM \neq 0$. There exists some $b \in L \setminus \operatorname{ann}_R(M)$, and so $bIM = 0$. This implies that $I\overline{b} = \overline{0}$, showing that $R/\operatorname{ann}_R(M)$ is an (A)-ring.
(2) ⇒ (1) Suppose that $R/\operatorname{ann}_R(M)$ is an (A)-ring. Let $I = (a_1, a_2, \ldots, a_n)$ be an ideal of R contained in $\mathrm{W}_R(M)$. Now, let $G = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n})$ be an ideal of $\mathrm{Z}(R/\operatorname{ann}_R(M))$. If $\overline{x} \in G$ for some $x \in R$, then there exist $u \in \operatorname{ann}_R(M)$ and $y \in I$ such that $x = u + y$. Since $y \in \mathrm{W}_R(M)$, we have $yM \neq M$ and $xM = (u + y)M = yM$. Since M is a comultiplication R-module, we have $xM = \operatorname{ann}_M(K)$ for some ideal K of R. Since $xM \neq M$, it follows that there exists some $b \in R \setminus \operatorname{ann}_R(M)$, and so $xbM = 0$. This implies that $\overline{xb} = \overline{0}$ for some $\overline{b} \neq \overline{0}$. Thus, we conclude that $G = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_n}) \subseteq \mathrm{Z}(R/\operatorname{ann}_R(M))$. Since $R/\operatorname{ann}_R(M)$ is an (A)-ring, there exists some $\overline{c} \in R/\operatorname{ann}_R(M)$ such that $G\overline{c} = \overline{0}$, which implies that $a_i c \in \operatorname{ann}_R(M)$ for all i with $c \notin \operatorname{ann}_R(M)$. Thus, we obtain that $cm \neq 0$ for some $m \in M$. Now, suppose for contradiction that $IM = M$. Then, $cM = cIM = 0$, which implies that $c \in \operatorname{ann}_R(M)$, a contradiction. Therefore, $IM \neq M$, and hence M satisfies the dual of Property (A). □

We conclude this section with the following proposition, which provides the module-theoretic version of this result.

Proposition 3.12. *Let R be a commutative ring and M be an R-module such that $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$. Then the following assertions are equivalent:*

(1) *M satisfies the dual of Property (A) as R-module;*

(2) *$\mathrm{q}_M(M)$ satisfies the dual of Property (A) as R-module;*

(2) *$\mathrm{q}_M(M)$ satisfies the dual of Property (A) as $\mathrm{q}_M(R)$-module.*

Proof. Apply [13, Corollary 2.10]. □

4 On the duals of strong Property (A)

We start this section with a theorem that is similar to Theorem 3.5.

Theorem 4.1. *Let R be an algebra commutative over a commutative ring S, and let E be an injective cogenerator for S-modules. For an R-module M, the following holds:*

(1) *M satisfies the dual of strong Property (A) if and only if M^* satisfies strong Property (A).*

(2) *M^* satisfies the dual of strong Property (A) if and only if M satisfies strong Property (A).*

Proof. The proof is similar to the proof of Theorem 3.5. □

Definition 4.2. The ring R is said to satisfy the dual of strong (A)-ring if R satisfies the dual of strong Property (A) as R-module.

Corollary 4.3. *Let R be a ring and E an injective cogenerator for R-modules. Then,*

(1) *R satisfies the dual of strong (A)-ring if and only if E satisfies strong Property (A).*

(2) *R satisfies strong (A)-ring if and only if E satisfies the dual of strong Property (A).*

Proof. Apply Theorem 4.1. □

The following theorem characterizes the conditions under which a module M satisfies the dual of the strong Property (A).

Theorem 4.4. *Let R be a ring and M an R-module. Consider the following conditions:*

(1) *M satisfies the dual of strong Property (A).*

(2) *M satisfies the dual of Property (A) and $\mathrm{W}_R(M)$ is an ideal of R.*

(3) *M satisfies the dual of Property (A) and $\mathrm{W}_R(M)$ is a prime ideal of R.*

(4) *M satisfies the dual of Property (A) and $\mathrm{q}_M(R)$ is a local ring.*

(5) *$\mathrm{q}_M(M)$ satisfies the dual of strong Property (A) as $\mathrm{q}_M(R)$-module.*

(6) *$\mathrm{q}_M(M)$ satisfies the dual of Property (A) as $\mathrm{q}_M(R)$-module and $\mathrm{q}_M(R)$ is a local ring.*

Then (1), (2) and (3) are equivalent. If further $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$, then (1), (2), (3), (4), (5) and (6) are equivalent.

For the proof this theorem, we need the following lemma.

Lemma 4.5. *Let R be a ring and M be an R-module. Consider the conditions:*

(1) *$\mathrm{W}_R(M)$ is an ideal of R.*

(2) *$\mathrm{W}_R(M)$ is a prime ideal of R.*

(3) *$\mathrm{q}_M(R)$ is a local ring.*

Then (1) $\Leftrightarrow$ (2) *and* (2) $\Rightarrow$ (3). *Furthermore, if* $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$, *then* (1), (2), *and* (3) *are equivalent.*

Proof. (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are straightforward. Now assume that $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$. Then (3) $\Rightarrow$ (2) follows from Proposition 2.12. □

Proof of Theorem 4.4. (1) $\Rightarrow$ (2) Assume that M satisfies the dual of strong Property (A). Thus, M also satisfies the dual of Property (A). Let $a, b \in \mathrm{W}_R(M)$. By hypothesis, there exists a submodule N of M such that $aM \subseteq N \neq M$ and $bM \subseteq N \neq M$. It follows that $(a-b)M \subseteq N \neq M$, which implies that $a - b \in \mathrm{W}_R(M)$. Hence, $\mathrm{W}_R(M)$ is an ideal of R.
(2) $\Rightarrow$ (1) Assume that M satisfies the dual of Property (A) and that $\mathrm{W}_R(M)$ is an ideal of R. Suppose that $r_1, \dots, r_n \in W_R(M)$. Since $\mathrm{W}_R(M)$ is an ideal of R, we have $(r_1, \dots, r_n) \subseteq W_R(M)$. As M satisfies the dual of Property (A), it follows that $(r_1, \dots, r_n)M \neq M$. This implies that M satisfies the dual of strong Property (A).
(2) $\Leftrightarrow$ (3) It holds by lemma 4.5.
Now, assume that $\mathrm{Z}_R(M) \subseteq \mathrm{W}_R(M)$. Then:
(3) $\Rightarrow$ (4) This follows from lemma 4.5.
(4) $\Rightarrow$ (5) Suppose that M satisfies the dual of Property (A) and that $\mathrm{q}_M(R)$ is a local ring. Let $G = (\frac{r_1}{s_1}, \frac{r_2}{s_2}, \dots, \frac{r_n}{s_n})$ be a finitely generated ideal of $\mathrm{q}_M(R)$ such that $\frac{r_i}{s_i} \in \mathrm{W}_{\mathrm{q}_M(R)}(\mathrm{q}_M(M))$, with $r_i \in \mathrm{W}_R(M)$ and $s_i \in S_M$ for each i, as stated in Proposition 2.12. Consider the finitely generated ideal $I = (r_1, r_2, \dots, r_n)$ of R, where $r_i \in \mathrm{W}_R(M)$ for each i. By lemma 4.5, $\mathrm{W}_R(M)$ is an ideal of R, so we have $I \subseteq \mathrm{W}_R(M)$. Since M satisfies the dual of Property (A), it follows that $IM \neq M$. Now, assume for contradiction that $Gq_M(M) = q_M(M)$. Let $m \in M \backslash IM$, then we can write $\frac{m}{1} = \sum_{i=1}^{i=n} \frac{r_i}{s_i}\frac{m_i}{t_i} = \sum_{i=1}^{i=n} \frac{r_i}{1}\frac{m_i}{s_i t_i}$, where $m_i \in M$ and $t_i \notin \mathrm{W}_R(M)$ for each i. Since $s_i t_i \notin \mathrm{W}_R(M)$, it follows that $s_i t_i M = M$, and so $m_i = s_i t_i m_i^{'}$ for some $m_i^{'} \in M$. Therefore, we have $\frac{m}{1} = \sum_{i=1}^{i=n} \frac{r_i m_i^{'}}{1}$, and there exists $t \notin \mathrm{W}_R(M)$ such that $t(m - \sum_{i=1}^{i=n} r_i m_i^{'}) = 0$. Since $t \notin \mathrm{Z}_R(M)$, we conclude that $m = \sum_{i=1}^{i=n} r_i m_i^{'} \in IM$, which is a contradiction. Hence, $Gq_M(M) \neq q_M(M)$. This implies that $q_M(M)$ satisfies the dual of strong Property (A) as a $\mathrm{q}_M(R)$-module.
(5) $\Rightarrow$ (6) The proof is similar to the proof (1) $\Rightarrow$ (2).
(6) $\Rightarrow$ (2) This follows from Proposition 3.12 and lemma 4.5. This completes the proof of the theorem. □

Note that $\mathrm{Z}_R(R) \subseteq \mathrm{W}_R(R)$ since $\mathrm{W}_R(R) = \bigcup\{m : m \in \max(R)\}$. Next, we state the ring-theoretic version of Theorem 4.4.

Proposition 4.6. *Let R be a commutative ring and E an injective cogenerator for R-modules. Then the following assertions are equivalent:*

(1) *R satisfies the dual of strong (A)-ring;*

(2) *E satisfies strong Property (A) ;*

(3) *$\mathrm{Z}_R(E)$ is an ideal of R;*

(4) *$\mathrm{W}_R(R)$ is an ideal of R;*

(5) *$\mathrm{W}_R(R)$ is a maximal ideal of R;*

(6) *R is a local ring;*

(7) *$\mathrm{q}_R(R)$ is a local ring;*

(8) *$\mathrm{q}_R(R)$ satisfies the dual of strong (A)-ring;*

(9) *$\widetilde{Coass}(R)$ has a unique maximal element;*

(10) *$\widetilde{Ass}(E)$ has a unique maximal element.*

For the proof this proposition, we need the following lemma.

Lemma 4.7 ([7], Theorem 2.4). *Let R be a ring and M an R-module. Then the following assertions are equivalent:*

(1) *M satisfies strong Property (A);*

(2) *M satisfies Property (A) and $\mathrm{Z}_R(M)$ is an ideal of R.*

Proof of Proposition 4.6. (1) $\Leftrightarrow$ (2) Apply Corollary 4.3.
(2) $\Leftrightarrow$ (3) This follows from Lemma 4.7 and Corollary 3.6.
(3) $\Leftrightarrow$ (4) Apply Theorem 2.8.
(4) $\Rightarrow$ (5) Assume that $\mathrm{W}_R(R)$ is an ideal of R. Since $W_R(R) = \bigcup\{m : m \in \max(R)\}$, we have $\mathrm{W}_R(R) = m$ for each maximal ideal m of R. Therefore, $\mathrm{W}_R(R)$ is a maximal ideal of R.
(5) $\Rightarrow$ (4) This is trivial.
(4) $\Leftrightarrow$ (6) and (6) $\Leftrightarrow$ (7) Apply Lemma 4.5.
(7) $\Leftrightarrow$ (8) Apply Theorem 4.4.
(5) $\Leftrightarrow$ (9) Since $\mathrm{W}_R(R) = \bigcup_{P \in C\widetilde{oass}(R)} P$ by [26, Theorem 2.15], it is easy to check that $\mathrm{W}_R(R)$ is a maximal ideal of R if and only if $C\widetilde{oass}(R)$ has a unique maximal element.
(5) $\Leftrightarrow$ (10) Note that $\mathrm{Z}_R(E) = W_R(R)$ by Corollary 2.9, and $\mathrm{Z}_R(E) = \bigcup_{P \in \widetilde{Ass}(E)} P$. Therefore, it is easy to check that $\mathrm{W}_R(R)$ is a maximal ideal of R if and only if $\widetilde{Ass}(E)$ has a unique maximal element. □

From the [26, Corollary 1.2 and Theorem 2.10], if R is Noetherian ring and M is a finitely generated R-module, then $\widetilde{Ass}(M) = \mathrm{Ass}(M)$ and $C\widetilde{oass}(R) = \mathrm{Coass}(M)$. Next, we can easily recover this result from Proposition 4.6, since in the Noetherian setting, the maximal primes of R are the coassociate primes of R.

Corollary 4.8. *Let R be a Noetherian ring and E an injective cogenerator for R-modules. Then the following assertions are equivalent:*

(1) *E satisfies strong Property (A);*

(2) *R is a local ring;*

(3) $\mathrm{Coass}(R)$ *has a unique maximal element;*

(4) $\mathrm{Ass}(E)$ *has a unique maximal element.*

Example 4.9. Let $\mathbb{Z}_n$ be the ring of integers modulo an integer $n \geq 1$. Let $n := up_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the decomposition of n into prime numbers, where $u \in \{-1, 1\}$. By Proposition 4.6, $\mathbb{Z}_n$ satisfies the dual of strong (A)-ring if and only if $n = up_i^{n_i}$, an associate of a power of a prime number p_i.

Our next result completely characterizes (A)-rings R that coincide with their total quotient rings and over which a strong (A)-ring coincide with the dual of strong (A)-ring. Note that among the (A)-rings that coincide with their total quotient rings, the class of zero-dimensional rings is included.

Corollary 4.10. *Let R be an (A)-ring with $Q(R) = R$ (for instance, a zero-dimensional ring). Then the following statements are equivalent:*

(1) *R is a strong (A)-ring;*

(2) *R satisfies the dual of strong (A)-ring;*

(3) *R is a local ring.*

Proof. (1) $\Leftrightarrow$ (2) Since $Q(R) = R$, by Proposition 2.11, we have $\mathrm{Z}_R(R) = \mathrm{W}_R(R)$. Since R satisfies the dual of the (A)-ring property and is an (A)-ring, by Proposition 4.6 and Lemma 4.7, R is a strong (A)-ring if and only if R satisfies the dual of strong (A)-ring.
(2) $\Leftrightarrow$ (3) Apply Proposition 4.6. □

Recall that an R-module M is said to be a reduced module if $r^2 m = 0$ for $r \in R$ and $m \in M$ implies that $rm = 0$. An R-module M is called a semisecond if $aM = a^2 M$ for each $a \in R$ [4]. Now, an R-module M is called a coreduced if $r^2 M \subseteq N$ implies that $rM \subseteq N$, where $r \in R$ and N is a completely irreducible submodule of M. Equivalently, M is a semisecond by [5, Theorem 2.13]. We now present the following proposition characterizing coreduced rings that satisfy the dual of strong Property (A).

Proposition 4.11. *Let R be a ring and E an injective cogenerator of R-modules. Then the following statements are equivalent:*

(1) *R is a coreduced and satisfy the dual of strong (A)-ring;*

(2) *E is a reduced and satisfy strong Property (A);*

(3) *R is a field.*

Proof. (1) $\Rightarrow$ (3) Suppose that R is coreduced and satisfies the dual of a strong (A)-ring. Then, by Proposition 4.6, R is local ring with maximal ideal m of R. Since R is a coreduced, it follows from [5, Theorem 2.13] that $aR = a^2R$ for each $a \in m$. By Nakayama's Lemma, we obtain $aR = 0$, which implies that $a = 0$ and hence $m = \{0\}$. It follows that R is a field.
(3) $\Rightarrow$ (2) is straightforward.
(2) $\Rightarrow$ (1) Suppose that E is a reduced and satisfies strong Property (A). Let $a \in R$ and I be a completely irreducible ideal of R such that $a^2R \subseteq I$. Since R/I is a cocyclic R-module, we see that R/I is a submodule of E. As E is reduced, we obtain $a(R/I) = 0$. Thus $aR \subseteq I$, and so R is coreduced. Since E satisfies strong Property (A), it follows from Proposition 4.6 that R satisfies the dual of a strong (A)-ring. □

Lemma 4.12. *Let R be a ring and E an injective cogenerator of R-modules. Consider the following conditions:*

(1) *R is an integral domain;*

(2) *E is coreduced and satisfy the dual of strong Property (A);*

(3) *R is reduced and a strong (A)-ring.*

Then (1) $\Rightarrow$ (2) *and* (2) $\Rightarrow$ (3).

Proof. (1) $\Rightarrow$ (2) Suppose that R is an integral domain. Since E is an injective R-module, we have $a^2E = aE$ four each $a \in R$. Thus, we conclude that E is semisecond, and hence E is coreduced. It is clear from the definition that an integral domain is a strong (A)-ring. Moreover, by Corollary 4.3 (2), E satisfies the dual of strong Property (A).
(2) $\Rightarrow$ (3) Suppose that E is coreduced and satisfies the dual of strong Property (A). Let $a \in R$ be such that $a^n = 0$ for some integer $n \geqslant 1$. Since E is coreduced, it follows that E is semisecond, and thus $aE = a^nE = 0$. Consequently, we conclude that $a \in \text{ann}_R(E) = \text{ann}_R(R) = 0$, implying that R is reduced. Since E satisfies the dual of strong Property (A), it follows from Corollary 4.3 (2) that R satisfies strong (A)-ring. □

In [10, Proposition 2.1], Dobbs and Shapiro proved that if R is a ring with only finitely many minimal prime ideals (for instance, a Noetherian ring or even a finite ring), then R is reduced and a strong (A)-ring if and only if R is an integral domain. Next, we easily recover this result from Lemma 4.12.

Corollary 4.13. *Let R be a ring with only finitely many minimal prime ideals, and let E be an injective cogenerator of R-modules. Then the following statements are equivalent:*

(1) *R is an integral domain;*

(2) *E is coreduced and satisfy the dual of strong Property (A);*

(3) *R is reduced and a strong (A)-ring.*

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