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IDEAL CONVERGENCE IN NEUTROSOPHIC n-NORMED LINEAR SPACES

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Abstract In this research paper, we present a thorough exploration of $\mathscr{I}$-convergence and $\mathscr{I}$-Cauchy sequence in neutrosophic n-normed linear spaces. Furthermore, we prove some interesting results associated with this notion. Additionally, we define the concept of $\mathscr{I}^*$-convergence and $\mathscr{I}^*$-Cauchy sequence and establish the connection between these notions within this specific framework.

1 Introduction

As a momentous generalization of ordinary convergence structure of sequences of real numbers, the concept of statistical convergence was first explored independently by Fast [10], Steinhaus [38] and Schoenberg [39]. One of its engrossing generalization known as $\mathscr{I}$-convergence has been described by Kostyrko et al. [24] where $\mathscr{I}$ stands for an ideal defined as a collection of subsets of the set of natural numbers which satisfies certain conditions. After that, this significant concept is being brought up in different directions by various researchers like [7, 8, 9, 12, 17, 19, 20, 21, 22, 27, 32, 33, 35, 36, 37, 42], and many more references therein.

Zadeh [46] is the first prominent pioneering of the introduction of fuzzy set theory as an extension of classical set theory. Since then, it is being developed and applied in various branches of engineering and science, namely population dynamics [4], control of chaos [11], fuzzy reasoning [15], fuzzy nonlinear dynamical system [18], fuzzy physics [30] etc. Later on, this concept has been intriguingly generalized into new notions as its extension like intuitionistic fuzzy set [1], interval valued fuzzy set [43], interval valued intuitionistic fuzzy set [2], vague fuzzy set [3]. As a comprehensive generalization of these concepts, Smarandache [41] defined a new idea named as neutrosophic set. Later on, Bera and Mahapatra explored the notion of neutrosophic soft linear space [5] and neutrosophic soft normed linear space [6]. In recent times, Kirişci and Şimşek [26] defined neutrosophic normed space and in this significant framework, different types of summability method have been investigated. The idea of 2-normed linear space as well as its momentous extension to n-normed linear space was introduced by Gähler [13, 14]. After that, this notion was intriguingly nurtured by many researchers like Kim and Cho [23], Malceski [31], Misiak [29], Gunawan and Mashadi [16], Tripathy and Borgogain [44, 45]. In 2023, Murtaza et al. [34] defined the excellent idea named as neutrosophic 2-normed linear space which is a momentous generalization of neutrosophic normed space. In very recent time, Kumar et al. [28] introduced the concept of neutrosophic n-normed linear space and studied convergence structure and defined Cauchy sequence within this space. In this article, we have defined and studied $\mathscr{I}$-convergence, $\mathscr{I}$-Cauchy sequence and prove some interesting results within this specific framework.

2 Preliminaries

In this section, we provide an overview of basic definitions and terminology which will be useful to describe our main results. Throughout the study $\mathbb{N}$ stands for the set of all natural numbers.

Definition 2.1. [24] A family $\mathscr{I}$ of subsets of a non empty set $\mathscr{X}$ is said to be an ideal in $\mathscr{X}$ if the following conditions hold:

(i) $\mathscr{A}, \mathscr{B} \in \mathscr{I}$ implies $\mathscr{A} \cup \mathscr{B} \in \mathscr{I}$;

(ii) $\mathscr{A} \in \mathscr{I}$ and $\mathscr{B} \subset \mathscr{A}$ implies $\mathscr{B} \in \mathscr{I}$.

An ideal $\mathscr{I}$ is called non trivial if $\mathscr{X} \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$.

Definition 2.2. [24] A non trivial ideal $\mathscr{I} \subset 2^{\mathscr{X}}$ is called admissible if $\{\{x\} : x \in X\} \subset \mathscr{I}$.

Definition 2.3. [24] A non empty family $\mathscr{F}$ of subsets of a non empty set $\mathscr{X}$ is called a filter in $\mathscr{X}$ if the following properties hold:

(i) $\emptyset \notin \mathscr{F}$;

(ii) $\mathscr{A}, \mathscr{B} \in \mathscr{F}$ implies $\mathscr{A} \cap \mathscr{B} \in \mathscr{F}$;

(iii) $\mathscr{A} \in \mathscr{F}$ and $\mathscr{A} \subset \mathscr{B}$ implies $\mathscr{B} \in \mathscr{F}$.

If $\mathscr{I} \subset 2^{\mathscr{X}}$ is a non trivial ideal then the class $\mathscr{F}(\mathscr{I}) = \{\mathscr{X} \setminus \mathscr{A} : \mathscr{A} \in \mathscr{I}\}$ is a filter on $\mathscr{X}$ which is called filter associated with the ideal $\mathscr{I}$ [24].

Definition 2.4. [24] An admissible ideal $\mathscr{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{\mathscr{A}_1, \mathscr{A}_2, \ldots\}$ belonging to $\mathscr{I}$ there exists a countable family of sets $\{\mathscr{B}_1, \mathscr{B}_2, \ldots\}$ such that the symmetric difference $\mathscr{A}_i \triangle \mathscr{B}_i$ is finite for each $i \in \mathbb{N}$ and $\bigcup_{i=1}^{\infty} \mathscr{B}_i \in \mathscr{I}$.

Definition 2.5. Let $\mathscr{K} \subset \mathbb{N}$. Then the natural density of $\mathscr{K}$, denoted by $\delta(\mathscr{K})$, is defined as

$$\delta(\mathscr{K}) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in \mathscr{K}\}|,$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

Definition 2.6. [40] A binary operation $\boxdot : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0, 1]$ is named to be a continuous t-norm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:

(i) $\boxdot$ is associative and commutative;

(ii) $\boxdot$ is continuous;

(iii) $\nu_1 \boxdot 1 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;

(iv) $\nu_1 \boxdot \nu_2 \leq \nu_3 \boxdot \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Definition 2.7. [40] A binary operation $\oplus : \mathscr{J} \times \mathscr{J} \to \mathscr{J}$, where $\mathscr{J} = [0, 1]$ is named to be a continuous t-conorm if for each $\nu_1, \nu_2, \nu_3, \nu_4 \in \mathscr{J}$, the below conditions hold:

(i) $\oplus$ is associative and commutative;

(ii) $\oplus$ is continuous;

(iii) $\nu_1 \oplus 0 = \nu_1$ for all $\nu_1 \in \mathscr{J}$;

(iv) $\nu_1 \oplus \nu_2 \leq \nu_3 \oplus \nu_4$ whenever $\nu_1 \leq \nu_3$ and $\nu_2 \leq \nu_4$.

Example 2.8. [25] The continuous t-norms are $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \boxdot \nu_2 = \nu_1.\nu_2$. On the other hand, continuous t-conorms are $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$ and $\nu_1 \oplus \nu_2 = \nu_1 + \nu_2 - \nu_1.\nu_2$.

Definition 2.9. [16] Let $n \in \mathbb{N}$ and $\mathscr{W}$ be a real vector space having dimension $d \geq n$ (d is finite or infinite). A real valued function $\|\cdot, \ldots, \cdot\|$ on $\underbrace{\mathscr{W} \times \mathscr{W} \times \ldots \times \mathscr{W}}_{n\ times} = \mathscr{W}^n$, gratifying the below four axioms:

(i) $\|(w_1, w_2, \ldots, w_n)\| = 0$ if and only if $w_1, w_2, \ldots, w_n$ are linearly dependent;

(ii) $\|(w_1, w_2, \ldots, w_n)\|$ remains invariant under any permutation of $w_1, w_2, \ldots, w_n$;

(iii) $\|(w_1, w_2, \ldots, w_{n-1}, \kappa w_n)\| = |\kappa| \, \|(w_1, w_2, \ldots, w_{n-1}, w_n)\|$ for $\kappa \in \mathbb{R}$ (set of real numbers);

(iv) $\|(w_1, w_2, \ldots, w_{n-1}, \tau + \omega)\| \leq \|(w_1, w_2, \ldots, w_{n-1}, \tau)\| + \|(w_1, w_2, \ldots, w_{n-1}, \omega)\|$.

is called an n-norm on $\mathscr{W}$ and the pair $(\mathscr{W}, \|\cdot, \ldots, \cdot\|)$ is named to be an n-normed linear space.

As an illustration of n-normed linear space we take $\mathscr{W} = \mathbb{R}^n$ equipped with the Euclidean norm

$$\|(w_1, w_2, \ldots, w_n)\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. For instance, we get $\|(w_1, w_2, \ldots, w_n)\| \geq 0$ in an n-normed linear space.

Definition 2.10. [28] Let $\mathscr{W}$ be a vector space over $\mathscr{F}$ and $\boxdot$ and $\oplus$ be continuous t-norm and t-conorm respectively. Let $\Im, \Re, \wp$ be the functions from $\mathscr{W}^n \times (0, \infty)$ to $[0, 1]$. Then a six tuple $(\mathscr{W}, \Im, \Re, \wp, \boxdot, \oplus)$ is named to be a neutrosophic n-normed linear space (in short Nn-NLS), $(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) \in \mathscr{W}^n \times (0, \infty) \to [0, 1]$, if the below conditions hold:

(i) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) + \Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) + \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) \leq 3$;

(ii) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) > 0$;

(iii) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 1$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;

(iv) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \ldots, w_n$;

(v) $\Im(w_1, w_2, \ldots, w_{n-1}, \kappa w_n; \zeta) = \Im\left(w_1, w_2, \ldots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|}\right)$, $\kappa \neq 0$ and $\kappa \in \mathscr{F}$;

(vi) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n + w_n'; \zeta + \tau)$
$\geq \Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) \boxdot \Im(w_1, w_2, \ldots, w_{n-1}, w_n'; \tau)$;

(vii) $\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is non-decreasing continuous in ζ;

(viii) $\lim_{\zeta \to \infty} \Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 1$ and $\lim_{\zeta \to 0} \Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 0$;

(ix) $\Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) > 0$;

(x) $\Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 0$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;

(xi) $\Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \ldots, w_n$;

(xii) $\Re(w_1, w_2, \ldots, w_{n-1}, \kappa w_n; \zeta) = \Re\left(w_1, w_2, \ldots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|}\right)$, $\kappa \neq 0$ and $\kappa \in \mathscr{F}$;

(xiii) $\Re(w_1, w_2, \ldots, w_{n-1}, w_n + w_n'; \zeta + \tau)$
$\leq \Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) \oplus \Re(w_1, w_2, \ldots, w_{n-1}, w_n'; \tau)$;

(xiv) $\Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;

(xv) $\lim_{\zeta \to \infty} \Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 0$ and $\lim_{\zeta \to 0} \Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 1$;

(xvi) $\wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) > 0$;

(xvii) $\wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 0$ if and only if w_j are linearly dependent, $1 \leq j \leq n$;

(xviii) $\wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is invariant under any permutation of $w_1, w_2, \ldots, w_n$;

(xix) $\wp(w_1, w_2, \ldots, w_{n-1}, \kappa w_n; \zeta) = \wp\left(w_1, w_2, \ldots, w_{n-1}, w_n; \frac{\zeta}{|\kappa|}\right)$, $\kappa \neq 0$ and $\kappa \in \mathscr{F}$;

(xx) $\wp(w_1, w_2, \ldots, w_{n-1}, w_n + w_n'; \zeta + \tau)$
$\leq \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) \oplus \wp(w_1, w_2, \ldots, w_{n-1}, w_n'; \tau)$;

(xxi) $\wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta)$ is non-increasing continuous in ζ;

(xxii) $\lim_{\zeta\to\infty} \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 0$ and $\lim_{\zeta\to 0} \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = 1$;

In the sequal, we shall use the notation $\mathscr{H}$ for neutrosophic n-normed linear space instead of $(\mathscr{W}, \Im, \Re, \wp, \boxdot, \oplus)$ and we denote $\mathscr{N}_n$ to mean neutrosophic n-norm on $\mathscr{H}$.

Example 2.11. [28] Let $(\mathscr{W}, \|w_1, w_2, \ldots, w_n\|)$ be an n-normed linear space. Also, let $\nu_1 \boxdot \nu_2 = \min(\nu_1, \nu_2)$ and $\nu_1 \oplus \nu_2 = \max(\nu_1, \nu_2)$ for every $\nu_1, \nu_2 \in [0, 1]$. If we define $\Im, \Re$ and $\wp$ as

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = \frac{\zeta}{\zeta + \|(w_1, w_2, \ldots, w_{n-1}, w_n)\|}$$

$$\Re(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = \frac{\|(w_1, w_2, \ldots, w_{n-1}, w_n)\|}{\zeta + \|(w_1, w_2, \ldots, w_{n-1}, w_n)\|}$$

$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_n; \zeta) = \frac{\|(w_1, w_2, \ldots, w_{n-1}, w_n)\|}{\zeta}.$$

Then $(\mathscr{W}, \Im, \Re, \wp, \boxdot, \oplus)$ is a neutrosophic n-normed linear space.

Definition 2.12. [28] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be convergent to $\upsilon \in \mathscr{W}$ with respect to $\mathscr{N}_n$ if for every $\sigma > 0, \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$ for all $k \geq k_0$. In this scenario, it is denoted as $\mathscr{N}_n - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{N}_n} \upsilon$.

Definition 2.13. [28] Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be Cauchy sequence with respect to $\mathscr{N}_n$ if for every $\sigma > 0, \zeta > 0$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_m; \zeta) < \sigma$ for all $k, m \geq k_0$.

Theorem 2.14. *[28] Let $\{w_k\}$ be a sequence in $\mathscr{W}$. Then $\{w_k\}$ is convergent in $(\mathscr{W}, \|\cdot, \ldots, \cdot\|)$ iff $\{w_k\}$ is convergent in $\mathscr{H}$ with respect to neutrosophic n-norm as defined in Example 2.11.*

3 Main Results

Throughout this section $\mathscr{I}$ stands for an admissible ideal of $\mathbb{N}$ unless otherwise stated. First we introduce the notion of $\mathscr{I}$-convergence in neutrosophic n-normed linear spaces.

Definition 3.1. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be $\mathscr{I}$-convergent to $\upsilon \in \mathscr{W}$ with respect to $\mathscr{N}_n$ (in short $\mathscr{I}(\mathscr{N}_n)$-convergence) if for every $\sigma \in (0, 1), \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ such that

$$\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \\ \text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}.$$

In this scenario, it is denoted as $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{I}(\mathscr{N}_n)} \upsilon$. And, υ is called $\mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$.

Example 3.2. Let $\mathscr{W} = \mathbb{R}^n$ with

$$\|(w_1, w_2, \ldots, w_n)\| = abs \left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix} \right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0, 1]$. We take Nn-NLS as defined in

Example 2.11. Let $\mathscr{I}$ be a class of subsets of $\mathbb{N}$ such that natural density of each subset is zero. Then $\mathscr{I}$ becomes a nontrivial admissible ideal. Now, we define a sequence $\{w_k\} \in \mathscr{W}$ by

$$w_k = \begin{cases} (1,0,\dots,0) = \mathbf{1}, \text{ if } k = i^2, \ i \in \mathbb{N} \\ (0,0,\dots,0) = \mathbf{0}, \text{ otherwise.} \end{cases}$$

Then for any $\sigma \in (0,1)$ and $\zeta > 0$, we have

$$\begin{aligned} &\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k; \zeta) \le 1-\sigma \text{ or } \Re(w_1, w_2, \dots, w_{n-1}, w_k; \zeta) \ge \sigma \text{ and} \\ &\wp(w_1, w_2, \dots, w_{n-1}, w_k; \zeta) \ge \sigma\} \\ =&\{k \in \mathbb{N} : \|(w_1, w_2, \dots, w_{n-1}, w_k)\| \ge \frac{\zeta\sigma}{1-\sigma} > 0 \text{ or } \|(w_1, w_2, \dots, w_{n-1}, w_k)\| \ge \zeta\sigma > 0\} \\ \subset&\{k \in \mathbb{N} : k = i^2, \ i \in \mathbb{N}\}. \end{aligned}$$

Since $\delta(\{k \in \mathbb{N} : k = i^2, \ i \in \mathbb{N}\}) = 0$, $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \mathbf{0}$.

From Definition 3.1, we can easily prove the following lemma.

Lemma 3.3. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then, for every $\sigma \in (0,1), \zeta > 0$ and $w_1, w_2, \dots, w_{n-1} \in \mathscr{W}$, the below properties are gratified:*

(i) $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$;

(ii) $\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \le 1-\sigma\} \in \mathscr{I}$, $\{k \in \mathbb{N} : \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\} \in \mathscr{I}$ *and* $\{k \in \mathbb{N} : \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \ge \sigma\} \in \mathscr{I}$;

(iii) $\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma$ *and* $\Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma$, $\wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\} \in \mathscr{F}(\mathscr{I})$;

(iv) $\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) > 1-\sigma\} \in \mathscr{F}(\mathscr{I})$, $\{k \in \mathbb{N} : \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\} \in \mathscr{F}(\mathscr{I})$ *and* $\{k \in \mathbb{N} : \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma\} \in \mathscr{F}(\mathscr{I})$;

(v) $\mathscr{I}(\mathscr{N}_n) - \lim \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) = 1$, $\mathscr{I}(\mathscr{N}_n) - \lim \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) = 0$ *and* $\mathscr{I}(\mathscr{N}_n) - \lim \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) = 0$.

Theorem 3.4. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-convergent, $\mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$ is unique.*

Proof. If possible, let $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_1$ and $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_2$ where $\upsilon_1 \ne \upsilon_2$. For a given $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxdot (1-\varpi) > 1-\sigma$ and $\varpi \oplus \varpi < \sigma$. For any $\zeta > 0$ and $w_1, w_2, \dots, w_{n-1} \in \mathscr{W}$, we define

$$\mathscr{M}_{\Im,1}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \Im\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \le 1-\varpi\right\};$$

$$\mathscr{B}_{\Im,2}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \Im\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \le 1-\varpi\right\};$$

$$\mathscr{M}_{\Re,1}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \Re\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \ge \varpi\right\};$$

$$\mathscr{B}_{\Re,2}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \Re\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \ge \varpi\right\};$$

$$\mathscr{M}_{\wp,1}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \wp\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_1; \frac{\zeta}{2}\right) \ge \varpi\right\};$$

$$\mathscr{B}_{\wp,2}(\varpi,\zeta) = \left\{k \in \mathbb{N} : \wp\left(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon_2; \frac{\zeta}{2}\right) \ge \varpi\right\}.$$

Since $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_1$, by Lemma 3.3 , each of $\mathscr{M}_{\Im,1}(\varpi,\zeta)$, $\mathscr{M}_{\Re,1}(\varpi,\zeta)$ and $\mathscr{M}_{\wp,1}(\varpi,\zeta)$ belongs to $\mathscr{I}$. Again, as $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_2$, each of $\mathscr{B}_{\Im,2}(\varpi,\zeta)$, $\mathscr{B}_{\Re,2}(\varpi,\zeta)$ and $\mathscr{B}_{\wp,2}(\varpi,\zeta)$

belongs to $\mathscr{I}$. Let $\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta) = \{\mathscr{M}_{\Im,1}(\varpi,\zeta) \cup \mathscr{B}_{\Im,2}(\varpi,\zeta)\} \cap \{\mathscr{M}_{\Re,1}(\varpi,\zeta) \cup \mathscr{B}_{\Re,2}(\varpi,\zeta)\} \cap \{\mathscr{M}_{\wp,1}(\varpi,\zeta) \cup \mathscr{B}_{\wp,2}(\varpi,\zeta)\}$. Then $\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta) \in \mathscr{I}$. Obviously $\mathbb{N} \setminus \mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta) \in \mathscr{F}(\mathscr{I})$. So, let $k \in \mathbb{N} \setminus \mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)$. Then, there arise three cases:

- $k \in \mathbb{N} \setminus (\mathscr{M}_{\Im,1}(\varpi,\zeta) \cup \mathscr{B}_{\Im,2}(\varpi,\zeta))$
- $k \in \mathbb{N} \setminus (\mathscr{M}_{\Re,1}(\varpi,\zeta) \cup \mathscr{B}_{\Re,2}(\varpi,\zeta))$
- $k \in \mathbb{N} \setminus (\mathscr{M}_{\wp,1}(\varpi,\zeta) \cup \mathscr{B}_{\wp,2}(\varpi,\zeta))$.

If $k \in \mathbb{N} \setminus (\mathscr{M}_{\Im,1}(\varpi,\zeta) \cup \mathscr{B}_{\Im,2}(\varpi,\zeta))$ then,

$$\begin{aligned}
&\Im(w_1,w_2,\ldots,w_{n-1},\upsilon_1-\upsilon_2;\zeta)\\
&\geq \Im\left(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right) \boxdot \Im\left(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon_2;\frac{\zeta}{2}\right)\\
&> (1-\varpi) \boxdot (1-\varpi)\\
&> 1-\sigma.
\end{aligned}$$

Since $\sigma \in (0,1)$ is arbitrary, $\Im(w_1,w_2,\ldots,w_{n-1},\upsilon_1-\upsilon_2;\zeta) = 1$ which yields $\upsilon_1 = \upsilon_2$. Using similar technique, for other two cases, we can prove the same. Hence, $\mathscr{I}(\mathscr{N}_n)$-limit of $\{w_k\}$ is unique. □

Theorem 3.5. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\mathscr{N}_n - \lim w_k = \upsilon$ then $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$.*

Proof. Let $\mathscr{N}_n - \lim w_k = \upsilon$. Then, for every $\sigma > 0, \zeta > 0$ and $w_1,w_2,\ldots,w_{n-1} \in \mathscr{W}$ there exists $k_0 \in \mathbb{N}$ such that $\Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) > 1-\sigma$ and $\Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) < \sigma$, $\wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) < \sigma$ for all $k \geq k_0$. Therefore, it is immediate that the set $\{k \in \mathbb{N} : \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \leq 1-\sigma$ or $\Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \geq \sigma$ and $\wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \geq \sigma\}$ is finite. Hence $\{k \in \mathbb{N} : \Im(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \leq 1-\sigma$ or $\Re(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \geq \sigma$ and $\wp(w_1,w_2,\ldots,w_{n-1},w_k-\upsilon;\zeta) \geq \sigma\} \in \mathscr{I}$. So, $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$. □

But, in general, the converse of the above theorem is not true which can be illustrated as given below.

Example 3.6. Let $\mathscr{W} = \mathbb{R}^n$ with

$$\|w_1,w_2,\ldots,w_n\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n}\\ \vdots & \ddots & \vdots\\ w_{n1} & \cdots & w_{nn}\end{vmatrix}\right)$$

where $w_i = (w_{i1},w_{i2},\ldots,w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \boxdot \nu_2 = \min\{\nu_1,\nu_2\}$ and $\nu_1 \oplus \nu_2 = \max\{\nu_1,\nu_2\}$ for every $\nu_1,\nu_2 \in [0,1]$. We take Nn-NLS as defined in Example 2.11. Let $\mathscr{I}$ be a class of subsets of $\mathbb{N}$ such that natural density of each subset is zero. Then $\mathscr{I}$ becomes a nontrivial admissible ideal. Now, we define a sequence $\{w_k\} \in \mathscr{W}$ by

$$w_k = \begin{cases} (k,0,\ldots,0), & \text{if } k = i^2,\ i \in \mathbb{N}\\ (0,0,\ldots,0) = \mathbf{0}, & \text{otherwise.}\end{cases}$$

Then $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \mathbf{0}$, but it is not $\mathscr{N}_n$-convergent.

Theorem 3.7. *Let $\mathscr{W}$ be a real vector space, $\{w_k\}$ and $\{l_k\}$ be two sequences in a Nn-NLS $\mathscr{H}$. Then, the below statements hold good:*

(i) If $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_1$ and $\mathscr{I}(\mathscr{N}_n) - \lim l_k = \upsilon_2$, $\mathscr{I}(\mathscr{N}_n) - \lim w_k + l_k = \upsilon_1 + \upsilon_2$;

(ii) If $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$, $\mathscr{I}(\mathscr{N}_n) - \lim \kappa w_k = \kappa\upsilon$, $\kappa \neq 0$;

(iii) If $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon_1$ and $\mathscr{I}(\mathscr{N}_n) - \lim l_k = \upsilon_2$, $\mathscr{I}(\mathscr{N}_n) - \lim w_k - l_k = \upsilon_1 - \upsilon_2$.

Proof. (i) Suppose that $\mathscr{I}(\mathscr{N}_n)-\lim w_k=\upsilon_1$ and $\mathscr{I}(\mathscr{N}_n)-\lim l_k=\upsilon_2$. For a given $\sigma\in(0,1)$, choose $\varpi\in(0,1)$ such that $(1-\varpi)\boxdot(1-\varpi)>1-\sigma$ and $\varpi\oplus\varpi<\sigma$. For any $\zeta>0$ and $w_1,w_2,\dots,w_{n-1}\in\mathscr{W}$, we take

$$\mathscr{M}_{\Im,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\le 1-\varpi\right\};$$

$$\mathscr{B}_{\Im,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Im\left(w_1,w_2,\dots,w_{n-1},l_k-\upsilon_2;\frac{\zeta}{2}\right)\le 1-\varpi\right\};$$

$$\mathscr{M}_{\Re,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Re\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\ge \varpi\right\};$$

$$\mathscr{B}_{\Re,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\Re\left(w_1,w_2,\dots,w_{n-1},l_k-\upsilon_2;\frac{\zeta}{2}\right)\ge \varpi\right\};$$

$$\mathscr{M}_{\wp,1}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\wp\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\ge \varpi\right\};$$

$$\mathscr{B}_{\wp,2}(\varpi,\zeta)=\left\{k\in\mathbb{N}:\wp\left(w_1,w_2,\dots,w_{n-1},l_k-\upsilon_2;\frac{\zeta}{2}\right)\ge \varpi\right\}.$$

Let $\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)=\{\mathscr{M}_{\Im,1}(\varpi,\zeta)\cup\mathscr{B}_{\Im,2}(\varpi,\zeta)\}\cap\{\mathscr{M}_{\Re,1}(\varpi,\zeta)\cup\mathscr{B}_{\Re,2}(\varpi,\zeta)\}\cap\{\mathscr{M}_{\wp,1}(\varpi,\zeta)\cup\mathscr{B}_{\wp,2}(\varpi,\zeta)\}$. Then, by Lemma 3.3, $\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)\in\mathscr{I}$. Let $k\in\mathbb{N}\setminus\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)$. Then,

$$\begin{aligned}&\Im(w_1,w_2,\dots,w_{n-1},(w_k+l_k)-(\upsilon_1+\upsilon_2);\zeta)\\&\ge\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\boxdot\Im\left(w_1,w_2,\dots,w_{n-1},l_k-\upsilon_2;\frac{\zeta}{2}\right)\\&>(1-\varpi)\boxdot(1-\varpi)\\&>1-\sigma,\end{aligned}$$

$$\begin{aligned}&\Re(w_1,w_2,\dots,w_{n-1},(w_k+l_k)-(\upsilon_1+\upsilon_2);\zeta)\\&\le\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon_1;\frac{\zeta}{2}\right)\oplus\Im\left(w_1,w_2,\dots,w_{n-1},l_k-\upsilon_2;\frac{\zeta}{2}\right)\\&<\varpi\oplus\varpi\\&<\sigma,\end{aligned}$$

and same inequality can be obtained for indeterminacy membership function $\wp$. This shows that $\{k\in\mathbb{N}:\Im(w_1,w_2,\dots,w_{n-1},(w_k+l_k)-(\upsilon_1+\upsilon_2);\zeta)\le 1-\sigma$ or $\Re(w_1,w_2,\dots,w_{n-1},(w_k+l_k)-(\upsilon_1+\upsilon_2);\zeta)\ge\sigma,$ and $\wp(w_1,w_2,\dots,w_{n-1},(w_k+l_k)-(\upsilon_1+\upsilon_2);\zeta)\ge\sigma\}\subset\mathscr{A}_{(\Im,\Re,\wp)}(\sigma,\zeta)$. Hence $\mathscr{I}(\mathscr{N}_n)-\lim w_k+l_k=\upsilon_1+\upsilon_2$.

(ii) Since $\mathscr{I}(\mathscr{N}_n)-\lim w_k=\upsilon$, for every $\sigma\in(0,1),\zeta>0$ and $w_1,w_2,\dots,w_{n-1}\in\mathscr{W}$ such that

$$\left\{k\in\mathbb{N}:\Im\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\frac{\zeta}{|\kappa|}\right)\le 1-\sigma \text{ or } \Re\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\frac{\zeta}{|\kappa|}\right)\right.$$
$$\left.\ge\sigma \text{ and } \wp\left(w_1,w_2,\dots,w_{n-1},w_k-\upsilon;\frac{\zeta}{|\kappa|}\right)\ge\sigma\right\}\in\mathscr{I},$$

i.e., $\{k\in\mathbb{N}:\Im(w_1,w_2,\dots,w_{n-1},\kappa w_k-\kappa\upsilon;\zeta)\le 1-\sigma$ or $\Re(w_1,w_2,\dots,w_{n-1},\kappa w_k-\kappa\upsilon;\zeta)\ge\sigma$ and $\wp(w_1,w_2,\dots,w_{n-1},\kappa w_k-\kappa\upsilon;\zeta)\ge\sigma\}\in\mathscr{I}$. Therefore $\mathscr{I}(\mathscr{N}_n)-\lim\kappa w_k=\kappa\upsilon$.

(iii) By (i) and (ii), we can easily prove our desired result.

□

Now we proceed with the notion of $\mathscr{I}^*$-convergence in a neutrosophic n-normed linear space

$\mathscr{H}$.

Definition 3.8. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be $\mathscr{I}^*$-convergent to $\upsilon \in \mathscr{W}$ with regards to $\mathscr{N}_n$ if there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$. In this case, we write $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$ or $w_k \xrightarrow{\mathscr{I}^*(\mathscr{N}_n)} \upsilon$ and υ is called $\mathscr{I}^*(\mathscr{N}_n)$-limit of $\{w_k\}$.

We establish the connection between $\mathscr{I}^*(\mathscr{N}_n)$ and $\mathscr{I}(\mathscr{N}_n)$-convergence.

Theorem 3.9. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$, $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$.*

Proof. Since $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$, there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$ i.e., for every $\sigma > 0, \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ there exists $p_0 \in \mathbb{N}$ such that $\Im(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) < \sigma$, $\wp(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) < \sigma$ for all $p \geq p_0$. So, $\{k_p \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) \leq 1 - \sigma$ or $\Re(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) \geq \sigma$ and $\wp(w_1, w_2, \ldots, w_{n-1}, w_{k_p} - \upsilon; \zeta) \geq \sigma\} \subset \{k_1, k_2, \ldots, k_{p_0-1}\}$. Let $\mathscr{G} = \mathbb{N} \setminus \mathscr{K}$. Then, $\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma$ or $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma$ and $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\} \subset \mathscr{G} \cup \{k_1, k_2, \ldots, k_{p_0-1}\}$. Since $\mathscr{I}$ is an admissible ideal, $\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma$ or $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma$ and $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}$. This shows that $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$. □

In general, the converse of Theorem 3.9 need not be true which can illustrated by the following example.

Example 3.10. Let $\mathscr{W} = \mathbb{R}^n$ with

$$\|(w_1, w_2, \ldots, w_n)\| = abs\left(\begin{vmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{vmatrix}\right)$$

where $w_i = (w_{i1}, w_{i2}, \ldots, w_{in}) \in \mathbb{R}^n$. We take continuous t-norm and t-conorm as $\nu_1 \boxdot \nu_2 = \min\{\nu_1, \nu_2\}$ and $\nu_1 \oplus \nu_2 = \max\{\nu_1, \nu_2\}$ for every $\nu_1, \nu_2 \in [0, 1]$. We consider the neutrosophic n-normed linear space defined as in Example 2.11. Let $\mathbb{N} = \bigcup_i \mathscr{D}_i$ be a decomposition of $\mathbb{N}$ such that for any $r \in \mathbb{N}$ each $\mathscr{D}_i$ contains infinitely many $i's$ where $i \geq r$ and $\mathscr{D}_i \cap \mathscr{D}_r = \emptyset$ whenever $i \neq r$. Let $\mathscr{I}$ be the class of all subsets of $\mathbb{N}$ which intersects only a finite number of $\mathscr{D}_i's$. Then $\mathscr{I}$ becomes a non trivial admissible ideal of $\mathbb{N}$. Now we define a sequence $\{w_k\} \in \mathscr{W}$ by $w_k = (\frac{1}{k}, 0, \ldots, 0) \in \mathbb{R}^n$ if $k \in \mathscr{D}_k$. Let $\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{R}^n$. Then for $\zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$, we have

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta) = \frac{\zeta}{\zeta + \|(w_1, w_2, \ldots, w_{n-1}, w_k)\|} \to 1$$

$$\Re(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta) = \frac{\|(w_1, w_2, \ldots, w_{n-1}, w_k)\|}{\zeta + \|(w_1, w_2, \ldots, w_{n-1}, w_k)\|} \to 0$$

$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k; \zeta) = \frac{\|(w_1, w_2, \ldots, w_{n-1}, w_k)\|}{\zeta} \to 0$$

as $k \to \infty$. Since $\mathscr{I}$ is an admissible ideal, therefore $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \mathbf{0}$.

Now, if possible, let $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \mathbf{0}$. Then there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p\to\infty} w_{k_p} = \mathbf{0}$. Since $\mathscr{K} \in \mathscr{F}(\mathscr{I})$, there is $\mathscr{G} \in \mathscr{I}$ such that $\mathbb{N} \setminus \mathscr{K} = \mathscr{G}$. Now by the construction of $\mathscr{I}$, there is $j \in \mathbb{N}$ such that $\mathscr{G} \subset \bigcup_{i=1}^{j} \mathscr{D}_i$. But then $\mathscr{D}_{j+1} \subset \mathscr{K}$ and therefore $w_{k_p} = (\frac{1}{j+1}, \underbrace{0, \ldots, 0}_{n-1 times})$ for infinitely many $k_p \in \mathscr{K}$ which contradicts $\mathscr{N}_n - \lim_{p\to\infty} w_{k_p} = \mathbf{0}$. Therefore $\{w_k\}$ is not $\mathscr{I}^*(\mathscr{N}_n)$-convergent to $\mathbf{0} \in \mathscr{W}$.

Then, question normally arises that under what condition the converse of Theorem 3.9 is true. We investigate it in the following theorem.

Theorem 3.11. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$ and $\mathscr{I}$ satisfies the condition (AP) then $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$.*

Proof. Suppose that $\mathscr{I}$ satisfies the condition (AP) and $\mathscr{I}(\mathscr{N}_n) - \lim w_k = \upsilon$. Then, for every $\sigma \in (0,1), \zeta > 0$ and $w_1, w_2, \dots, w_{n-1} \in \mathscr{W}$ such that

$$\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \geq \sigma\} \in \mathscr{I}.$$

Define

$$\mathscr{A}_j = \left\{k \in \mathbb{N} : 1 - \frac{1}{j} \leq \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) < 1 - \frac{1}{j+1} \text{ or } \frac{1}{j+1} < \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \leq \frac{1}{j} \text{ and } \frac{1}{j+1} < \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \leq \frac{1}{j}\right\}.$$

Clearly, $\{\mathscr{A}_1, \mathscr{A}_2, \dots\}$ is countable and pairwise disjoint and each $\mathscr{A}_j \in \mathscr{I}$. Since $\mathscr{I}$ satisfies the condition (AP), there exists a countable family $\{\mathscr{B}_1, \mathscr{B}_2, \dots\}$ of subsets of $\mathbb{N}$ belonging to $\mathscr{I}$ and $\mathscr{A}_i \triangle \mathscr{B}_i$ is finite for each i and $\mathscr{G} = \cup_i \mathscr{B}_i \in \mathscr{I}$. Now from the associated filter of $\mathscr{I}$ there is $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ such that $\mathscr{K} = \mathbb{N} \setminus \mathscr{G}$. It is sufficient to prove the theorem that the subsequence $\{w_k\}_{k \in \mathscr{K}}$ is convergent to υ with regard to $\mathscr{N}_n$. Let $\varpi \in (0,1)$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} < \varpi$. Then, it is immediate that

$$\begin{aligned}
&\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \varpi \text{ or } \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \geq \varpi \\
&\text{and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \geq \varpi\} \\
&\subset \left\{k \in \mathbb{N} : \Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \leq 1 - \frac{1}{k_0} \text{ or } \Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta)\right. \\
&\left. \geq \frac{1}{k_0} \text{ and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) \geq \frac{1}{k_0}\right\} \subset \bigcup_{i=1}^{k_0+1} \mathscr{A}_i.
\end{aligned}$$

Since $\mathscr{A}_i \triangle \mathscr{B}_i$, $i = 1, 2, \dots, k_0 + 1$, are finite, there is $p_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{k_0+1} \mathscr{B}_i\right) \cap \{k \in \mathbb{N} : k \geq p_0\} = \left(\bigcup_{i=1}^{k_0+1} \mathscr{A}_i\right) \cap \{k \in \mathbb{N} : k \geq p_0\}. \tag{3.1}$$

If $k \geq p_0$ and $k \in \mathscr{K}$, $k \notin \bigcup_{i=1}^{k_0+1} \mathscr{B}_i$. So, by 3.1, $k \notin \bigcup_{i=1}^{k_0+1} \mathscr{A}_i$. Therefore, for every $k \geq p_0$ and $k \in \mathscr{K}$ we get

$$\begin{aligned}
\Im(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) &> 1 - \varpi \\
\Re(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) &< \varpi \\
\text{and } \wp(w_1, w_2, \dots, w_{n-1}, w_k - \upsilon; \zeta) &< \varpi.
\end{aligned}$$

Since $\varpi \in (0,1)$ is arbitrary, we have $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$. This completes the proof. □

Theorem 3.12. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then, the following conditions are equivalent:*

(i) $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$;

(ii) There exist two sequences $\{t_k\}$ and $\{l_k\}$ in $\mathscr{W}$ such that $w_k = t_k + l_k$, $t_k \xrightarrow{\mathscr{N}_n} \upsilon$ and $\{k \in \mathbb{N} : l_k \neq \mathbf{0}\} \in \mathscr{I}$ where $\mathbf{0}$ is the zero element in $\mathscr{W}$.

Proof. First suppose that (i) holds. Then, there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and $\mathscr{N}_n - \lim_{p\to\infty} w_{k_p} = \upsilon$. So, for every $\sigma > 0, \zeta > 0$ and

$w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ we get

$$\Im(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) > 1 - \sigma \tag{3.2}$$
$$\Re(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma \tag{3.3}$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \zeta) < \sigma, \tag{3.4}$$

whenever $k \in \mathscr{K}$. Define the sequences $\{t_k\}$ and $\{l_k\}$ in $\mathscr{W}$ as follows

$$t_k = \begin{cases} w_k, \text{ if } k \in \mathscr{K} \\ \upsilon, \text{ if } k \in \mathscr{K}^c \end{cases} \quad \text{and } l_k = w_k - t_k, \ \forall k \in \mathbb{N}. \tag{3.5}$$

Then for each $k \in \mathscr{K}^c$,

$$\Im(w_1, w_2, \ldots, w_{n-1}, t_k - \upsilon; \zeta) = 1 > 1 - \sigma$$
$$\Re(w_1, w_2, \ldots, w_{n-1}, t_k - \upsilon; \zeta) = 0 < \sigma$$
$$\wp(w_1, w_2, \ldots, w_{n-1}, t_k - \upsilon; \zeta) = 0 < \sigma.$$

Therefore, using (3.2), (3.3), (3.4) and 3.5, we get $t_k \xrightarrow{\mathscr{N}_n} \upsilon$. From (3.5), we have $\{k \in \mathbb{N} : t_k \neq w_k\} \subset \mathscr{K}^c \implies \{k \in \mathbb{N} : t_k - w_k \neq \mathbf{0}\} \subset \mathscr{K}^c \implies \{k \in \mathbb{N} : l_k \neq \mathbf{0}\} \subset \mathscr{K}^c$. Therefore, $\{k \in \mathbb{N} : l_k \neq \mathbf{0}\} \in \mathscr{I}$.

Let the condition (ii) holds. Then, it is obvious that $\{k \in \mathbb{N} : l_k = \mathbf{0}\} \in \mathscr{F}(\mathscr{I})$ must be infinite. Let $\{k \in \mathbb{N} : l_k = \mathbf{0}\} = \mathscr{K}$. Since $t_k \xrightarrow{\mathscr{N}_n} \upsilon$ and $t_k = w_k$ for $k \in \mathscr{K}$, the subsequence $\{w_k\}_{k \in \mathscr{K}}$ must be $\mathscr{N}_n$-convergent to υ. Therefore $\mathscr{I}^*(\mathscr{N}_n) - \lim w_k = \upsilon$. This completes the proof. □

Now, we proceed with the notion of $\mathscr{I}$-Cauchyness in neutrosophic n-normed linear spaces.

Definition 3.13. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be $\mathscr{I}$-Cauchy sequence with regard to $\mathscr{N}_n$ (i.e. $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence) if for every $\sigma \in (0,1), \zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ there exists a natural number $k_0 = k_0(\sigma)$ such that

$$\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma$$
$$\text{and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\} \in \mathscr{I}.$$

Definition 3.14. Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. Then $\{w_k\}$ is named to be $\mathscr{I}^*$-Cauchy sequence with regard to $\mathscr{N}_n$ (i.e. $\mathscr{I}^*(\mathscr{N}_n)$-Cauchy sequence) if there exists a set $\mathscr{K} = \{k_1 < k_2 < \cdots < k_p < \cdots\} \subset \mathbb{N}$ such that $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ and the subsequence $\{w_{k_p}\}$ is an ordinary Cauchy sequence with regard to $\mathscr{N}_n$.

Theorem 3.15. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\{w_k\}$ is $\mathscr{I}^*(\mathscr{N}_n)$-Cauchy sequence, it is $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence.*

Theorem 3.16. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\mathscr{I}$ satisfies the condition (AP) and $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence, it is $\mathscr{I}^*(\mathscr{N}_n)$-Cauchy sequence.*

The Theorem 3.15 and 3.16 can be proved in the line of Theorem 3.9 and 3.11 respectively. So, we omit details.

Now we proceed with the investigations of relation ship between $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence and $\mathscr{I}(\mathscr{N}_n)$-convergence of a sequence.

Theorem 3.17. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-convergent, it is $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence.*

Proof. Let $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-convergent to υ. For a given $\sigma \in (0,1)$, choose $\varpi \in (0,1)$ such that $(1-\varpi) \boxdot (1-\varpi) > 1-\sigma$ and $\varpi \oplus \varpi < \sigma$. Then for any $\zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$, each of the following sets

$$\mathscr{A}_1 = \left\{k \in \mathbb{N} : \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > 1 - \varpi\right\}$$
$$\mathscr{A}_2 = \left\{k \in \mathbb{N} : \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi\right\}$$
$$\text{and } \mathscr{A}_3 = \left\{k \in \mathbb{N} : \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \varpi\right\}$$

belongs to $\mathscr{F}(\mathscr{I})$. Let $\mathscr{B} = \mathscr{A}_1 \cap \mathscr{A}_2 \cap \mathscr{A}_3$. Then $\mathscr{B} \in \mathscr{F}(\mathscr{I})$. So, let $k \in \mathscr{B}$. Choose a fixed $k_0 \in \mathscr{B}$. Then

$$\begin{aligned}
&\Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \\
&\geq \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \boxdot \Im\left(w_1, w_2, \ldots, w_{n-1}, w_{k_0} - \upsilon; \frac{\zeta}{2}\right) \\
&> (1 - \varpi) \boxdot (1 - \varpi) \\
&> (1 - \sigma)
\end{aligned}$$

and

$$\begin{aligned}
&\Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \\
&\leq \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \oplus \Re\left(w_1, w_2, \ldots, w_{n-1}, w_{k_0} - \upsilon; \frac{\zeta}{2}\right) \\
&< \varpi \oplus \varpi \\
&< \sigma.
\end{aligned}$$

Similarly we have $\wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma$. Therefore,

$$\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) > 1 - \sigma \text{ and } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma\} \in \mathscr{F}(\mathscr{I}).$$

Hence, $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence. □

Theorem 3.18. *Let $\{w_k\}$ be a sequence in a Nn-NLS $\mathscr{H}$. If $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence, it is $\mathscr{I}(\mathscr{N}_n)$-convergent.*

Proof. Let $\{w_k\}$ be $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence but not $\mathscr{I}(\mathscr{N}_n)$-convergent. Then for $\sigma \in (0, 1)$, $\zeta > 0$ and $w_1, w_2, \ldots, w_{n-1} \in \mathscr{W}$ there exists $k_0 = k_0(\sigma) \in \mathbb{N}$ such that $\mathscr{K} \in \mathscr{I}$ where

$$\mathscr{K} = \{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 1 - \sigma \text{ or } \Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma \text{ and } \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \geq \sigma\}.$$

And, $\mathscr{M} \in \mathscr{I}$ where $\mathscr{M} = \left\{k \in \mathbb{N} : \Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > 1 - \sigma \text{ and } \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma,\ \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma\right\}$. Since

$$\begin{aligned}
&\Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \\
&\geq 2\Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) \\
&> 1 - \sigma
\end{aligned}$$

and

$$\Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 2\Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma$$

$$\wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) \leq 2\wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \sigma,$$

if

$$\Im\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) > \frac{1-\sigma}{2}$$

$$\text{and } \Re\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \frac{\sigma}{2}; \wp\left(w_1, w_2, \ldots, w_{n-1}, w_k - \upsilon; \frac{\zeta}{2}\right) < \frac{\sigma}{2}.$$

This yields $\{k \in \mathbb{N} : \Im(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) > 1 - \sigma$ and $\Re(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma,\ \wp(w_1, w_2, \ldots, w_{n-1}, w_k - w_{k_0}; \zeta) < \sigma\} \in \mathscr{I}$ which means $\mathscr{K}^c \in \mathscr{I}$ that implies $\mathscr{K} \in \mathscr{F}(\mathscr{I})$ by which we arrive at a contradiction. Hence, $\{w_k\}$ is $\mathscr{I}(\mathscr{N}_n)$-convergent. □

Definition 3.19. A Nn-NLS is named to be $\mathscr{I}$-complete with regard to $\mathscr{N}_n$ (in short $\mathscr{I}(\mathscr{N}_n)$-complete) if every $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence is $\mathscr{I}(\mathscr{N}_n)$-convergent.

Remark 3.20. In the light of Theorem 3.18, we see every Nn-NLS is $\mathscr{I}\mathscr{N}_n)$-complete.

Conclusion and future developments

In this research paper, we have introduced $\mathscr{I}(\mathscr{N}_n)$ -convergence and $\mathscr{I}(\mathscr{N}_n)$-Cauchy sequence and obtained that every neutrosophic n-normed linear space is $\mathscr{I}(\mathscr{N}_n)$-complete. In future, based on this research work, one can generalize this notion to different types of convergence. Also, this idea can be used in the field of convergence related problems in many branches of science and engineering.

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