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SOLUTION OF THE DIOPHANTINE EQUATION $p^x + (2p+1)^y = z^2$ WHERE p IS A LUCASIAN PRIME WITH $p > 5$

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Abstract In this paper, an exponential Diophantine equation $p^x + (2p+1)^y = z^2$ is explored for the presence of integer solutions or absence of solutions under the assumption that the sum of the exponents x and y is $1, 2, 3$. Here we consider p and $2p+1$ to be prime numbers and obtained the result that there exist an integer solution for the equation when $p > 5$ and p is considered to be equal to $4n+3$ where $n \in \mathbb{Z}$.The primes that we consider in this paper are said to be the Lucasian primes.

1 Introduction

Number theory is a branch of pure mathematics devoted primarily to the study of the integers and integer-valued functions. A Diophantine equation in Mathematics is a polynomial equation with two or more unknowns and integer coefficients, with the only interesting solutions being those with integer coefficients. The sum of two or more degree one monomials is a constant in a linear Diophantine equation.
When unknowns can take the form of exponents, a Diophantine equation is said to be exponential. Many mathematicians have worked on the challenging problem of resolving Diophantine equations of type $a^x + b^y = c^z$, where $x, y, z \in$ N.[1, 6, 7, 8, 9, 10] Some authors used elementary congruence and Catalan's conjecture to find the full solution to this equation for small values of a, b and c. [2, 3, 4, 5]

Lucasian prime is a prime number p such that $p \cong 3(mod 4)$ with $2p+1$ prime. Some Lucasian primes are $3, 11, 23, 83, 131, 179, 191, 239, 251, 359, 419, 431, 443, 491, 659, ..$.

In this paper, an exponential Diophantine equation with three unknowns $p^x + (2p+1)^y = z^2$ is discussed where p and $(2p+1)$ are prime numbers expressed as $p = 4n+3$ and $p > 5$ .Such primes are called the Lucasian primes and obtained a result that there is an integer solution for the equation under the assumption that the sum of the exponents x and y is $1, 2, 3$.

2 Main Results

Theorem 2.1. *Let p be a Lucasian prime with $p > 5$ and x, y, z are positive integers.If $x + y = 1, 2, 3$ then the Diophantine equation $p^x + (2p+1)^y = z^2$ has a non negative integer solution at $(x, y, z) = (2, 1, p+1)$.*

Proof. Consider the equation

$$p^x + (2p+1)^y = z^2 \tag{2.1}$$

Case 1 Assume $x = 0$ and $y = 1$

These two chosen x and y values lead (2.1) to the following equation with two variables p and z.
$2p + 2 = z^2 \implies 8(n+1) = z^2$

When $n = 2k^2 - 1, k \in \mathbb{Z}$, $z^2 = 16k^2$, $\implies z = \pm 4k, k \in \mathbb{Z}$.The same value for n gives $p = 8k^2 - 1$, which is not a prime number for all $k \in \mathbb{Z}$ and $2p + 1 = 16k^2 - 1$ which is not a prime number for all $k \in \mathbb{Z}$.Therefore no integer solution exist when $x = 0$ and $y = 1$.

Case 2 Assume $x = 1$ and $y = 0$

These two selected values of x and y leads (2.1) to the following equations with two variables p and z.
$p + 1 = z^2 \implies 4(n+1) = z^2$

When $n = k^2 - 1$,where $k \in \mathbb{Z}$,$z^2 = 4k^2 \implies z = \pm 2k$ where $k \in \mathbb{Z}$.The same value for n gives $p = 4k^2 - 1$, which is not a prime number for all $k \in \mathbb{Z}$ and is not greater than 5 and $2p + 1 = 8k^2 - 1$ which is not a prime number for all $k \in \mathbb{Z}$.Therefore no integer solution exist when $x = 1$ and $y = 0$.

Case 3 Assume $x = 0$ and $y = 2$

These two selected values of x and y leads (2.1) to the following equations with two variables p and z.
$1 + (2p+1)^2 = z^2 \implies (2p+1)^2 = z^2 - 1$

The square of an integer minus one can never be a square, hence the above mentioned postulation is always untrue.Therefore no integer solution exist when $x = 0$ and $y = 2$.

Case 4 Consider $x = 1$ and $y = 1$

The selected values of x and y leads (2.1) to the following equations with two variables p and z.
$p + (2p+1)^2 = z^2 \implies 12n + 10 = z^2$

For any value for n we are not geting a perfect square, which shows that there exist no integer solution exist when $x = 1$ and $y = 1$.

Case 5 Consider $x = 2$ and $y = 0$

The choosen values of x and y leads (2.1) to the following equations with two variables p and z.
$p^2 + 1 = z^2 \implies p^2 = z^2 - 1$

The square of an integer minus one can never be a square, hence the above mentioned postulation is always untrue.Therefore no integer solution exist when $x = 2$ and $y = 0$.

Case 6 Consider $x = 0$ and $y = 3$

The assumed values of x and y leads (2.1) to the following equations with two variables p and z.
$1 + (2p+1)^3 = z^2 \implies (8n+7)^3 = z^2 - 1 \implies r^3 = z^2 - 1 \implies r = 2, z = \pm 3.$

The value of n that corresponds to the value of r mentioned above gives $n = -\frac{5}{8} \notin \mathbb{Z}$ Thus the values obtained for p and $2p+1$ are not prime numbers.Therefore no integer solution exist when $x = 0$ and $y = 3$.

Case 7 Assume $x = 1$ and $y = 2$

The assumed values of x and y leads (2.1) to the following equations with two variables p and z.
$p + (2p+1)^2 = z^2 \implies 64n^2 + 116n + 52 = z^2 \implies (4u-29)(4u+29) = (4z + 8\sqrt{13})(4z - 8\sqrt{13})$.

Consider the fractional form of the above equation as

$$\frac{4u+29}{4z+8\sqrt{13}} = \frac{4u-29}{4z-8\sqrt{13}} = \frac{a}{b} \neq 0$$

Convert the above equation into double equations and then solving it by the method of cross multiplication we obtain,

$$u = \frac{16\sqrt{13}ab - 29(b^2+a^2)}{4(b^2-a^2)} \implies n = \frac{16\sqrt{13}ab + 58b^2}{32(b^2-a^2)}$$
$$z = \frac{8\sqrt{13}(b^2+a^2) - 58ab}{4(b^2-a^2)}.$$

This shows that z is not an integer solution for any selection of a and b.Therefore no integer solution exist when $x = 1$ and $y = 2$.

Case 8 Consider $x = 2$ and $y = 1$

The considerd values of x and y leads (2.1) to the following equations with two variables p and z.

$p^2 + (2p+1) = z^2 \implies (p+1)^2 = z^2 \implies z = \pm(p+1)$
Thus there exist an integer solution for (2.1) at $x = 2$, $y = 1$ and $z = \pm(p+1)$, where p is lucasian prime and $p > 5$.

Case 9 Consider $x = 3$ and $y = 0$
The assumed values of x and y leads (2.1) to the following equations with two variables p and z.
$p^3 + 1 = z^2 \implies (4n+3)^3 + 1 = z^2 \implies r^3 = z^2 - 1 \implies r = 2, z = \pm 3.$
The value of n that corresponds to the value of r mentioned above gives $n = -\frac{1}{4} \notin \mathbb{Z}$ Thus the values obtained for $p = 2$,a prime number which is not greater than 5.Therefore no integer solution exist when $x = 3$ and $y = 0$.

□

3 Remarkable Observations:

The exponential Diophantine equation $p^x + (2p+1)^y = z^2$ has integer solutions when p is considered to be a Lucasian prime with $p > 5$. Some of the solutions for the above equations are $(p, 2p+1, x, y, z) = (11, 23, 2, 1, 12), (23, 47, 2, 1, 24), (131, 263, 2, 1, 132)$,
$(179, 359, 2, 1, 180)(191, 383, 2, 1, 192), (239, 479, 2, 1, 240)$ etc.

4 Conclusion

In this paper, we have shown an exponential Diophantine equation $p^x + (2p+1)^y = z^2$ has an integer solution when p is considered to be a Lucasian prime with $p > 5$. In this way, one search integer solutions for various exponential equations with base as any other prime numbers and $x + y > 3$.

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