

# On the Diophantine equations $5^x + 3 \cdot p^y = z^2$ where $p \equiv 1, 2, 3, 4 \pmod{5}$ is a prime number

Fernando Neres

Communicated by Ayman Badawi

MSC 2010 Classifications: 11A07; 11A41; 11D61.

Keywords and phrases: Diophantine equation, nonlinear Diophantine equation, prime number, factorization, congruences, non-negative integer solutions.

**Abstract** In this paper, we investigate the solvability of the Diophantine equation

$$5^x + 3 \cdot p^y = z^2 \quad (0.1)$$

where  $p \neq 5$  is a prime number and  $x, y$  and  $z$  are non-negative integers. We prove that: if  $p \equiv 1, 4 \pmod{5}$  is an odd prime number, then the only solution  $(p, x, y, z)$  of Eq. (0.1) is  $(p, 0, 0, 2)$ ; if  $p \equiv 2 \pmod{5}$  and  $y \equiv 0 \pmod{4}$ , then only solutions  $(p, x, y, z)$  of Eq. (0.1) are  $(2, 0, 4, 7)$  and  $(p, 0, 0, 2)$ ; if  $p \equiv 2 \pmod{5}$  and  $y \equiv 2 \pmod{4}$ , then Eq. (0.1) has no solution  $(p, x, y, z)$  in non-negative integers; and finally, if  $p \equiv 3 \pmod{5}$  is an odd prime number and  $y$  is even, then the only solution  $(p, x, y, z)$  of Eq. (0.1) is  $(p, 0, 0, 2)$ . In addition, we give several examples to illustrate the results obtained and leave some open problems about this study for future research.

## 1 Introduction

The study of Diophantine equations is a subject that challenges and enthruses many mathematicians, especially number theorists. A Diophantine equation is an equation of the form

$$f(x_1, x_2, \dots, x_n) = d \quad (1.1)$$

where  $d$  is an integer number,  $f$  is an  $n$ -ary function with  $n \geq 2$  and whose unknowns  $x_1, x_2, \dots, x_n$  are integer numbers [2]. Equation (1.1) receives special names for certain types of the function  $f$ , e.g., Eq. (1.1) is a *linear Diophantine equation* [2] when  $f$  is a polynomial of degree 1, Eq. (1.1) is a *quadratic Diophantine equation* [3] when  $f$  is a polynomial of degree 2 and Eq. (1.1) is an *exponential Diophantine equation* [7, 21] when  $f$  is a linear combination of exponential functions restricted to the set of integers.

The basic problems concerning a Diophantine equation that are investigated by the researchers are as follows: (i) Is the equation solvable? (ii) If it is solvable, is the number of its solutions finite or infinite? (iii) If it is solvable, determine all of its solutions [2]. In general, the solvability/unsolvability of Diophantine equations is investigated from *elementary methods* of number theory such as factorization, congruences and mathematical induction [2, 13, 10, 12]; or *advanced methods* that combine techniques of algebraic number theory [24] and/or transcendental number theory [16, 4]. In this work, we explore only a few elementary methods to investigate the solvability/unsolvability of a particular class of Diophantine equations.

The Diophantine equations have been studied by mathematicians since antiquity, and over the years, this study has been deepened by renowned mathematicians such as Fermat, Euler, Kummer, Siegel, and Wiles [3]. Some Diophantine equations have great prominence in the literature for the complexity of investigating their solvability and/or for the importance they have for the development of the theory, such as Fermat equation [24], Pell equation [5, 11] and Catalan equation [6, 15, 23]. The research on Diophantine equations remains active and very rich in contemporary mathematics. In recent decades, it has been noted that a large number of

papers have been published by researchers interested in studying Diophantine equations of the form

$$r \cdot a^x \pm s \cdot b^y = t \cdot z^2 \quad (1.2)$$

where  $a$  and/or  $b$  are prime numbers and  $r$ ,  $s$  and  $t$  are positive integers.

In 2007, Acu [1] showed that the only solutions  $(x, y, z)$  in non-negative integers of the Diophantine equation  $2^x + 5^y = z^2$  are  $(3, 0, 3)$  and  $(2, 1, 3)$ . In 2011, Suvarnamani [25] studied the solvability of the Diophantine equations  $2^x + p^y = z^2$  where  $p$  is a prime number. In 2013, Chotchaisthit [9] proved that  $(p, x, y, z) = (7, 0, 1, 3)$  and  $(p, x, y, z) = (3, 2, 2, 5)$  are the only solutions of the Diophantine equations  $p^x + (p + 1)^y = z^2$ , where  $p$  is a Mersenne prime. In 2017, Nathanson [17] proved that the Diophantine equation  $x^n - y^n = z^{n+1}$  has infinitely many positive integer solutions. In 2018, Rabago [20] studied the solvability of the Diophantine equations  $4^x - p^y = z^2$  and  $4^x - p^y = 3z^2$  when  $p$  is a prime number.

Still in 2018, Neres [18] proved that the Diophantine equation  $p^x + (p + 8)^y = z^2$  has no solution  $(x, y, z)$  in positive integers when  $p > 3$  and  $p + 8$  are primes. In 2019, Laipaporn et al. [14] determined all solutions of the Diophantine equation  $3^x + p \cdot 5^y = z^2$  in non-negative integers when  $p$  is a prime number not equal to 2 or 5. In addition, they presented sufficient conditions for the unsolvability of this equation in non-negative integers. In 2021, using elementary techniques, Singha [22] solved the Diophantine equations  $8^x + p^y = z^2$  and  $(8^n)^x + p^y = z^2$  where  $p \not\equiv 1 \pmod{8}$  is an odd prime number and  $n \geq 2$  is an integer number.

In 2022, Thongnak et al. [26, 27] showed that  $(x, y, z) = (2, 0, 10)$  is the only solution of the Diophantine equation  $11 \cdot 3^x + 11^y = z^2$  in non-negative integers. Moreover, these same authors also proved that the equation  $5^x - 2 \cdot 3^y = z^2$  is unsolvable in non-negative integers. Also in 2022, Buosi et al. [8] found the solutions  $(x, y, z)$  in non-negative integers of the Diophantine equations  $p^x - 2^y = z^2$ , where  $p = k^2 + 4$  are prime numbers with  $k \geq 3$  an integer. Finally, in 2023, Porto et al. [19] studied the solvability of the Diophantine equations  $p \cdot 3^x + p^y = z^2$  where  $p$  is a prime number.

Following this same line of research, we consider in this paper a particular subclass of the class of Diophantine equations given by Eq. (1.2), in which  $r = 1$ ,  $a = 5$ ,  $s = 3$  and  $b = p \neq 5$  is a prime number. We study the solvability/unsolvability of Eq. (0.1) using only the basic properties of factorization, divisibility, and congruence. We determine all solutions of Eq. (0.1) when  $p \equiv 1, 4 \pmod{5}$  is an odd prime number; we present a sufficient condition for Eq. (0.1) to be solvable (resp. unsolvable) when  $p \equiv 2 \pmod{5}$ , and we also present a sufficient condition for Eq. (0.1) to be solvable when  $p \equiv 3 \pmod{5}$  is an odd prime number.

## 2 Preliminaries

In this section, we establish the notation and present some results that will be useful for the presentation of the main results of this work.

**Lemma 2.1.** *Let  $p$  be a prime number. Then, the only solutions  $(p, y, z)$  of the Diophantine equation  $1 + 3 \cdot p^y = z^2$  in non-negative integers are  $(2, 4, 7)$ ,  $(5, 1, 4)$  and  $(p, 0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $y$  and  $z$  such that  $1 + 3 \cdot p^y = z^2$ . So,  $3p^y = (z - 1)(z + 1)$ . Hence, as  $p$  is prime, then there are non-negative integers  $\eta$  and  $\delta$  such that  $\eta \leq \delta$ ,  $\eta + \delta = y$  and either

$$p^\eta = z - 1 \quad \text{and} \quad 3p^\delta = z + 1 \quad (2.1)$$

or

$$3p^\eta = z - 1 \quad \text{and} \quad p^\delta = z + 1 \quad (2.2)$$

First, consider  $p^\eta = z - 1$  and  $3p^\delta = z + 1$ . So,  $p^\eta(3p^{\delta-\eta} - 1) = 3p^\delta - p^\eta = 2$ . **Claim 1:**  $\eta = 0$ . In fact, if  $\eta \geq 1$ , then  $p = 2$ ,  $\eta = 1$  and  $3 \cdot 2^{\delta-1} - 1 = 1$ , which is impossible.

Thus,  $3p^\delta - 1 = 2$ , i.e.  $\delta = 0$ . So,  $y = 0$  and, by Eqs. (2.1), one obtains that  $z = 2$ . Now, admit that  $3p^\eta = z - 1$  and  $p^\delta = z + 1$ . So,  $p^\eta(p^{\delta-\eta} - 3) = p^\delta - 3p^\eta = 2$ . If  $\eta = 0$ , then  $p = 5$  and  $\delta = 1$ . Thus,  $y = 1$  and, by Eqs. (2.2),  $z = 4$ . If  $\eta \geq 1$ , then  $p = 2$ ,  $\eta = 1$  and  $2^{\delta-1} - 3 = 1$ , i.e.  $\delta = 3$ . So,  $y = 4$  and, by Eqs. (2.2),  $z = 7$ .  $\square$

**Corollary 2.2.** *Let  $p \neq 2$  be a prime number. Then, the only solutions  $(p, y, z)$  of the Diophantine equation  $1 + 3 \cdot p^y = z^2$  in non-negative integers are  $(5, 1, 4)$  and  $(p, 0, 2)$ .*

**Corollary 2.3.** *Let  $p \neq 5$  be a prime number. Then, the only solutions  $(p, y, z)$  of the Diophantine equation  $1 + 3 \cdot p^y = z^2$  in non-negative integers are  $(2, 4, 7)$  and  $(p, 0, 2)$ .*

**Corollary 2.4.** *Let  $p$  be a prime number such that  $p \neq 2$  and  $p \neq 5$ . Then, the only solution  $(p, y, z)$  of the Diophantine equation  $1 + 3 \cdot p^y = z^2$  in non-negative integers is  $(p, 0, 2)$ .*

Let  $a, b_1, \dots, b_k$  and  $m$  be non-negative integers with  $m \geq 1, k \leq m$  and  $b_i \leq m - 1$ , for each  $i \in \{1, \dots, k\}$ . Henceforth, we write  $a \equiv b_1, \dots, b_k \pmod{m}$  to mean that the remainder of the division of  $a$  by  $m$  is exactly one of the  $b_i$ 's, i.e.  $a \equiv b_i \pmod{m}$  for exactly one  $i \in \{1, \dots, k\}$ .

**Lemma 2.5.** *Let  $p$  be a prime number. Then, the only solution  $(x, z)$  of the Diophantine equation  $5^x + 3 = z^2$  in non-negative integers is  $(0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $x$  and  $z$  such that  $5^x + 3 = z^2$ . If  $x = 0$ , then  $z = 2$ . Let  $x \geq 1$ . So,  $5^x + 3 \equiv 3 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 = z^2$ . □

### 3 Main results

In this section, we present the main results on the solvability of Eq. (0.1). We start by considering the class of these equations in which  $p$  is an odd prime number satisfying  $p \equiv 1 \pmod{5}$ .

**Theorem 3.1.** *Let  $p$  be an odd prime number such that  $p \equiv 1 \pmod{5}$ . Then, the only solution  $(p, x, y, z)$  of the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  in non-negative integers is  $(p, 0, 0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $x, y$  and  $z$  satisfying  $5^x + 3 \cdot p^y = z^2$ . Two possible cases can be considered:

Case 1:  $x = 0$  or  $y = 0$ . In this case, by Corollary 2.4 and Lemma 2.5, one has that  $(p, x, y, z) = (p, 0, 0, 2)$ .

Case 2:  $x \geq 1$  and  $y \geq 1$ . Since  $p \equiv 1 \pmod{5}$  and  $5^x \equiv 0 \pmod{5}$ , then  $5^x + 3p^y \equiv 3 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 \cdot p^y = z^2$ . □

**Corollary 3.2.** *Let  $p$  be a prime number such that  $p \equiv 1 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  has no solution  $(p, x, y, z)$  in positive integers.*

**Corollary 3.3.** *Let  $p$  be a prime number such that  $p \equiv 1 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^4$  has no solution  $(p, x, y, z)$  in positive integers.*

**Example 3.4.**  $(x, y, z) = (0, 0, 2)$  is the only solution in non-negative integers of the Diophantine equations:

$$\begin{aligned}
 5^x + 3 \cdot 11^y &= z^2 \\
 5^x + 3 \cdot 31^y &= z^2 \\
 5^x + 3 \cdot 41^y &= z^2 \\
 5^x + 3 \cdot 61^y &= z^2 \\
 5^x + 3 \cdot 71^y &= z^2 \\
 5^x + 3 \cdot 101^y &= z^2 \\
 5^x + 3 \cdot 131^y &= z^2
 \end{aligned}$$

**Lemma 3.5.** *Let  $p$  be a prime number and  $y$  be a positive integer such that  $p \equiv 2 \pmod{5}$  and  $y \equiv 0, 2 \pmod{4}$ . Then,*

$$p^y \equiv 1 \pmod{5}, \text{ if } y \equiv 0 \pmod{4} \text{ or } 4 \pmod{5}, \text{ if } y \equiv 2 \pmod{4}.$$

*Proof.* Straightforward. □

In the following theorem, we determine all solutions of Eq. (0.1) when  $p \equiv 2 \pmod{5}$  and  $y \equiv 0 \pmod{4}$ .

**Theorem 3.6.** *Let  $p$  be a prime number such that  $p \equiv 2 \pmod{5}$ . Then, the only solutions  $(p, x, y, z)$  of the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  in non-negative integers with  $y \equiv 0 \pmod{4}$  are  $(2, 0, 4, 7)$  and  $(p, 0, 0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $x, y$  and  $z$  satisfying  $5^x + 3 \cdot p^y = z^2$  with  $y \equiv 0 \pmod{4}$ . Two possible cases can be considered:

Case 1:  $x = 0$  or  $y = 0$ . In this case, by Corollary 2.3 and Lemma 2.5, one has that  $(p, x, y, z) \in \{(2, 0, 4, 7), (p, 0, 0, 2)\}$ .

Case 2:  $x \geq 1$  and  $y \geq 1$ . Since  $p \equiv 2 \pmod{5}$  and  $y \equiv 0 \pmod{4}$ , so, by Lemma 3.5, it follows that  $5^x + 3p^y \equiv 3 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 \cdot p^y = z^2$ . □

**Corollary 3.7.** *Let  $p$  be a prime number such that  $p \equiv 2 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  has no solution  $(p, x, y, z)$  in positive integers with  $y \equiv 0 \pmod{4}$ .*

**Corollary 3.8.** *Let  $p$  be a prime number such that  $p \equiv 2 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^4$  has no solution  $(p, x, y, z)$  in positive integers with  $y \equiv 0 \pmod{4}$ .*

**Example 3.9.** The only solutions  $(x, y, z)$  of the Diophantine equation  $5^x + 3 \cdot 2^y = z^2$  in non-negative integers with  $y \equiv 0 \pmod{4}$  are  $(0, 0, 2)$  and  $(0, 4, 7)$ .

**Example 3.10.** Consider the following Diophantine equations:

$$\begin{aligned} 5^x + 3 \cdot 7^y &= z^2 \\ 5^x + 3 \cdot 17^y &= z^2 \\ 5^x + 3 \cdot 37^y &= z^2 \\ 5^x + 3 \cdot 47^y &= z^2 \\ 5^x + 3 \cdot 67^y &= z^2 \\ 5^x + 3 \cdot 97^y &= z^2 \\ 5^x + 3 \cdot 107^y &= z^2. \end{aligned}$$

The only solution  $(x, y, z)$  of these equations in non-negative integers with  $y \equiv 0 \pmod{4}$  is  $(0, 0, 2)$ .

In the next result, we present a sufficient condition for the unsolvability of Eq. (0.1) when  $p \equiv 2 \pmod{5}$ .

**Theorem 3.11.** *Let  $p$  be a prime number such that  $p \equiv 2 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  has no solution  $(p, x, y, z)$  in non-negative integers with  $y \equiv 2 \pmod{4}$ .*

*Proof.* Suppose there are non-negative integers  $x, y$  and  $z$  satisfying  $5^x + 3 \cdot p^y = z^2$  with  $y \equiv 2 \pmod{4}$ . We can divide the discussion into two possible cases:

Case 1:  $x = 0$  and  $y \geq 1$ . Since  $y \equiv 2 \pmod{4}$ , then  $y \neq 0$  and  $y \neq 4$ . Hence, by Corollary 2.3, one concludes that the Diophantine equation  $1 + 3 \cdot p^y = z^2$  has no solution  $(p, y, z)$  in non-negative integers (contradiction!).

Case 2:  $x \geq 1$  and  $y \geq 1$ . Since  $p \equiv 2 \pmod{5}$  and  $y \equiv 2 \pmod{4}$ , so, by Lemma 3.5, it follows that  $5^x + 3p^y \equiv 2 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 \cdot p^y = z^2$ . □

**Corollary 3.12.** *Let  $p$  be a prime number such that  $p \equiv 2 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^4$  has no solution  $(p, x, y, z)$  in non-negative integers with  $y \equiv 2 \pmod{4}$ .*

**Example 3.13.** The Diophantine equations

$$\begin{aligned}
 5^x + 3 \cdot 2^y &= z^2 \\
 5^x + 3 \cdot 7^y &= z^2 \\
 5^x + 3 \cdot 17^y &= z^2 \\
 5^x + 3 \cdot 37^y &= z^2 \\
 5^x + 3 \cdot 47^y &= z^2 \\
 5^x + 3 \cdot 67^y &= z^2 \\
 5^x + 3 \cdot 97^y &= z^2 \\
 5^x + 3 \cdot 107^y &= z^2
 \end{aligned}$$

has no solution  $(x, y, z)$  in non-negative integers with  $y \equiv 2 \pmod{4}$ .

**Lemma 3.14.** *Let  $y$  be a positive integer and  $p$  be an odd prime number such that  $p \equiv 3 \pmod{5}$ . Then,*

$$p^y \equiv 1, 4 \pmod{5}, \text{ if } y \equiv 0 \pmod{2}, 3 \pmod{5}, \text{ if } y \equiv 1 \pmod{2}.$$

*Proof.* Straightforward. □

Next, we show that Eq. (0.1) has a unique solution when  $p \equiv 3 \pmod{5}$  and  $y$  is even.

**Theorem 3.15.** *Let  $p$  be an odd prime number such that  $p \equiv 3 \pmod{5}$ . Then, the only solution  $(p, x, y, z)$  of the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  in non-negative integers with  $y \equiv 0 \pmod{2}$  is  $(p, 0, 0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $x, y$  and  $z$  satisfying  $5^x + 3 \cdot p^y = z^2$  with  $y \equiv 0 \pmod{2}$ . Two possible cases can be considered:

Case 1:  $x = 0$  or  $y = 0$ . In this case, by Corollary 2.4 and Lemma 2.5, one has that  $(p, x, y, z) = (p, 0, 0, 2)$ .

Case 2:  $x \geq 1$  and  $y \geq 1$ . Since  $p \equiv 3 \pmod{5}$  and  $y \equiv 0 \pmod{2}$ , so, by Lemma 3.14, it follows that  $5^x + 3p^y \equiv 2, 3 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 \cdot p^y = z^2$ . □

**Corollary 3.16.** *Let  $p$  be an odd prime number such that  $p \equiv 3 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  has no solution  $(p, x, y, z)$  in positive integers with  $y \equiv 0 \pmod{2}$ .*

**Corollary 3.17.** *Let  $p$  be an odd prime number such that  $p \equiv 3 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^4$  has no solution  $(p, x, y, z)$  in positive integers with  $y \equiv 0 \pmod{2}$ .*

**Example 3.18.** Consider the following Diophantine equations:

$$\begin{aligned}
 5^x + 3 \cdot 3^y &= z^2 \\
 5^x + 3 \cdot 13^y &= z^2 \\
 5^x + 3 \cdot 23^y &= z^2 \\
 5^x + 3 \cdot 43^y &= z^2 \\
 5^x + 3 \cdot 53^y &= z^2 \\
 5^x + 3 \cdot 73^y &= z^2 \\
 5^x + 3 \cdot 83^y &= z^2 \\
 5^x + 3 \cdot 103^y &= z^2.
 \end{aligned}$$

The only solution  $(x, y, z)$  of these equations in non-negative integers with  $y \equiv 0 \pmod{2}$  is  $(0, 0, 2)$ .

**Lemma 3.19.** *Let  $y$  be a positive integer and  $p$  be an odd prime number. If  $p \equiv 4 \pmod{5}$ , then  $p^y \equiv 1, 4 \pmod{5}$ .*

*Proof.* Straightforward. □

We finished this paper, showing that  $(p, x, y, z) = (p, 0, 0, 2)$  is the only solution of Eq. (0.1) when  $p \equiv 4 \pmod{5}$ .

**Theorem 3.20.** *Let  $p$  be an odd prime number such that  $p \equiv 4 \pmod{5}$ . Then, the only solution  $(p, x, y, z)$  of the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  in non-negative integers is  $(p, 0, 0, 2)$ .*

*Proof.* Suppose there are non-negative integers  $x, y$  and  $z$  such that  $5^x + 3 \cdot p^y = z^2$ . Two possible cases can be considered:

Case 1:  $x = 0$  or  $y = 0$ . In this case, using again Corollary 2.4 and Lemma 2.5, one concludes that  $(p, x, y, z) = (p, 0, 0, 2)$ .

Case 2:  $x \geq 1$  and  $y \geq 1$ . Since  $p \equiv 4 \pmod{5}$ , then, by Lemma 3.19, it follows that  $5^x + 3p^y \equiv 2, 3 \pmod{5}$ , but  $z^2 \equiv 0, 1, 4 \pmod{5}$ , which contradicts the equality  $5^x + 3 \cdot p^y = z^2$ . □

**Corollary 3.21.** *Let  $p$  be an odd prime number such that  $p \equiv 4 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^2$  has no solution  $(p, x, y, z)$  in positive integers.*

**Corollary 3.22.** *Let  $p$  be an odd prime number such that  $p \equiv 4 \pmod{5}$ . Then, the Diophantine equation  $5^x + 3 \cdot p^y = z^4$  has no solution  $(p, x, y, z)$  in positive integers.*

**Example 3.23.** Consider the following Diophantine equations:

$$\begin{aligned} 5^x + 3 \cdot 19^y &= z^2 \\ 5^x + 3 \cdot 29^y &= z^2 \\ 5^x + 3 \cdot 59^y &= z^2 \\ 5^x + 3 \cdot 79^y &= z^2 \\ 5^x + 3 \cdot 89^y &= z^2 \\ 5^x + 3 \cdot 109^y &= z^2 \\ 5^x + 3 \cdot 139^y &= z^2 \\ 5^x + 3 \cdot 149^y &= z^2. \end{aligned}$$

The only solution  $(x, y, z)$  of these equations in non-negative integers is  $(0, 0, 2)$ .

## 4 Conclusion remarks

In this work, we investigate the Diophantine equation (0.1) using some elementary techniques of number theory and produce some results on the solvability of such an equation. We can summarize the main contributions of this paper as follows:

- We show that Eq. (0.1) is solvable whenever  $p$  is an odd prime number satisfying  $p \equiv 1, 4 \pmod{5}$ , in this case,  $(p, x, y, z) = (p, 0, 0, 2)$  is its only solution in non-negative integers;
- We prove that Eq. (0.1) is solvable whenever  $p \equiv 2 \pmod{5}$  and  $y \equiv 0 \pmod{4}$ , in this case, its only solutions  $(p, x, y, z)$  in non-negative integers are  $(2, 0, 4, 7)$  and  $(p, 0, 0, 2)$ ;
- We prove that Eq. (0.1) is unsolvable whenever  $p \equiv 2 \pmod{5}$  and  $y \equiv 2 \pmod{4}$ ;
- We show that Eq. (0.1) is solvable whenever  $y$  is even and  $p$  is an odd prime number satisfying  $p \equiv 3 \pmod{5}$ , in this case,  $(p, x, y, z) = (p, 0, 0, 2)$  is its only solution in non-negative integers.

However, some questions of this study remain unanswered: (i) Is Equation (0.1) solvable or unsolvable when  $p \equiv 2 \pmod{5}$  and  $y \equiv 1, 3 \pmod{4}$ ? (ii) What can we say about the solvability/unsolvability of Eq. (0.1) when  $y$  is odd and  $p$  is an odd prime number satisfying  $p \equiv 3 \pmod{5}$ ?

## References

- [1] D. Acu, *On a Diophantine equation*, General Mathematics, (2007).
- [2] T. Andreescu, D. Andrica and I. Cucurezeanu, *An Introduction to Diophantine Equations: A Problem-Based Approach*, Birkhäuser Boston, (2010).
- [3] T. Andreescu and D. Andrica, *Quadratic Diophantine Equations*, Springer New York, (2015).
- [4] A. Baker, *Transcendental Number Theory*, Cambridge University Press, (2022).
- [5] E. J. Barbeau, *Pell's Equation*, Springer New York, (2003).
- [6] Y. F. Bilu, Y. Bugeaud and M. Mignotte, *The Problem of Catalan*, Springer International Publishing, (2014).
- [7] J. L. Brenner and L. Foster, *Exponential Diophantine equations*, Pacific Journal of Mathematics, **101(2)**, 263–301, (1982).
- [8] M. Buosi, A. Lemos, A. L. P. Porto and D. F. G. Santiago, *On the Exponential Diophantine Equation  $p^x - 2^y = z^2$  with  $p = k^2 + 4$  a Prime Number*, Palestine Journal of Mathematics, **11(4)**, 130–135, (2022).
- [9] S. Chotchaisthit, *On the Diophantine equation  $p^x + (p + 1)^y = z^2$  where  $p$  is a Mersenne prime*, International Journal of Pure and Applied Mathematics, **88(2)**, 1314–3395, (2013).
- [10] G. Effinger and G. L. Mullen, *Elementary Number Theory*, Chapman and Hall/CRC, (2021).
- [11] M. J. Jacobson and H. C. Williams, *Solving the Pell Equation*, Springer New York, (2008).
- [12] G. A. Jones and J. M. Jones, *Elementary Number Theory*, Springer London, (1998).
- [13] T. Koshy, *Elementary Number Theory with Applications*, 2nd Edition. Elsevier Science & Technology, (2007).
- [14] K. Laipaporn, S. Wananiyakul and P. Khachorncharoenkul, *On the Diophantine Equation  $3^x + p \cdot 5^y = z^2$* , Walailak Journal of Science and Technology (WJST), **16(9)**, 647–653, (2019).
- [15] P. Mihăilescu, *Primary cyclotomic units and a proof of Catalans conjecture*, Journal für die reine und angewandte Mathematik (Crelles Journal), **2004(572)**, 167–195, (2004).
- [16] S. Natarajan and R. Thangadurai, *Pillars of Transcendental Number Theory*, Springer Singapore, (2020).
- [17] M. B. Nathanson, *On a Diophantine equation of M. J. Karama*, Palestine Journal of Mathematics, **6(2)**, 524–527, (2017).
- [18] F. Neres, *On the Solvability of the Diophantine Equation  $p^x + (p + 8)^y = z^2$  when  $p > 3$  and  $p + 8$  are Primes*, Annals of Pure and Applied Mathematics, **18(1)**, 9–13, (2018).
- [19] A. Porto, M. Buosi and G. Ferreira, *On the Exponential Diophantine Equation  $p \cdot 3^x + p^y = z^2$  with  $p$  a Prime Number*, Annals of Pure and Applied Mathematics, **29(1)**, 13–19, (2023).
- [20] J. F. T. Rabago, *On the Diophantine Equation  $4^x - p^y = 3z^2$  where  $p$  is a Prime*, Thai Journal of Mathematics, **16(3)**, 643–650, (2018).
- [21] T. N. Shorey and R. Tijdeman, *Exponential Diophantine Equations*, Cambridge University Press, (1986).
- [22] B. Singha, *Non-negative Solutions of the Nonlinear Diophantine Equation  $(8^n)^x + p^y = z^2$  for Some Prime Number  $p$* , Walailak Journal of Science and Technology (WJST), **18(16)**, (2021).
- [23] V. G. Sprindžuk, *Classical Diophantine Equations*, Springer Berlin Heidelberg, (1993).
- [24] I. Stewart and D. Tall, *Algebraic Number Theory and Fermat's Last Theorem*, Chapman and Hall/CRC, (2015).
- [25] A. Suvarnamani, *Solutions of the Diophantine equation  $2^x + p^y = z^2$* , International Journal of Mathematical Sciences and Applications, **1(3)**, 1415–1419, (2011).
- [26] S. Thongnak, W. Chuayjan and T. Kaewong, *On the Diophantine Equation  $11 \cdot 3^x + 11^y = z^2$  where  $x$ ,  $y$  and  $z$  are Non-Negative Integers*, Annals of Pure and Applied Mathematics, **25(1)**, 51–54, (2022).
- [27] S. Thongnak, W. Chuayjan and T. Kaewong, *On the Exponential Diophantine Equation  $5^x - 2 \cdot 3^y = z^2$* , Annals of Pure and Applied Mathematics, **25(2)**, 109–112, (2022).

## Author information

Fernando Neres, Departamento de Ciência e Tecnologia, Universidade Federal Rural do Semi-Árido, Caraúbas, Rio Grande do Norte, Brazil.

E-mail: fernandoneres@ufersa.edu.br

Received: 2024-10-21

Accepted: 2025-05-05