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An Accelerated Fourth-Order Fitted Numerical Approach for Singularly Perturbed Reaction-Diffusion Problems of Small Time Lag

A.W. Demsie, A.A. Tiruneh and E.G. Tsega

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Coressponding Author: A.W. Demsie

Abstract. In this paper, we considered a class of singularly perturbed parabolic reaction-diffusion problems involving a small time lag at the reaction term, common in applications such as in biological and chemical reactions, population growth and epidemiology. The presence of a small diffusion parameter $\varepsilon, (0 < \varepsilon \ll 1)$ give rise to sharp boundary layers near the end points of the spatial domain, where the classical numerical methods fail to give accurate solutions. To address these challenges, we formulated and analyzed an accelerated fourth-order fitted numerical scheme integrating the Crank-Nicolson and non-standard finite differences on uniform meshes. The various stability and convergence analysis demonstrated that the proposed scheme is uniformly convergent with respect to the perturbation parameter ε. Numerical experiments with three standard test problems revealed that the proposed scheme performs better than some existing approaches in the literature and verify the theoretical findings.

1 Introduction

Delay differential equations (DDEs) arise in the modeling of a large variety of real life situations in engineering and science, for instance, physiological processes [1], the population dynamics [2], diseases spread [3], diffusion in polymers [4], epidemiology [5], and hydrodynamics of liquid helium [6]. Singularly perturbed delay differential equations (SPDDEs) are differential equations in which its highest order derivative term is multiplied by a small perturbation parameter $\varepsilon(0 < \varepsilon \ll 1)$ and has at least one delay term $\delta > 0$. These types of differential equations mostly arise in various practical phenomena , such as biological and chemical reactions, population growth [7], epidemiology [8]. Examples of delays include the time taken for a signal to travel to the controlled object, driver reaction time, the time for the body to produce red blood cells, and cell division time in the dynamics of viral exhaustion or persistence [9], In particular, time delays are natural components of the dynamic processes of biology, ecology, physiology, economics, epidemiology, and mechanics [10].

The most captivating aspect about SPDDEs is that the solutions contain boundary and/or interior layers (the regions in which there are steep gradients) when the perturbation parameter ε approaches zero. These features describe that the classical numerical approaches are not suitable for accurately and successfully resolving these problems [11]. This emphasises the need for parameter independent numerical techniques [12]. As a result, parameter-uniformly convergent algorithms have been developed [13, 14, 15, 16, 17]. The fitted operator methods, fitted mesh methods, domain decomposition methods, and grid equi-distribution methods are some examples of parameter independent numerical methods.

According to Wang refereed in [18] a simplified mathematical model for the control system of a furnace used to process metal sheets is represented by

$$\psi_t - \varepsilon\psi_{xx} = \omega(g(\psi(x,t-\delta)))\psi_x(x,t) + C\left[f(\psi(x,t-\delta) - \psi(x,t)))\right] \tag{1.1}$$

where the temperature distribution in a metal sheet ψ is moving at an instantaneous material strip velocity ω and source specified by the function f. Moreover, both the velocity ω and the source function f are dynamically adapted by a controlling device monitoring the current temperature distribution. The finite speed of the controller, however, introduces a fixed delay of length δ. When $\delta = 0$, this problem becomes a thermal problem without time delay.

The establishment of the model equation in (1.1) has given the opportunity for various scholars to establish parameter-uniformly convergent methods for the numerical treatment of singularly perturbed time delay convection-diffusion-reaction problems (SPTDCDRPs). Woldaregay et al [19] proposed a novel numerical scheme for solving SPTDCDRPs. The uniform convergence analysis of of the scheme proved that the formulated scheme converges uniformly with a linear order of convergence. Hassan et al [20] presented a parameter-uniform numerical method to solve SPTDCDRPs. The proposed method is shown to be parameter uniform convergent, unconditionally stable, and linear in order of accuracy.
Another significant mathematical model presented in [21] for a class of deterministic partial differential equations is the logistic equation of the form:

$$\psi_t(x,t) - \varepsilon\psi_{xx}(x,t) = \psi(x,t)(1 - \psi(x,t-\delta)) \tag{1.2}$$

which arise in mathematical ecology for the evolution of a population with density function $\psi(x,t)$ which depends on the population at an earlier time, $t-\delta$ rather than t. The delay δ can arise from a great variety of causes, such as duration of gestation, hatching period, and slow replacement of food supplies. Thus, the density function $\psi(x,t)$ depends on average past population $\psi(x,t-\delta)$. The initial and final spatial spread of the favoured population are given by $\psi(0,t)$ and $\psi(1,t)$, respectively.

Using the above logistic time delay model equation (1.2), various scholars have doing significant research works. For example, Negero [22] designed a robust fitted numerical scheme for singularly perturbed parabolic time delay reaction-diffusion problems (SPTDRDPs). Both the time and spatial derivatives are discretised via implicit Euler and exponentially fitted cubic-B spline method on a uniform meshes respectively. Furthermore, the author applied the Richardson extrapolation technique to accelerate the order of convergence of the proposed method to second-order accuracy. Singh et al [23] devised a domain decomposition method to solve SPTDRDPs up on discretizing the proposed problem using backward Euler scheme in the time direction and central difference scheme in the spatial direction. The proposed method is shown to be uniformly convergent, having almost second order in space and first order in time. Gelu et al [24] investigated a uniformly convergent collocation method for resolving SPTDRDPs. The problem is discretised by the implicit Euler method on a uniform mesh in time and the extended cubic B-spline collocation method on a Shishkin mesh in space directions. The proposed method is shown to be ε uniformly convergent of first order in time and almost second order in space with the logarithmic factors. Ayele et al [25] constructed a numerical scheme to solve SPTDRDPs. For the discretization of the time derivative, the authors used the Crank-Nicolson method and a hybrid scheme, which is a combination of compact difference scheme and the central difference scheme on a special type of Shishkin mesh in the spatial direction. Tiruneh et al [26] developed and analyzed a nonstandard fitted operator method for large time delay SPTDRDPs. The problem domain is discretised using the Crank-Nicolson method in the time direction and a nonstandard finite difference in the spatial directions on a uniform mesh. Govindarao [27] presented a parameter uniformly convergent method for a SPTDRDPs initial-boundary-value problem. Furthermore, using the Richardson extrapolation technique, the proposed method proved that it is a fourth-order convergent in both time and spatial variables. Duressa et al [28] contributed a higher-order parameter uniformly convergent method for a SPTDRDPs. The proposed problem discretised the time derivative via the Crank-Nicolson method on uniform meshes and the space derivative with central difference on Shishkin meshes.

The literature underscores the significance of the proposed problem through demonstrating that it is still in the earliest stages of investigation. Furthermore, the existing approaches typically show lower convergence orders and less accuracy. This is essentially motivated us to formulate and analyze a higher-order fitted numerical scheme that is highly accurate, swift, and easy to implement for solving singularly perturbed reaction-diffusion problems involving a small time lag. We discretised problem using the Crank-Nicolson and nonstandard finite difference methods on uniform meshes for the temporal and spatial directions, respectively. The proposed scheme is verified to be fourth order accurate in both the temporal and spatial directions using various types of convergence analyses.

Notations: Throughout this paper, C and its subscripts denote generic positive constants independent of the perturbation parameter ε and mesh sizes h and Δt. Also, $||.||$ denotes the standard supremum norm, defined as $||f|| = \sup_{(x,t)\in\overline{\mathcal{D}}_{xt}} |f(x,t)|$ for a function f defined on some domain $\overline{\mathcal{D}}_{xt}$.

2 Problem Description

In this paper, we consider a class of singularly perturbed parabolic reaction-diffusion problems involving small time lag subject to the initial and interval boundary conditions of the form:

$$\begin{cases} \mathcal{L}_{\varepsilon,\delta}\psi(x,t) \equiv: f(x,t), \quad \forall (x,t)\in\mathcal{D}_{xt} = \mathcal{D}_x\times\mathcal{D}_t = (0,1)\times(0,T], T>0., \\ \begin{cases} \psi(x,t) = \mathcal{V}_0(x,t) & \mu_0 = [0,1]\times[-\delta,0], \\ \psi(0,t) = \mathcal{V}_l(t) & \mu_l = \{(0,t) : 0\le t\le T\}, \\ \psi(1,t) = \mathcal{V}_r(t)) & \mu_r = \{(1,t) : 0\le t\le T\}. \end{cases} \end{cases} \tag{2.1}$$

where $\mathcal{L}_{\varepsilon,\delta}\psi(x,t) = \psi_t(x,t) - \varepsilon\psi_{xx}(x,t) + a(x,t)\psi(x,t) + b(x,t)\psi(x,t-\delta)$ is a differential operator and $\varepsilon, (0 < \varepsilon \ll 1)$, is a singular perturbation parameter such that, $\delta < \varepsilon$ is a small delay parameter which satisfies the relation $\delta = O(\varepsilon)$. For the existence of a unique solution, the functions $a(x,t), b(x,t), f(x,t), \mathcal{V}_0(x,t), \mathcal{V}_l(t)$ and $\mathcal{V}_r(t)$ are assumed to be sufficiently smooth, bounded, and independent of the perturbation parameter ε and satisfy

$$a(x,t) + b(x,t) \ge \alpha > 0, b(x,t) \le 0, \quad \forall (x,t)\in\overline{\mathcal{D}}_{xt}. \tag{2.2}$$

When the parameter ε approaches zero, the solution of problem (2.1) exhibits twin boundary layers near the end points of the spacial domain depending on the sign of the coefficient of the reaction term $a(x,t)$. For $a(x,t) > 0$ the twin boundary layers occur in the neighborhood of μ_l and μ_r near $x = 0$ and $x = 1$ of width $O(\sqrt{\varepsilon})$.

Following the approach in [19] and assuming $\delta = O(\varepsilon)$ in the sense that $\delta \le \sigma\varepsilon$ for some constant σ, typically with $\sigma < 1$ ensuring the higher-order terms in the Taylor series expansion remain negligible. Using this, we approximate the delay term about the point (x,t) as follows:

$$\psi(x,t-\delta) = \psi(x,t) - \delta\psi_t(x,t) + O(\delta^2). \tag{2.3}$$

Substituting Eq.(2.4) in to Eq.(2.1), we get a transformed equation of the form:

$$\begin{cases} \mathcal{L}_{C_\varepsilon}\psi \equiv \psi_t(x,t) - C_\varepsilon(x,t)\psi_{xx}(x,t) + A(x,t)\psi(x,t) = F(x,t), \forall (x,t)\in\mathcal{D}_{xt}, \\ \begin{cases} \psi(x,0) = \mathcal{S}_0(x) & \forall x\in\mu_b = \left\{(x,0) : x\in\overline{\mathcal{D}}_x\right\}, \\ \psi(0,t) = \mathcal{S}_l(0,t) & \forall (x,t)\in\mu_l = \left\{(x,t) : x=0, t\in\overline{\mathcal{D}}_t\right\}, \\ \psi(1,t) = \mathcal{S}_r(1,t)) & \forall (x,t)\in\mu_l = \left\{(x,t) : x=1, t\in\overline{\mathcal{D}}_t\right\}. \end{cases} \end{cases} \tag{2.4}$$

where $\mathcal{L}_{C_\varepsilon}\psi = \psi_t(x,t) - C_\varepsilon\psi_{xx}(x,t) + A(x,t)\psi(x,t)$ is the differential operator, $C_\varepsilon(x,t), (0 < C_\varepsilon(x,t) \ll 1)$ is the perturbation parameter to the transformed problem and the coefficients

$$\begin{cases} Q(x,t) = 1/(1-\delta b(x,t)) & \forall (x,t)\in\mathcal{D}_{xt}, \\ C_\varepsilon(x,t) = \varepsilon Q(x,t) & \forall (x,t)\in\mathcal{D}_{xt}, \\ A(x,t) = (a(x,t)+b(x,t))Q(x,t) & \forall (x,t)\in\mathcal{D}_{xt}, \\ F(x,t) = Q(x,t)f(x,t) & \forall (x,t)\in\mathcal{D}_{xt}. \end{cases} \tag{2.5}$$

For small value of the delay parameter δ, problem (2.1) and (2.4) are asymptotically equivalent, since the difference between the two equations is $O(\delta^2)$. Furthermore, for the existence of a unique solution, the functions $A(x,t), F(x,t), \mathcal{S}_0(x), \mathcal{S}_l(t), \mathcal{S}_r(t)$ are assumed to be sufficiently smooth, bounded, and independent of the perturbation parameter C_ε and satisfy

$$A(x,t) \ge \beta > 0 \quad \forall (x,t)\in\overline{\mathcal{D}}_{xt}. \tag{2.6}$$

2.1 Bounds on the analytical Solution and Its Derivatives:

The existence and uniqueness of the solution of problem (2.4) can be established by assuming the data is Hölder continuous on the domain $\overline{\mathcal{D}}_{xt}$ [23, 25]. The required compatibility condition at the corner points $(0,0)$ and $(1,0)$ are defined as $\mathcal{S}_0(0,0) = \mathcal{S}_l(0)$, $\mathcal{S}_0(1,0) = \mathcal{S}_r(1)$ and satisfy

$$\begin{cases} \frac{\partial \mathcal{S}_l(0)}{\partial t} - C_\varepsilon \frac{\partial^2 \mathcal{S}_0(0,0)}{\partial x^2} + A(0,0)\mathcal{S}_0(0,0) = F(0,0), \\ \frac{\partial \mathcal{S}_r(1)}{\partial t} - C_\varepsilon \frac{\partial^2 \mathcal{S}_0(1,0)}{\partial x^2} + A(1,0)\mathcal{S}_0(1,0) = F(1,0). \end{cases} \tag{2.7}$$

where $\mathcal{S}_0(x,t),\ \mathcal{S}_l(x,t)$ and $\mathcal{S}_r(x,t)$ are supposed to be sufficiently smooth functions. Based on the above suitable conditions on the initial data and the initial-boundary value problem (2.4) has unique solution $\psi(x,t) \in C^{(4,2)}\left(\overline{\mathcal{D}}_{xt}\right)$ which exhibits a twin boundary layers each of width $O(\sqrt{C_\varepsilon})$ at x = 0 and x = 1. These conditions guarantee the existence of a constant C independent of the perturbation parameter C_ε such that

$$\begin{cases} |\psi(x,t) - \mathcal{S}_0(x)| \leq Ct,\ (x,t) \in \mathcal{D}_x, \\ |\psi(x,t)| \leq C,\ \ (x,t) \in \overline{\mathcal{D}}_{xt}. \end{cases} \tag{2.8}$$

Proof: Interested reader can refer the proof in [15].

Lemma 2.1. *(Continuous Maximum Principle). Assume that $\phi(x,t)$ is any sufficiently smooth function satisfying $\phi(x,t) \geq 0, \forall (x,t) \in \partial\mathcal{D}_{xt} = \overline{\mathcal{D}}_{xt} \setminus \mathcal{D}_{xt}$ and $\mathcal{L}_{C_\varepsilon}\phi(x,t) \geq 0, \forall (x,t) \in \mathcal{D}_{xt}$ then, $\phi(x,t) \geq 0, \forall (x,t) \in \overline{\mathcal{D}}_{xt}$, where the differential operator is defined by $\mathcal{L}_{C_\varepsilon}\phi(x,t) = \phi_t(x,t) - C_\varepsilon\phi_{xx} + A(x,t)\phi(x,t)$*

Proof. Suppose that $(x^*,t^*) \in \mathcal{D}_{xt}$ such that $\phi(x^*,t^*) = min_{(x,t)\in\overline{\mathcal{D}}_{xt}}\phi(x,t) < 0$.
From this condition and elementary concepts of calculus we have $\phi_t(x^*,t^*) = 0$,
$\phi_x(x^*,t^*) = 0$ and $\phi_{xx}(x^*,t^*) \geq 0$.
Using all the above conditions, we have defined the differential operator as follows

$$\begin{aligned} \mathcal{L}_{C_\varepsilon}\phi(x^*,t^*) &= \phi_t(x^*,t^*) - C_\varepsilon\phi_{xx}(x^*,t^*) + A(x^*,t^*)\phi(x^*,t^*), \\ &= -C_\varepsilon\phi_{xx}(x^*,t^*) + A(x^*,t^*)\phi(x^*,t^*), \\ &\leq 0 \quad ,\ since, \phi_{xx}(x^*,t^*) \geq 0,\ \phi(x^*,t^*) < 0\ and\ A(x^*,t^*) > 0. \end{aligned}$$

This implies that $\mathcal{L}_{C_\varepsilon}\phi(x^*,t^*) \leq 0$ which contradicts to our assumption $\mathcal{L}_\varepsilon\phi(x,t) \geq 0$.
Consequently, we conclude that the minimum of $\phi(x,t)$ is non negative. □

Lemma 2.2. *(Uniform Stability Estimate). If $\psi(x,t) \in C^{2,1}(\overline{\mathcal{D}}_{xt})$ be the solution of the continuous problem of equation (2.4) ,then it satisfies the bound*

$$|\psi(x,t)| \leq \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(x)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\}.$$

Proof. To prove this Lemma, we defined two barrier functions $\pi^\pm$ as follows

$$\pi^\pm(x,t) = \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(x)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\} \pm \psi(x,t),$$

When $t = 0$, we have

$$\begin{aligned} \pi^\pm(x,0) &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(x)|, |\mathcal{S}_l(0)|, |\mathcal{S}_r(0)|\right\} \pm \psi(x,0), \\ &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(x)|, |\mathcal{S}_l(0)|, |\mathcal{S}_r(0)|\right\} \pm S_0(x) \geq 0. \end{aligned}$$

When $x = 0$, we have

$$\begin{aligned} \pi^\pm(0,t) &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(0)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\} \pm \psi(0,t), \\ &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(0)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\} \pm S_l(t) \geq 0. \end{aligned}$$

When $x = 1$, we have

$$\begin{aligned} \pi^\pm(1,t) &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(1)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\} \pm \psi(1,t), \\ &= \beta^{-1}|F|| + max\left\{|\mathcal{S}_0(0)|, |\mathcal{S}_l(t)|, |\mathcal{S}_r(t)|\right\} \pm S_r(t) \geq 0. \end{aligned}$$

Having all these on the domain $\mathcal{D}_{xt}$, we have

$$\begin{aligned}\mathcal{L}_{C_\varepsilon}\pi^{\pm}(x,t) &= \frac{\partial\pi^{\pm}(x,t)}{\partial t} - C_\varepsilon\frac{\partial^2\pi^{\pm}(x,t)}{\partial x^2} + A(x,t)\pi^{\pm}(x,t),\\ &= A(x,t)\beta^{-1}||F|| + A(x,t)\times max\{|\mathcal{S}_0(x)|,|\mathcal{S}_l(t)|,|\mathcal{S}_r(t)|\}\\ &+ \beta^{-1}||F|| \pm \mathcal{L}_{C_\varepsilon}\psi(x,t),\\ &= A(x,t)\times max\{|\mathcal{S}_0(x)|,|\mathcal{S}_l(t)|,|\mathcal{S}_r(t)|\}\,\beta^{-1}||F|| \pm F(x,t)\geq 0.\end{aligned}$$

From Lemma (2.1) it follows that $\pi^{\pm}(x,t)\geq 0,\ \forall(x,t)\in\overline{\mathcal{D}}_{xt}$. □

Lemma 2.3. *Let $\psi(x,t)$ be the solution of the problem (2.4) at the time level m. Then we have*

$$\left|\psi^{(i,m)}(x,t)\right| \leq C\left[1 + C_\varepsilon^{-i/2}\left(exp(-\sqrt{\frac{\beta}{C_\varepsilon}}x + exp(-\sqrt{\frac{\beta}{C_\varepsilon}}(1-x))\right)\right]. \tag{2.9}$$

where $0\leq i+2m\leq 4$ and β is the lower bound of $A(x,t)$ as shown in $Eq.$(2.6).

Proof: Interested reader can refer the proof in [29] and [30].

3 Domain Discretization

The main concern of this section is to discretize the proposed problem using Crank-Nicolson and NSFD methods in the temporal and spatial directions on uniform meshes respectively. Finally, we integrate the two semi-discretized problems to get the full discrete problem of (2.4).

3.1 Temporal Semi-discretisation:

Divide the domain $[0,T]$ in to M equal number of mesh intervals with step length k gives

$$\overline{D}_t^M = \left\{t_m : t^m = m\Delta t, m = 0,1,2,3,\dots M \text{ and } \Delta t = \frac{T}{M}\right\}. \tag{3.1}$$

Using the Crank Nicolson approach in problem (2.4) and the steps in [26], we obtain the time semi-discretized problem of the following form:

$$\begin{cases}\frac{\psi^{m+1}-\psi^m}{\Delta t} = C_\varepsilon(\frac{\psi^{m+1}_{xx}+\psi^m_{xx}}{2}) - \frac{((A\psi)^{m+1}+(A\psi)^m)}{2} + \frac{F^{m+1}+F^m}{2},\\ \begin{cases}\psi(x,0) = \mathcal{S}_0(x),\ \forall x\in\mathcal{D}_x,\\ \psi(0,t_{m+1}) = \mathcal{S}_l(t^{m+1}), 0\leq m\leq M-1,\\ \psi(1,t_{m+1}) = \mathcal{S}_r(1,t^{m+1})), 0\leq m\leq M-1.\end{cases}\end{cases} \tag{3.2}$$

After some rearrangements, we get

$$\begin{cases}-C_\varepsilon\psi^{m+1}_{xx}(x) + (2/\Delta t + A^{m+1})\psi^{m+1}(x) = g^m(x),\\ \begin{cases}\psi(x,0) = \mathcal{S}_0(x)\forall x\in\mathcal{D}_x,\\ \psi(0,t_{m+1}) = \mathcal{S}_l(t^{m+1}),\ 0\leq m\leq M-1,\\ \psi(1,t_{m+1}) = \mathcal{S}_r(1,t^{m+1}), 0\leq m\leq M-1.\end{cases}\end{cases} \tag{3.3}$$

where $g^m(x) = C_\varepsilon\psi^m_{xx}(x) + (2/\Delta t - A^m(x))\psi^m(x) + F^{m+1}(x) + F^m(x)$.
Using problem (3.3) at $(m+1)^{th}$ time level in-terms of spatial variables, we can check that the maximum principle and stability estimate of the temporal semi-discretization of the proposed scheme as follows

$$\begin{cases}\mathcal{L}^*Y :\equiv -C_\varepsilon Y^{''}(x) + W(x)Y(x) = g^m(x),\ \forall x\ \in\overline{\mathcal{D}}_x,\\ Y(0) = S_l(t_{m+1}),\ Y(1) = S_r(t_{m+1}),\ \forall t^{m+1}\in\overline{\mathcal{D}}_t^M.\end{cases} \tag{3.4}$$

where $\mathcal{L}^*$ is a discrete operator , $Y(x) = \psi^{m+1}(x),\ W(x) = (2/\Delta t + A^{m+1}(x))$.

Lemma 3.1. *(Semi-discrete Maximum Principle). If $\phi^{m+1}(x) \in C^2(\overline{\mathcal{D}}_x)$ is the solution of the semi-discrete problem (3.4) such that $\phi^{m+1}(0) \geq 0, \psi^{m+1}(1) \geq 0$ on $\partial\overline{\mathcal{D}}_x = \overline{\mathcal{D}}_x \setminus \mathcal{D}_x$ and $\mathcal{L}^*\phi^{m+1}(x) \geq 0,\ \forall x \in \mathcal{D}_x$, then $\phi^{m+1}(x) \geq 0,\ \forall x \in \overline{\mathcal{D}}_x$.*

Proof. Let $x^* \in \overline{\mathcal{D}}_x$ such that $\phi^{m+1}(x^*) = min_{x\in\overline{\mathcal{D}}_x}\phi^{m+1}(x) < 0$ and since $\phi^{n+1}(0) \geq 0,\ \phi^{m+1}(1) \geq 0$ then $x^* \notin \{0,1\}$ that is $x^* \in \mathcal{D}_x$.
Moreover, since $\phi_x^{m+1}(x^*) = 0$ and $\phi_{xx}^{m+1}(x^*) \geq 0$ we have that

$$\begin{aligned}\mathcal{L}^*_{C_\varepsilon}\phi^{m+1}(x^*) &= -C_\varepsilon\phi_{xx}^{m+1}(x^*) + W(x^*)\phi^{m+1}(x^*),\\ &\leq 0.\end{aligned}$$

This implies that $\mathcal{L}^*_{C_\varepsilon}\phi^{m+1}(x) \leq 0$, which contradicts to our assumption $\mathcal{L}^*_{C_\varepsilon}\phi^{m+1}(x) \geq 0$. Consequently, it is proved that the minimum of $\phi^{m+1}(x)$ is non-negative. □

Lemma 3.2. *(Semi-discrete Stability Estimate). Let $Y(x)$ be the solution of the semi-discrete problem* (3.4)*.Then we have*

$$||Y(x)|| \leq \frac{1}{\alpha}||\mathcal{L}^*Y(x)|| + ||Y(x)||_{\mathcal{D}_x},\ W^*(x) \geq \alpha > 0. \tag{3.5}$$

Proof: For the proof of this lemma, the reader can refer Lemma (2.2) of section 2.
The local truncation error of the semi-discrete problem (3.3) is given by $e^{m+1} \equiv \psi(x,t_m) - \psi_i^{m+1}$, where ψ_i^{m+1} is the computed solution of the proposed boundary value problem. Moreover, this error measures the contribution of each time step to the global error of the temporal semi-discretization.

Lemma 3.3. *(Local Error Estimate). Suppose that*

$$\left|\frac{\partial^i\psi(x,t)}{\partial t^i}\right| \leq C, \forall(x,t) \in \overline{\mathcal{D}}_x \text{ and } 0 \leq i \leq 2$$

then the local error estimate in the temporal direction is given by

$$||e^{m+1}||_\infty \leq C_1\Delta t^3. \tag{3.6}$$

Proof. Using the Taylor series expansion about $(x, t^{m+1/2})$, we have

$$\psi^{m+1} = \psi(x,t^{m+1/2}) + \frac{\Delta t}{2}\psi_t(x,t^{m+1/2}) + \frac{\Delta t^2}{8}\psi_{tt}(x,t^{m+1/2}) + O(\Delta t^3), \tag{3.7}$$

$$\psi^{m} = \psi(x,t^{m+1/2}) - \frac{\Delta t}{2}\psi_t(x,t^{m+1/2}) + \frac{\Delta t^2}{8}\psi_{tt}(x,t^{m+1/2}) + O(\Delta t^3), \tag{3.8}$$

Now on subtracting Eq.(3.7) from Eq.(3.8) we get

$$\frac{\psi^{m+1} - \psi^m}{\Delta t} = \psi_t(x,t^{m+1/2}) + O(\Delta t^2). \tag{3.9}$$

Substituting Eq.(3.9) in Eq.(2.1) we get

$$\begin{aligned}\frac{\psi^{m+1} - \psi^m}{\Delta t} &= \varepsilon\psi_{xx}(x,t^{m+\frac{\Delta t}{2}}) - (A\psi)(x,t^{m+\frac{\Delta t}{2}}) \\ &\quad + F(x,t^{m+\frac{\Delta t}{2}}) + O\Delta t^2.\end{aligned} \tag{3.10}$$

where

$$\begin{cases} F(x,t^{m+1/2}) = \frac{F(x,t^{m+1}) + F(x,t^m)}{2}, \\ \psi(x,t^{m+1/2}) = \frac{\psi(x,t^{m+1}) + \psi(x,t^m)}{2}, \\ A(x,t^{m+1/2}) = \frac{A(x,t^{m+1}) + A(x,t^m)}{2}. \end{cases} \tag{3.11}$$

The solution to the local error is

$$\begin{cases} \mathcal{L}^* e^{m+1} = O(\Delta t^3), \\ e^{m+1}(0) = 0, \ e^{m+1}(1) = 0. \end{cases} \tag{3.12}$$

Thus, using the maximum principle, the required result is satisfied.
Hence, $||e^{m+1}||_\infty \leq C_1 \Delta t^3$ as required. □

To illustrate the temporal direction's global error estimate, we represent E^{m+1} as the accumulation of the scheme's local errors at the $(m+1)^{th}$ time level.

Lemma 3.4. *(Global Error Estimate): based on the assumption of Lemma (3.3) the global error estimate of the semi-discretization at $(m+1)^{th}$ time level is given by*

$$||E^{m+1}||_\infty \leq C_2 \Delta t^2. \tag{3.13}$$

Proof. Since the global error estimate of the $(m+1)^{th}$ time level be given by

$$\begin{aligned} ||E^{m+1}||_\infty &= ||\sum_{s=1}^{m+1} e^s||_\infty, \text{ since, } (m+1) \leq T/\Delta t, \\ &\leq ||e^1||_\infty + ||e^2||_\infty + \cdots + ||e^{m+1}||_\infty, \qquad (3.14) \\ &\leq C_1 (m+1) \Delta t^3, \text{ using Lemma (3.3)}, \\ &\leq C_1 (T) \Delta t^2, \quad \text{since } (m+1)\Delta t \leq T, \\ &\leq C_2 \Delta t^2, \ C_2 = C_1 T. \qquad (3.15) \end{aligned}$$

where C_2 is a positive real constant independent of the perturbation parameter C_ε. □

From the proof it is seen that the Crank Nicolson method is a second order parameter uniform convergent method.

3.2 Spatial semi-discretization:

Divide the domain $[0, 1]$ in to N equal number of mesh intervals of step length h gives

$$\overline{\mathcal{D}}_x^N = \left\{ x_0 = 0, x_i = x_{i-1} + h, x_N = 1, \ i = 1, 2, 3, \ldots N-1 \text{ and } h = \frac{1}{N} \right\}. \tag{3.16}$$

Non-Standard Finite Difference Method (NSFDM): The fundamental idea of the NSFDM is to use complicated positive functions instead of the denominator of the finite difference approximation of the derivatives. To design an exact finite difference scheme, we considered the homogeneous differential equations with constant coefficients, which correspond to problem (3.3) as mentioned by Mickens in [31]. Furthermore, the theory of difference equations gives us

$$\begin{aligned} -C_\varepsilon \left[\frac{\psi_{i+1}^{m+1} - 2\psi_i^{m+1} + \psi_{i-1}^{m+1}}{\eta_{i,m+1}^2} \right] + W_i^{m+1} \psi_i^{m+1} &= C_\varepsilon \left[\frac{\psi_{i+1}^m - 2\psi_i^m + \psi_{i-1}^m}{\eta_{i,m}^2} \right] \\ &- W_i^m \psi_i^m + F_i^{m+1} + F_i^m, \qquad (3.17) \end{aligned}$$

with the discrete initial and boundary conditions

$$\begin{cases} \psi_i^0 = 0, \ i = 0, 1, 2, \ldots, N, \\ \psi_0^{m+1} = \psi_N^{m+1} = 0, \ m = 0, 1, 2, \ldots, M-1. \end{cases} \tag{3.18}$$

where the denominator function $\eta_{i,m+1}^2$ to be determined from the next procedures. Furthermore, we consider the homogeneous part of Eq.(3.17) with constant coefficients, where the constant coefficients are the lower bounds of the coefficients $W^{m+1}(x_i) \geq w^* > 0$, we get

$$-C_\varepsilon \frac{d}{dx^2} \psi^{m+1} + w^* \psi^{m+1} = 0. \tag{3.19}$$

Then the two linearly independent solutions of Eq.(3.19) are given by

$$exp(\lambda_1 x) \text{ and } exp(\lambda_1 x),\ \lambda_1, \lambda_2 = \pm\sqrt{\frac{w^*}{C_\varepsilon}}. \tag{3.20}$$

Since a linear combination of two linearly independent solutions is also the solution of the same problem, then $\psi_i^{n+1} = c_1 exp(\lambda_1 x_i) + c_2 exp(\lambda_2 x_i)$ is the solution of Eq.(3.19).Now using the theory of difference equations in [31], the following set of equations can be obtained by taking the consecutive points,

$$\begin{vmatrix} \psi_{i-1}^{n+1} & \exp(\lambda_1 x_{i-1}) & \exp(\lambda_2 x_{i-1}) \\ \psi_i^{n+1} & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ \psi_{i+1}^{n+1} & \exp(\lambda_1 x_{i+1}) & \exp(\lambda_2 x_{i+1}) \end{vmatrix} = 0. \tag{3.21}$$

Rearranging Eq.(3.21) we get

$$\begin{vmatrix} \psi_{i-1}^{n+1} & \exp(\lambda_1 (x_i - h)) & \exp(\lambda_2 (x_i - h)) \\ \psi_i^{n+1} & \exp(\lambda_1 x_i) & \exp(\lambda_2 x_i) \\ \psi_{i+1}^{n+1} & \exp(\lambda_1 (x_i + h)) & \exp(\lambda_2 (x_i + h)) \end{vmatrix} = 0. \tag{3.22}$$

Factorizing the above equation results

$$(exp(\lambda_1 x_i) + exp(\lambda_2 x_i)) \begin{vmatrix} \psi_{i-1}^{n+1} & \exp(\lambda_1 (x_i - h)) & \exp(\lambda_2 (x_i - h)) \\ \psi_i^{n+1} & 1 & 1 \\ \psi_{i+1}^{n+1} & \exp(\lambda_1 (x_i + h)) & \exp(\lambda_2 (x_i + h)) \end{vmatrix} = 0. \tag{3.23}$$

On substituting the values of λ_1 and λ_2 from the determinant of Eq.(3.23) and simplifying the terms by using the double angle formula for hyperbolic functions ,we obtain the exact difference scheme as follows:

$$\psi_{i-1}^{n+1} - 2\cosh\left(\sqrt{\frac{w^*}{C_\varepsilon}}\right)\psi_i^{n+1} + U_{i+1}^{n+1} = 0.$$

Also after some rearrangement for $i = 1, 2, \ldots, N-1$, we get

$$-C_\varepsilon \frac{\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1,j+1}}{4\left(\frac{C_\varepsilon}{w^*}\right)\sinh^2\left(\frac{1}{2}h\sqrt{\frac{w^*}{C_\varepsilon}}\right)} + W_i^{m+1}\psi_{i+1}^{n+1} = 0. \tag{3.24}$$

As a result, the denominator function for the second order derivative approximation can be found as

$$\eta^2(C_\varepsilon, h) = 4\left(\frac{C_\varepsilon}{w^*}\right)\sinh^2\left(\frac{1}{2}h\sqrt{\frac{w^*}{C_\varepsilon}}\right). \tag{3.25}$$

For $i = 1, 2, 3, \ldots, N-1,\ m = 0, 1, 2, \ldots, M-1$, we defined the general variable coefficient form of the denominator function as follows:

$$\eta_{i,m+1}^2 = 4\left(\frac{C_\varepsilon}{W_i^{m+1}}\right)\sinh^2\left(\frac{1}{2}h\sqrt{\frac{W_i^{m+1}}{C_\varepsilon}}\right). \tag{3.26}$$

Using a similar approach as in the $(m+1)^{th}$ time level, we determined that the m^{th} time level denominator function for variable coefficient as follows:

$$\eta_{i,m}^2 = 4\left(\frac{C_\varepsilon}{W_i^m}\right)\sinh^2\left(\frac{1}{2}h\sqrt{\frac{W_i^m}{C_\varepsilon}}\right). \tag{3.27}$$

3.3 The Fully Discrete Problem:

In this section, we combined the temporal and spatial semi-discretized results by substituting Eq.(3.26) and Eq.(3.27) in to Eq.(3.17) as follows

$$\begin{aligned} -C_\varepsilon \frac{\left[\psi_{i+1}^{m+1} - 2\psi_i^{m+1} + \psi_{i-1}^{m+1}\right]}{\eta_{i,m+1}^2} + W_i^{m+1}\psi_i^{m+1} &= C_\varepsilon \frac{\left[\psi_{i+1}^{m} - 2\psi_i^{m} + \psi_{i-1}^{m}\right]}{\eta_{i,m}^2} \\ &- W_i^m \psi_i^m + H_i^m. \end{aligned} \tag{3.28}$$

Simplifying these equations, we get the tri-diagonal system of equations subject to the initial and boundary conditions of the following form:

$$\begin{cases} \mathcal{L}_{C_\varepsilon}^{N,M} :\equiv R_i^- \psi_{i-1}^{m+1} + R_i^c \psi_i^{m+1} + R_i^+ \psi_{i+1}^{m+1} = G_i^m, \\ \begin{cases} \psi_i^0 = \mathcal{S}_0(x_i), \quad i = 0, 1, 2, \ldots, N-1, \\ \psi_0^{m+1} = \mathcal{S}_l^{m+1}(0), \ m = 0, 1, 2, \ldots, M-1, \\ \psi_N^{m+1} = \mathcal{S}_r^{m+1}(1), \ m = 0, 1, 2, \ldots, M-1. \end{cases} \end{cases} \tag{3.29}$$

where the coefficients of the tri-diagonal system are to be determined for $i = 1, 2, 3, \ldots, N-1,$ and $m = 0, 1, 2, \ldots, M-1$ using the subsequent equations

$$\begin{cases} R_i^- = \frac{-C_\varepsilon}{\eta_{i,m+1}^2}, \\ R_i^c = \left(\frac{2C_\varepsilon}{\eta_{i,m+1}^2} + \frac{2}{\Delta t} + A_i^{m+1}\right), \\ R_i^+ = \frac{-C_\varepsilon}{\eta_{i,m+1}^2}, \\ G_i^m = C_\varepsilon \frac{\left[\psi_{i+1}^m - 2\psi_i^m + [\psi_{i-1}^m\right]}{\eta_{i,m}^2} - W_i^m \psi_i^m + F_i^m + F_i^{m+1}. \end{cases} \tag{3.30}$$

Since the boundary values of ψ_0^{m+1} and ψ_N^{m+1} are given from Eq.(2.4), we found the remaining values of ψ_i^{m+1}, for $i = 1, 2, 3, \ldots, N-1$ from the tri-diagonal system in Eq. (3.29) by means of the matrix inverse method with the help of Mat-Lab 2023.

Lemma 3.5. *(Discrete Maximum Principle). If ϕ_i^{m+1} be any mesh function defined on $\overline{\mathcal{D}}_{xt}^{N,M}$ such that $\phi_0^{m+1} \geq 0$, $\phi_N^{m+1} \geq 0$ and $\mathcal{L}^{N,M}\psi_i^{M+1} \geq 0, \forall i = 1, 2, 3, ..., N-1$, then $\phi_i^{m+1} \geq 0$, $\forall i = 0, 1, 2, ..., N$.*

Proof. Suppose that there exists a mesh point $(i^*, m+1)$ for $i^* \in \{1, 2, 3, \ldots, N-1\}$ such that $\phi_{i^*}^{m+1} = min_{0 \leq i \leq N} \phi_i^{m+1}$ and assume that $\phi_{i^*}^{m+1} < 0$, $\phi_{i^*+1}^{m+1} - \psi_{i^*}^{m+1} > 0$, $\phi_{i^*-1}^{m+1} - \phi_{i^*}^{m+1} > 0$ and $L_{C_\varepsilon}^{N,M}\phi_i^{M+1} \geq 0$. Since $i^* \notin \{0, N\}$,then for $i^* \in \{1, 2, 3, \ldots, N-1\}$ we have that

$$\begin{aligned} \mathcal{L}_{C_\varepsilon}^{N,M}\phi_{i^*}^{m+1} &= -C_\varepsilon \left[\frac{\phi_{i^*+1}^{m+1} - 2\phi_{i^*}^{m+1} + \phi_{i^*-1}^{m+1}}{\eta_{i^*}^2}\right] + W_i^{m+1}\phi_{i^*}^{m+1}, \\ &= -C_\varepsilon \left[\frac{(\phi_{i^*+1}^{m+1} - \phi_{i^*}^{m+1}) + (\phi_{i^*-1}^{m+1} - \phi_{i^*}^{m+1})}{\eta_{i^*}^2}\right] + W_i^{m+1}\phi_{i^*}^{m+1} \leq 0. \end{aligned}$$

This implies that $\mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1} \leq 0$, which contradicts our assumption $\mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1} \geq 0$. Hence, $\phi_i^{m+1} \geq 0, \forall i = 1, 2, 3, ..., N$ as required. □

Lemma 3.6. *(Discrete Stability Estimate). If ϕ_i^{m+1} is any mesh function such that $\phi_0^{m+1} = 0 = \phi_N^{m+1}$, then it satisfies the bound*

$$\left|\phi_i^{m+1}\right| \leq \beta^{-1} \max_{1 \leq j \leq N-1} \left|\mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1}\right|, for\ 1 \leq i \leq N-1.$$

Proof. Let $Q = \beta^{-1}\max_{1 \leq i \leq N-1}\left|\mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1}\right|$ and $(\pi^\pm)_i^{m+1}$ be the barrier functions defined by $(\pi^\pm)_i^{m+1} = Q \pm \phi^{m+1}$.

It is clear that $(\pi^{\pm})_0^{m+1} = (\pi^{\pm})_N^{m+1} = Q > 0$. Also, for $1 \le i \le N-1$, we have

$$\begin{aligned}
\mathcal{L}_{C_\varepsilon}^{N,M}\left(\pi^{\pm}\right)_i^{m+1} &= -C_\varepsilon\left[\frac{Q \pm \phi_{i+1}^{m+1} - 2\left(Q \pm \phi_i^{m+1}\right) + Q \pm \phi_{i-1}^{m+1}}{\eta_i^2}\right] \\
&+ \left(\frac{1}{\Delta t} + A_i^{m+1}\right)\left(Q \pm \phi_i^{m+1}\right), \\
&= \left(\frac{1}{\Delta t} + A_i^{m+1}\right) Q \pm \mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1}, \\
&= \beta^{-1}\left(\frac{1}{\Delta t} + A_i^{m+1}\right)\max\left|\mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1}\right| \pm \mathcal{L}_{C_\varepsilon}^{N,M}\phi_i^{m+1}.
\end{aligned}$$

Since, $\left(\frac{1}{\Delta t} + A_i^{m+1}\right) > \beta$, we have $L_{C_\varepsilon}^{N,M}(\pi^{\pm})_i^{m+1} \ge 0$. Thus by Discrete Maximum Principle in Lemma (3.5) we obtain $(\pi^{\pm})_i^{m+1} \ge 0,\ \ 0 \le i \le N$.
Hence, we proved the uniform stability estimate of the discrete scheme at $(m+1)^{th}$ time level as required. □

Lemma 3.7. *For a fixed mesh and for all integers j we have that*

$$\lim_{C_\varepsilon \to 0}\max_{1\le i\le N-1}\frac{exp\left(-Cx_i/\sqrt{C_\varepsilon}\right)}{C_\varepsilon^{j/2}} = 0 \ \ \textit{and} \ \ \lim_{C_\varepsilon \to 0}\max_{1\le i\le N-1}\frac{exp\left(-C(1-x_i)/\sqrt{C_\varepsilon}\right)}{C_\varepsilon^{j/2}} = 0.$$

Proof: Interested reader can refer the proof in [32] of Lemma 5.3.

4 Convergence Analysis

Omitting the time level index for the sake of simplicity and supposing ψ_i^m and $\psi^m(x_i)$ are the computed and analytical solutions of the considered problem respectively, then the local truncation error of the proposed nonstandard finite difference method from Eq.(3.29) is defined by

$$\mathcal{L}_{C_\varepsilon}^{N}\left(\psi_i^m - \psi^m(x_i)\right) = \left(\mathcal{L}_{x,C_\varepsilon} - \mathcal{L}_{x,C_\varepsilon}^{N}\right)\psi^m(x_i). \tag{4.1}$$

Using the nonstandard finite difference method

$$\mathcal{L}_{C_\varepsilon}^{N,M}\left(\psi_i^m - \psi^m(x_i)\right) = -C_\varepsilon\psi_i^{''} + C_\varepsilon\frac{(\psi_{i+1} - 2\psi_i + \psi_{i-1})}{\eta_{i,m}^2}. \tag{4.2}$$

By using the Taylor series expansions of u_{i-1} and u_{i+1} from Eq.(4.2) we get

$$\mathcal{L}_{C_\varepsilon}^{N,M}\left(\psi_i^m - \psi^m(x_i)\right) = -C_\varepsilon\psi_i^{''} + \frac{C_\varepsilon}{\eta_{i,m}^2}\left(h^2\psi^{''}(\zeta_i) + \frac{h^4}{12}u^{(4)}(\zeta_i)\right)\ \zeta_i \in (x_{i-1}, x_{i+1}). \tag{4.3}$$

Furthermore, the truncated Taylor series expansion of $1/\eta_i^2$ is given by

$$\frac{1}{\eta_{i,m}^2} = \left(\frac{1}{h^2} - \frac{W_i^m}{12C_\varepsilon} + \frac{W_i^m h^2}{240C_\varepsilon^2}\right), \quad \text{where, } (W_i^m = A_i^m - 2/k). \tag{4.4}$$

Substituting Eq.(4.4) in to Eq.(4.3) and simplifying it using Lemma (2.3) and (3.7) we get

$$\mathcal{L}_{C_\varepsilon}^{N,M}\left(\psi_i^m - \psi^m(x_i)\right) \le Ch^2.$$

Hence, the absolute error is defined by

$$|\psi_i^m - \psi^m(x_i)| \le Ch^2. \tag{4.5}$$

where C is independent of the perturbation parameter ε and the step size h. Based on the above results we stated the following theorems.

Theorem 4.1. *Let ψ_i^m be the numerical solution of Eq.(3.29) and $\psi(x_i, t^m)$ be the solution of the problem in Eq.(2.4) both at time level m,then*

$$\max_{0\le i\le N,\ 0\le m\le M} |\psi_i^m - \psi(x_i, t^m)| \le Ch^2. \tag{4.6}$$

Hence, Theorem 4.1 shows that the NSFDM is a second order uniformly convergent scheme in space variable. The subsequent theorem is the combined results of Lemma (3.2) and Theorem (4.1).

Theorem 4.2. *Let ψ_i^m be the m^{th} time level approximate solution of Eq.(3.29) and $\psi(x_i, t^m)$ be the solution of Eq.(2.4). Then*

$$\max_{0\le i\le N,\ 0\le m\le M} |\psi_i^m - \psi(x_i, t^m)| \le C(h^2 + \Delta t^2). \tag{4.7}$$

From this we concluded that the proposed NSFD and Crank Nicolson methods are second order accurate uniformly convergent methods before extrapolation.

5 Richardson Extrapolation Technique

The Richardson extrapolation technique is used to accelerate the order of the method. Using theorem (4.2) at the m^{th} time level, we have

$$\left|\psi_i^m - \overline{\psi}_i^m\right| \le C(h^2 + \Delta t^2). \tag{5.1}$$

where ψ_i^m and $\overline{\psi}_i^m$ are approximate solutions on the mesh intervals $\mathcal{D}^{N,M}$ and $\mathcal{D}^{2N,2M}$ respectively so that that C is a parameter independent constant [32].
Moreover, since $\overline{\psi}_i^m$ is obtained by doubling the mesh $\mathcal{D}^{N,M}$ then, the mesh sizes $h \neq 0$ and $\Delta t \neq 0$ in Eq.(5.1) becomes

$$\psi_i^m(x_i, t_m) - \overline{\psi}_i^m(x_i, t^m) \le C(h^2 + \Delta t^2) + R^{N,M},\ (x_i, t^m) \in \mathcal{D}^{N,M}. \tag{5.2}$$

Again using the mesh sizes $h/2,\ \Delta t/2 \neq 0$ Eq.(5.2) becomes

$$\psi_i^m(x_i, t^m) - \overline{\psi}_i^m(x_i, t^m) \le C\left(\frac{h^2}{4} + \frac{\Delta t^2}{4}\right) + R^{2N,2M},\ (x_i, t^m) \in \mathcal{D}^{2N,2M}. \tag{5.3}$$

where the remainders $R^{N,M}$ and $R^{2N,2M}$ are $O(h^4 + \Delta t^4)$. Taking the difference of Eq.(5.3) and Eq.(5.2) we get the subsequent general extrapolation formula.

$$(\psi_i^m)^{ext} = \frac{4\overline{\psi}_i^m - \psi_i^m}{3}. \tag{5.4}$$

where $(\psi_i^m)^{ext}$ is the approximate solution obtained from the Richardson extrapolation technique with truncated error of the form

$$\left|\psi_i^m - (\psi_i^m)^{ext}\right| \le C(h^4 + \Delta t^4). \tag{5.5}$$

From these discussions, we generalized the following Theorem for the general purpose of the Richardson extrapolation technique.

Theorem 5.1. *Let $(\psi_i^m)^{ext}$ be the approximate solution of Eq(3.29) obtained after the Richardson extrapolation technique applied and ψ_i^m be the approximate solution of the proposed model equation (2.4).Then,the maximum absolute error encountered is defined by*

$$\left|\psi_i^m - (\psi_i^m)^{ext}\right| \le C(h^4 + \Delta t^4). \tag{5.6}$$

From the overall constructions and discussions, we concluded that using Richardson extrapolation technique accelerates the proposed method to fourth order uniformly convergent scheme in both spacial and temporal directions.

6 Numerical Examples and Illustrations

One of the major challenge in modeling problems is to obtain their analytical solutions due to a variety of factors including higher in dimension, non-linearity, and other model problem parameters. This troubles the analysis of the error estimation. The maximum absolute error of the approximate solutions is computed via doubling mesh principle to overcome around these challenges.

Example 6.1. Consider the singularly perturbed parabolic reaction diffusion problem as in [33] and [34].

$$\begin{cases} \psi_t - \varepsilon\psi_{xx} + \frac{1+x^2}{2}\psi + \psi(x,t-\delta) = t^3, & (x,t) \in (0,1)\times(0,2], \\ \psi(x,t) = 0, & (x,t) \in [0,1]\times[-\delta,0], \\ \psi(0,t) = 0, \psi(1,t) = 0. & t \in (0.2]. \end{cases}$$

Example 6.2. Consider the unsteady 1D singularly perturbed parabolic reaction-diffusion problem of the form in [22].

$$\begin{cases} \psi_t - \varepsilon\psi_{xx} + (1.1+x^2)\psi + \psi(x,t-\delta) = t^3, & (x,t) \in (0,1)\times(0,2], \\ \psi(x,t) = 0, & (x,t) \in [0,1]\times[-\delta,0], \\ \psi(0,t) = 0, \psi(1,t) = 0. & t \in (0.2]. \end{cases}$$

Example 6.3. Consider the singularly perturbed parabolic reaction diffusion problem as in [33].

$$\begin{cases} \psi_t - \varepsilon\psi_{xx} + x^2\psi = t^3 - \psi(x,t-\delta), & (x,t) \in (0,1)\times(0,2], \\ \psi(x,t) = 0, & (x,t) \in [0,1]\times[-\delta,0], \\ \psi(0,t) = 0, \psi(1,t) = 0, & t \in (0.2]. \end{cases}$$

Solution: Since the analytical solutions are unknown, we apply the doubling mesh technique to find the maximum absolute errors and the order of convergences.
Before Extrapolation:
Let $\psi^{N,M}(x_i,t^m)$ and $\psi^{2N,2M}(x_i,t^m)$ be the approximate solutions with mesh lengths $(h, \Delta t \neq 0)$ and $(h/2, \Delta t/2 \neq 0)$ respectively, then

- The maximum absolute error is given by

$$E_\varepsilon^{N,M} = \max_{0\le i,m\le N,M} \left|\psi^{N,M}(x_i,t^m) - \psi^{2N,2M}(x_{2i},t^{2m})\right|. \tag{6.1}$$

- The $\varepsilon-$ uniform maximum absolute error is given by

$$E^{N,M} = \max_{\varepsilon} E_\varepsilon^{N,M}. \tag{6.2}$$

- Corresponding ε-uniform order of convergence is $R^{N,M} = log_2\left(\frac{E^{N,M}}{E^{2N,2M}}\right)$.

After Extrapolation: Similarly, letting $\psi^{2N,2M}(x_i,t^m)$ and $(\psi_i^m)^{ext}$ be the approximate solutions with mesh lengths $(h/2, \Delta t/2 \neq 0)$ and $(h/4, \Delta t/4 \neq 0)$ respectively, then

- The maximum absolute error is given by

$$E_{ext}^{N,M} = \max_{0\le i,m\le N,M} \left|\psi^{2N,2M}(x_{2i},t^{2m}) - (\psi_i^m)^{ext}(x_i,t^m)\right|. \tag{6.3}$$

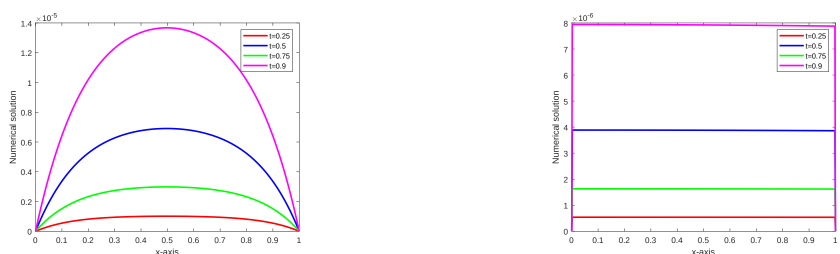
- The $\varepsilon-$ uniform maximum absolute error is given by

$$E_{ext}^{N,M} = \max_{\varepsilon} E_{\varepsilon,ext}^{N,M}. \tag{6.4}$$

- Corresponding ε, ext-uniform order of convergence is $R_{ext}^{N,M} = log_2\left(\frac{E_{ext}^{N,M}}{E^{2N,2M}}\right)$.

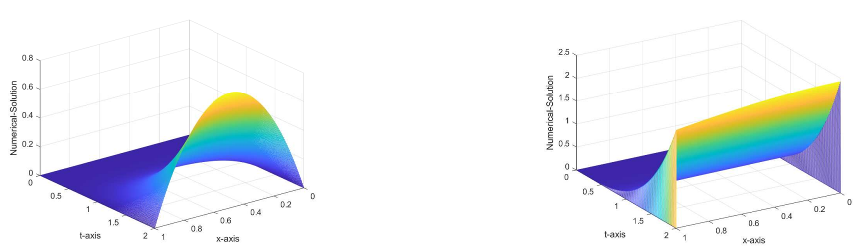
Table 1: Example 6.1, maximum absolute error $E^{N,M}$,corresponding rates of convergence $R^{N,M}$ and comparisons of the methods in [33] and [34] for the relation $\delta = 0.5 * \varepsilon$.

$\epsilon \downarrow$ N, M $\rightarrow$	64,20	128,40	256,80
2^{-22}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
2^{-24}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
2^{-26}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
2^{-28}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
2^{-30}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
2^{-32}	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
	2.0032	2.0004	
$E^{N,M}$	$1.3350e-03$	$3.3370e-04$	$8.3423e-05$
$R^{N,M}$	2.0032	2.0004	
The method in [33]			
$E^{N,M}$	$8.9288e-02$	$2.4218e-02$	$4.9511e-03$
$R^{N,M}$	1.8824	2.2903	
The method in [34]			
$E^{N,M}$	$1.7554e-001$	$2.6693e-001$	$3.1039e-001$
$R^{N,M}$	-0.6047	-0.2176	



(a) $\varepsilon = 10^{-0}$ (b) $\varepsilon = 10^{-22}$

Figure 1: 2D solution graph for Example 6.1 with values $N = 256$, $M = 32$.



(a) $\varepsilon = 10^{-0}$ (b) $\varepsilon = 10^{-22}$

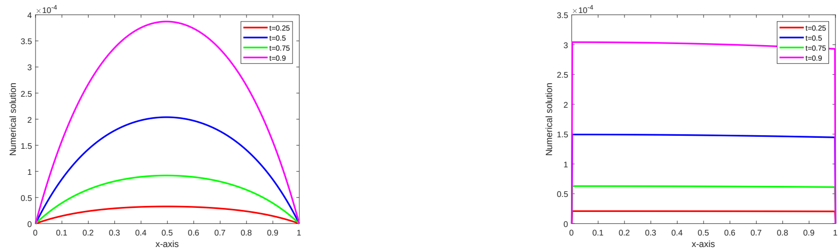
Figure 2: 3D solution graph for Example 6.1 with values $N = 256$, $M = 32$.

Table 2: Example 6.1, maximum absolute error $E^{N,M}$ and corresponding rates of convergent $R^{N,M}$ for $\delta = 0.5 * \varepsilon$ before and after Extrapolation technique.

$\epsilon \downarrow$ N, M $\rightarrow$	32,4	64,8	128,16	256,32
Before Extrapolation				
10^{-6}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-7}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-8}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-9}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-10}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-11}	$3.3615e-02$	$8.3573e-03$	$2.0864e-03$	$5.2144e-04$
	2.0080	2.0020	2.0004	
10^{-12}	$3.3615e-02$	$8.3573e-03$	$2.0865e-03$	$5.2145e-04$
	2.0080	2.0020	2.0005	
$E^{N,M}$	$3.3615e-02$	$8.3573e-03$	$4.7296e-03$	$5.2145e-04$
$R^{N,M}$	2.0080	2.0020	2.0004	
The method in [22]				
$E^{N,M}$	$9.9391e-02$	$5.9082e-02$	$3.2060e-02$	$1.6730e-02$
$R^{N,M}$	0.7504	0.8820	0.9425	
After Extrapolation				
10^{-6}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-7}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-8}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-9}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-10}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-11}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
10^{-9}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
	4.0760	4.0190	3.751	
$E_{ext}^{N,M}$ 10^{-9}	$6.8250e-06$	$4.0466e-07$	$2.4961e-08$	$1.8745e-09$
$R_{ext}^{N,M}$	4.0760	4.0190	3.751	
The method in [22]				
$E_{ext}^{N,M}$	$4.5049e-03$	$1.1675e-03$	$3.0348e-04$	$7.5922e-05$
$R_{ext}^{N,M}$	1.9481	1.9437	1.9990	

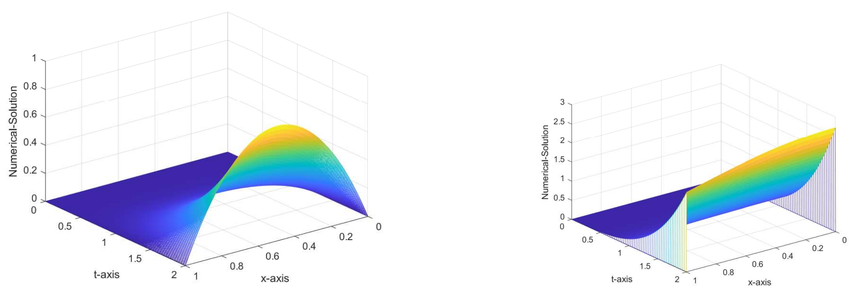
Table 3: Example 6.2, maximum absolute error $E^{N,M}$ and corresponding rates of convergence $R^{N,M}$ for $\delta = 0.5 * \varepsilon$.

$\epsilon \downarrow$ N, M $\rightarrow$	32,4	64,8	128,16	256,32
Before Extrapolation				
10^{-6}	$5.0134e-02$	$1.2510e-02$	$3.1257e-03$	$7.8137e-04$
	2.0027	2.0008	2.0001	
10^{-7}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
10^{-8}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
10^{-9}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
10^{-10}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
10^{-11}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
10^{-12}	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
	2.0027	2.0007	2.0002	
$E^{N,M}$	$5.0134e-02$	$1.2510e-02$	$3.1260e-03$	$7.8140e-04$
$R^{N,M}$	2.0027	2.0007	2.0002	
The method in [22]				
$E^{N,M}$	6.7384e − 02	3.9822e − 02	2.1533e − 03	1.1184e − 03
$R^{N,M}$	0.7588	0.8870	0.9452	
After Extrapolation				
10^{-6}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-7}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-8}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-9}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-10}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-11}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
10^{-12}	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
	4.0818	4.0209	4.0052	
$E_{ext}^{N,M}$	$6.8718e-06$	$4.0581e-07$	$2.4998e-08$	$1.5567e-09$
$R_{ext}^{N,M}$	4.0818	4.0209	4.0052	
The method in [22]				
$E_{ext}^{N,M}$	3.5282e − 03	9.1961e − 04	3.3499e − 04	5.9612e − 05
$R_{ext}^{N,M}$	1.9398	1.9684	1.9789	



(a) $\varepsilon = 10^{-0}$ (b) $\varepsilon = 10^{-18}$

Figure 3: 2D solution graph for Example 6.2 with values $N = 64, M = 8$.



(a) $\varepsilon = 10^{-0}$ (b) $\varepsilon = 10^{-18}$

Figure 4: 3D solution graph for Example 6.2 with values $N = 64, M = 8$.

7 Discussions

To assess the effectiveness and applicability of the proposed scheme, we provide three model examples. Based on the Examples in 6.1, 6.2 and 6.3 the maximum absolute errors and corresponding order of convergences are presented in Tables 1, 2 and 3 respectively. For a fixed value of the perturbation parameter ε and a decrease in the values of the step sizes h and Δt across a row, it is seen that the maximum absolute errors decrease. This is to mean that the accuracy of the method increases. On the other hand, for fixed values of step sizes h, Δt and a decrease in the values of the perturbation parameter ε down a column, we observed that the values of the maximum absolute errors are in a uniform pattern in each column and the order of convergence approaches to two before and four after Richardson extrapolation is used as seen in the Tables 2 and 3. These results confirmed that the proposed scheme is a higher-order parameter-uniformly convergent method. In addition to this, the 2D and 3D plots depicted in Example 6.1, Figures 1 and 2 and Example 6.2, Figures 4 and 3 elucidated that a decrease in the values of the perturbation parameter ε leads the problem to attain twin boundary layers near $x = 0$ and $x = 1$ as discoursed in the analytical aspects of the proposed problem. As shown in Table 4, for fixed value of the step sizes h and Δt, the perturbation parameter $\varepsilon = 0.1$, $\varepsilon = 0.01$ and a decrease in the values of the delay parameter δ, it is seen that the increase in the maximum absolute error down a column. On the other hand, for fixed value of the delay parameter and a decrease in the step sizes across a row it is observed that the maximum absolute error decreases. From these situations it is understood that the decrease the values of the delay parameter δ increases the maximum absolute error. To handle such problems it is better to chose the value of the delay term near the perturbation parameter ε but exclusively ε it self. From the analysis and discussions presented in this work, it follows that the theoretical analysis are in agreement with the results of the model examples. Overall, from the comparisons Tables 1, 2 and 3, we see that the present scheme outperforms the existing approaches in [22], [33] and [34], particularly in terms of accuracy and order of convergences.

Table 4: Example 6.3, $E^{N,M}$ and $R^{N,M}$ with fixed value of the perturbation parameter ε and vary the retarded term δ such that $\delta = \beta * \varepsilon$. Where, $\beta = 1/2, 1/4, 1/8$ and $1/10$.

$\varepsilon = 0.1$				
$\delta \downarrow$ N, M $\rightarrow$	32,4	64,8	128,16	256,32
0.0500	$2.0604e-02$	$5.1506e-03$	$1.3074e-03$	$3.3796e-04$
	2.0001	1.9781	1.9517	
0.0250	$2.1044e-02$	$5.2591e-03$	$1.3342e-03$	$3.4461e-04$
	2.0005	1.9788	1.9530	
0.0125	$2.1259e-02$	$5.3122e-03$	$1.3474e-03$	$3.4786e-04$
	2.0007	1.9791	1.9536	
0.0100	$2.1454e-02$	$5.3605e-03$	$1.3594e-03$	$3.5082e-04$
	2.0008	1.9794	1.9542	
$E^{N,M}$	$2.1454e-02$	$5.3605e-03$	$1.3594e-03$	$3.5082e-04$
$R^{N,M}$	2.0008	1.9794	1.9542	

.

$\varepsilon = 0.01$				
$\delta \downarrow$ N, M $\rightarrow$	32,4	64,8	128,16	256,32
0.00500	$4.7190e-02$	$1.1776e-02$	$2.9553e-03$	$7.4724e-04$
	2.0026	1.9945	1.9837	
0.00250	$4.7221e-02$	$1.1784e-02$	$2.9573e-03$	$7.4771e-04$
	2.0026	1.9945	1.9837	
0.00125	$4.7236e-02$	$1.1788e-02$	$2.9582e-03$	$7.4794e-04$
	2.0025	1.9945	1.9837	
0.00100	$4.7239e-02$	$1.1789e-02$	$2.9584e-03$	$7.4798e-04$
	2.0025	1.9945	1.9838	
$E^{N,M}_{ext}$	$4.7239e-02$	$1.1789e-02$	$2.9584e-03$	$7.4798e-04$
$R^{N,M}_{ext}$	1.9945	2.0025	1.9838	

8 Conclusion

In this paper, we proposed an accelerated fourth-order fitted numerical scheme for singularly perturbed parabolic reaction-diffusion problem of small time lag. The problem under consideration is discretised by means of Crank-Nicolson and NSFD methods in the temporal and spatial directions on uniform meshes, respectively. The numerous stability and convergence analyses confirmed that the proposed scheme is fourth-order accurate in both the spatial and temporal directions. Further, the numerical illustrations and discussions demonstrated that the theoretical findings and the results of the practical model examples are in agreement, and the formulated scheme outperforms some existing approaches in the literature. As further work of the research, we recommend the proposed scheme to higher-dimensional singularly perturbed problems.

Data Availability

No external data are used in this article.

Conflicts of Interest

The authors declare there are no potential conflicts of interest.

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Author information

A.W. Demsie, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: `amarew924@gmail.com`

A.A. Tiruneh, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: `awoke248@yahoo.com`

E.G. Tsega, Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia.
E-mail: `endalebdumath2016@gmail.com`

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