

ON A NONLINEAR DEGENERATE ELLIPTIC UNILATERAL PROBLEM

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Abstract. The present work deals on the existence of a solution for a nonlinear unilateral problem of the form: $-div(\nu_1 a(x, u, \nabla u)) + \Theta(x, u, \nabla u)\nu_2 = f$ in the framework of the weighted Sobolev space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$, where ν_1, ν_2 are two \mathcal{A}_p -weights and $f \in L^1(\Omega)$. Although $\Theta(x, \tau, \zeta)$ is a nonlinear term having a growth condition with respect to ζ and non-growth concerning τ , but it fulfills a sign condition on τ .

1 Introduction

Let Ω be a bounded smooth domain in \mathbb{R}^N ($N \geq 2$) and ν_1, ν_2 two weight functions belonging to the \mathcal{A}_p -weight class (see section 2). Let $A(u) = -div(\nu_1 a(x, u, \nabla u))$ be a Leray-Lions operator acting from the weighted Sobolev space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ into its dual $W^{-1,p'}(\Omega, \nu_1, \nu_2)$ ($p' = \frac{p}{p-1}, 1 < p < \infty$).

The purpose of this work is to show the existence of a solution for the nonlinear degenerate elliptic unilateral problem :

$$\begin{cases} -div(\nu_1 a(x, u, \nabla u)) + \Theta(x, u, \nabla u)\nu_2 = f \text{ in } D'(\Omega), \\ u \in W_0^{1,p}(\Omega, \nu_1, \nu_2), \Theta(x, u, \nabla u) \in L^1(\Omega, \nu_2), \end{cases}$$

where Θ is a nonlinear degenerate lower-order term including some natural growth with respect to $|\nabla u|$. Regarding u , we suppose that there are no growth restrictions, however it satisfies the sign condition, and the data f is in $L^1(\Omega)$.

Our objective is to involve sufficient conditions on a, Θ ensuring the existence of at least one solution for the nonlinear degenerate elliptic problem with Dirichlet boundary condition:

$$\begin{cases} u \in K_\phi, \Theta(x, u, \nabla u) \in L^1(\Omega, \nu_2), \\ \langle A(u), T_k(v - u) \rangle + \int_\Omega \nu_2 \Theta(x, u, \nabla u) T_k(v - u) \, dx \geq \langle f, T_k(v - u) \rangle \\ \text{for all } v \in K_\phi \text{ and all } k > 0, \end{cases} \quad (1.1)$$

where ϕ is a measurable function, $K_\phi = \{v \in W_0^{1,p}(\Omega, \nu_1, \nu_2), v \geq \phi \text{ a.e. in } \Omega\}$.

When $\phi = -\infty$, the existence has been shown in [8, 9] for the equation case where f belongs to $W^{-1,p'}(\Omega, \nu_1, \nu_2)$ and $L^1(\Omega)$ respectively.

In a recent work [5], Akdim et al. established an existence result by showing that the positive and negative parts of the approximate solution converge strongly in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ whenever

the second term $f \in W^{-1,p'}(\Omega, \nu_1, \nu_2)$. Azroul et al. studied in [6] the previous problem in the weighted Sobolev space $W_0^{1,p}(\Omega, \nu)$, where ν is a weight function that satisfies some hypothesis and $f \in L^1(\Omega)$. In this direction, we can also refer to results obtained in [1, 2, 4] in both weighted and unweighted Sobolev spaces.

Let us mention that the Muckenhoupt \mathcal{A}_p class of weights play a crucial role in various applied mathematical fields. In fluid dynamics, particularly in the study of the Navier-stocks equations, these weights are used to model fluid flow through irregular media, ensuring well-posedness in complex geometries, as explored by Fabes et.al [11]. In image processing, weights are applied in edge, accounting for non-uniform lighting and noise variations across an image, as demonstrated by Mallat [16].

Our goal in this work is to discuss the existence for a weak solution to the unilateral elliptic problem (1.1) based on the strong convergence for the approximate problem in the new intermediate space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ with $W_0^{1,p}(\Omega, \nu_1) \subset W_0^{1,p}(\Omega, \nu_1, \nu_2) \subset W_0^{1,p}(\Omega, \nu_2)$ where $\nu_1, \nu_2 \in \mathcal{A}_p$ with $\nu_2 \leq \nu_1$ (see section 3). For that, we assume that the term Θ satisfies in addition some coercivity conditions.

Let us note that many ideas used here have been adapted from [5, 6] to our case and where $f \in L^1(\Omega)$.

Finally, let us shortly state the contents of this paper. The next section is devoted to some preliminary knowledges that will be used in the sequel. In section 3, we specify the assumptions ensuring that the nonlinear degenerate elliptic problem (1.1) admits at least one solution, and then we prove the main results. Our results are illustrated at the end by an example.

2 Variational Framework

Throughout this paper, let ν be a locally integrable function in \mathbb{R}^N ($0 < \nu(x) < \infty$). Also, note that every weight ν rises to a measure on the measurable subsets of \mathbb{R}^N , denoted by μ . Then, we have for a measurable sets $F \subset \mathbb{R}^N$, $\mu(F) = \int_F \nu(x)dx$.

Next, we introduce some properties of \mathcal{A}_p -weights which was initially presented in [17] by Muckenhoupt, where the author showed these are the only class of weights such that the Hardy-Littlewood maximal operator is bounded from the weighted Lebesgue space into itself and thus plays a very important role in harmonic analysis (see [20]).

2.1 Muckenhoupt class \mathcal{A}_p

Definition 2.1. [17, 18] If there is a positive constant $A_1 = A_1(p, \nu)$ (known as the \mathcal{A}_p constant of ν) that satisfies for all ball $B \subset \mathbb{R}^N$ and $1 < p < \infty$,

$$\left(\frac{1}{|B|} \int_B \nu(x)dx \right) \left(\frac{1}{|B|} \int_B \nu(x)^{1/(1-p)}dx \right)^{p-1} \leq A_1,$$

then the weight ν is said to belong to the Muckenhoupt class \mathcal{A}_p or to be an \mathcal{A}_p -weight.

$|B|$ means the Lebesgue measure in \mathbb{R}^N of B .

If $1 < q < p < \infty$, then $\mathcal{A}_q \subset \mathcal{A}_p$ and the \mathcal{A}_q constant of a weight $\nu \in \mathcal{A}_p$ is equal to the \mathcal{A}_p constant of ν (see [20]).

Example 2.2. [19, 20]

- For $1 < p < \infty$, assume that $\nu(x_1) = |x_1|^{\sigma_1}, x_1 \in \mathbb{R}^N$, then ν is an \mathcal{A}_p weight if $-N < \sigma_1 < N(p - 1)$.
- If ν is a weight and there exist two positive constants C_1 and D_1 such that $C_1 \leq \nu(x) \leq D_1$ for a.e. $x \in \mathbb{R}^N$, then $\nu \in \mathcal{A}_p$ for $1 < p < \infty$.

Proposition 2.3. [13] Let F be a measurable subset of a ball $B \subset \mathbb{R}^N$.

If $\nu \in \mathcal{A}_p$ with $1 < p < \infty$, then

$$\left(\frac{|F|}{|B|} \right)^p \leq A_1 \frac{\mu(F)}{\mu(B)}, \tag{2.1}$$

where A_1 is the constant of the weight ν .

Remark 2.4.

- Let $\nu \in \mathcal{A}_p$, recall that a measure μ is doubling if there exists a positive constant D such that $\mu(B(x_1, 2r_1)) \leq D \mu(B(x_1, r_1))$ for every ball $B = B(x_1, r_1) \subset \mathbb{R}^N$.

Proposition 2.5. [13] Let F be a measurable subset of a ball B in \mathbb{R}^N . If $\nu \in \mathcal{A}_p$, then there are $0 < q \leq 1$ and $C_{N,p} > 0$ such that

$$\frac{\mu(F)}{\mu(B)} \leq C_{N,p} \left(\frac{|F|}{|B|} \right)^q.$$

As a direct consequence of Proposition 2.3 and Proposition 2.5, we get the following result

- $|F| = 0$ if and only if $\mu(F) = 0$. Thus, if $(u_n)_n$ is a sequence of functions defined in B and $u_n \rightarrow u$ μ - a.e. then $u_n \rightarrow u$ a.e.

For more information on \mathcal{A}_p - weights, we refer to the works [12, 13, 14, 19, 20].

2.2 Weighted Sobolev spaces

To deal with the problem (1.1), basic results are required for both the weighted Lebesgue space $L^p(\Omega, \nu)$ and the weighted Sobolev space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

Definition 2.6. Let ν be an \mathcal{A}_p - weight . For $1 \leq p < \infty$, the weighted Lebesgue space $L^p(\Omega, \nu)$ is defined as the set of measurable functions u on Ω such that

$$\|u\|_{L^p(\Omega, \nu)} = \left(\int_{\Omega} |u(x)|^p \nu(x) dx \right)^{\frac{1}{p}} < \infty.$$

Remark 2.7. [20] If $\nu \in \mathcal{A}_p$, $1 < p < \infty$, then $L^p(\Omega, \nu) \subset L^1_{loc}(\Omega)$ for every open set Ω . That makes sense to talk about weak derivatives of functions in $L^p(\Omega, \nu)$.

Definition 2.8. Let $1 < p < \infty$, and ν_1, ν_2 two \mathcal{A}_p -weights. The weighted Sobolev space $W^{1,p}(\Omega, \nu_1, \nu_2)$ is defined as follows:

$$W^{1,p}(\Omega, \nu_1, \nu_2) = \left\{ u \in L^p(\Omega, \nu_2) : \frac{\partial u}{\partial x_j} \in L^p(\Omega, \nu_1) \text{ for } 1 \leq j \leq n \right\},$$

with the norm

$$\|u\|_{W^{1,p}(\Omega, \nu_1, \nu_2)} = \left(\int_{\Omega} |u|^p \nu_2 dx + \sum_{j=1}^N \int_{\Omega} \left| \frac{\partial u}{\partial x_j} \right|^p \nu_1 dx \right)^{\frac{1}{p}}.$$

The space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ is defined as the closure of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{W^{1,p}(\Omega, \nu_1, \nu_2)}$. $W^{1,p}(\Omega, \nu_1, \nu_2)$ and $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ are Banach spaces.

Remark 2.9.

Note that if $\nu_2 \leq \nu_1$, then

$$W_0^{1,p}(\Omega, \nu_1) \subset W_0^{1,p}(\Omega, \nu_1, \nu_2) \subset W_0^{1,p}(\Omega, \nu_2).$$

For more details about the weighted Sobolev space $W^{1,p}(\Omega, \nu)$ with $\nu \in \mathcal{A}_p$, we refer to [13, 17, 20].

Theorem 2.10. [11]

Let ν be an \mathcal{A}_p -weight, $1 < p < \infty$. Then there exist $\theta_1, \phi_1 > 0$ such that for all $u \in W_0^{1,p}(\Omega, \nu)$ and $1 \leq \eta_1 \leq \frac{N}{N-1} + \phi_1$, we have

$$\|u\|_{L^{\eta_1}(\Omega, \nu)} \leq \theta_1 \|\nabla u\|_{L^p(\Omega, \nu)}. \tag{2.2}$$

Using Theorem 2.2 in [10], we give the following theorem related to a compactness result which we will be needed later.

Theorem 2.11. (Compact embedding for \mathcal{A}_p - weights)
 For any $\nu \in \mathcal{A}_p$ and $1 < p < \infty$, the following injection is compact

$$W^{1,p}(\Omega, \nu) \hookrightarrow L^p(\Omega, \nu).$$

Definition 2.12. The dual space of $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ is the space $[W_0^{1,p}(\Omega, \nu_1, \nu_2)]^* = W^{-1,p'}(\Omega, \nu_1, \nu_2)$, defined by

$$[W_0^{1,p}(\Omega, \nu_1, \nu_2)]^* = \left\{ f = f_0 - \operatorname{div} F, F = (f_1, \dots, f_N), \frac{f_0}{\nu_2} \in L^{p'}(\Omega, \nu_2), \frac{f_j}{\nu_1} \in L^{p'}(\Omega, \nu_1), j = 1, \dots, N \right\}.$$

Remark 2.13. If $f \in [W_0^{1,p}(\Omega, \nu_1, \nu_2)]^*$ and $g \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$, we denote

$$\langle f, g \rangle = \int_{\Omega} f_0 g dx + \sum_{j=1}^N \int_{\Omega} f_j \frac{\partial g}{\partial x_j} dx,$$

$$\|f\|_* = \left\| \frac{f_0}{\nu_2} \right\|_{L^{p'}(\Omega, \nu_2)} + \sum_{j=1}^N \int_{\Omega} \left\| \frac{f_j}{\nu_1} \right\|_{L^{p'}(\Omega, \nu_1)} dx,$$

$$|\langle f, g \rangle| \leq \|f\|_* \|g\|_{W_0^{1,p}(\Omega, \nu_1, \nu_2)}.$$

Let us now recall some crucial results on the convergence in the spaces $L^p(\Omega, \nu_1)$ and $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ whenever $\nu_1, \nu_2 \in \mathcal{A}_p$.

Theorem 2.14. [15] Let Ω be a bounded open set in \mathbb{R}^N and if $u_n \rightarrow u$ in $L^p(\Omega, \nu)$, then there exist a subsequence (u_{n_k}) and a function $\Phi \in L^p(\Omega, \nu)$ ($1 < p < \infty$) such that

- i) $u_{n_k}(x) \rightarrow u(x)$ as $n_k \rightarrow \infty$ μ -a.e. on Ω ;
- ii) $|u_{n_k}(x)| \leq \Phi(x)$ μ -a.e. on Ω .

Definition 2.15. For $k > 0, s \in \mathbb{R}$, define the truncation function T_k as follows

$$T_k(s) = \max(-k, \min(k, s)).$$

Lemma 2.16. [3] Let $u_n \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$ be such that $(u_n)_n$ converges weakly to u in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$, then $T_k(u_n)$ converges weakly to $T_k(u)$ in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

Lemma 2.17. [2] Let $g \in L^r(\Omega, \nu)$ and $g_n \in L^r(\Omega, \nu)$, where ν is a weight function on Ω and (g_n) satisfies $\|g_n\|_{r,\nu} \leq c$ ($0 < r < \infty$). If $g_n(x) \rightarrow g(x)$ a.e. in Ω , then $g_n(x) \rightharpoonup g$ weakly in $L^r(\Omega, \nu)$.

Lemma 2.18. [2] For every $u \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$, we have $T_k(u) \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$ and $T_k(u)$ converges strongly to u in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

3 Main result

3.1 Basic Assumptions

Let Ω be a bounded smooth domain in \mathbb{R}^N , $1 < p < \infty$ and ν_1, ν_2 two \mathcal{A}_p - weights. Let ϕ be a measurable function with values on $\overline{\mathbb{R}}$ such that

$$\phi^+ \in W_0^{1,p}(\Omega, \nu_1, \nu_2) \cap L^\infty(\Omega), \tag{3.1}$$

define the convex and non empty set:

$$K_\phi = \left\{ v \in W_0^{1,p}(\Omega, \nu_1, \nu_2), v \geq \phi \text{ a.e. in } \Omega \right\}.$$

Note that by (3.1), we have $K_\phi \neq \emptyset$.

To focus on the existence of a solution for the nonlinear degenerate elliptic unilateral problem (1.1), we present our basic assumptions as follows

Assumptions (A1)

We assume that $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory-vector function satisfying for a.e. $x \in \Omega$, for all $\tau \in \mathbb{R}$ and ζ, η in \mathbb{R}^N with $\zeta \neq \eta$, the conditions:

$$| a(x, \tau, \zeta) | \leq \beta (h(x) + |\tau|^{p-1} + |\zeta|^{p-1}), \tag{3.2}$$

$$[a(x, \tau, \zeta) - a(x, \tau, \eta)](\zeta - \eta) > 0, \text{ (Strict monoticity)} \tag{3.3}$$

$$a(x, \tau, \zeta) \cdot \zeta \geq \alpha |\zeta|^p, \text{ (Coercivity)} \tag{3.4}$$

where $h(x) \geq 0, h \in L^{p'}(\Omega, \nu_1)$ and $\alpha > 0, \beta > 0$ are constants.

Assumptions (A2)

Θ is a Carathéodory function satisfying :

$$\Theta(x, \tau, \zeta)\tau \geq 0, \tag{3.5}$$

$$| \Theta(x, \tau, \zeta) | \leq d(|\tau|)(h_1(x) + |\zeta|^p), \tag{3.6}$$

$$| \Theta(x, \tau, \zeta) | \geq \rho_2 \frac{\nu_1}{\nu_2} |\zeta|^p, \text{ for } |\tau| > \rho_1, \tag{3.7}$$

where $d : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous increasing function, $h_1 \in L^1(\Omega, \nu_2)$ ($h_1 \geq 0$), ρ_1, ρ_2 are positive constants.

Finally, the data f is such that

$$f \in L^1(\Omega). \tag{3.8}$$

3.2 Existence result

Before giving our existence result, we propose the following Lemma, whose proof is analogous to Lemma 5 in [7].

Lemma 3.1. *Assume that (3.2)-(3.4) are satisfied. Let ν_1, ν_2 be two \mathcal{A}_p - weights ($1 < p < \infty$), and let (u_n) be a sequence of $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ such that*

i) $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$ and μ_1 -a.e. in Ω ,

ii) $\int_{\Omega} \left(\nu_1(a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)) \right) \nabla(u_n - u) dx \rightarrow 0$ with $n \rightarrow \infty$.

Then,

$$u_n \rightarrow u \text{ strongly in } W_0^{1,p}(\Omega, \nu_1, \nu_2).$$

We are now in position to give our main results.

Theorem 3.2. *In addition to $(A_1) - (A_2)$, suppose that (3.1) is verified. If $f \in L^1(\Omega)$ and $\nu_1, \nu_2 \in \mathcal{A}_p$ are such that $\nu_2 \leq \nu_1$, then the problem (1.1) admits at least one solution.*

3.3 Proof of Theorem 3.2

The proof of our principal result will be given in several steps.

Step 1: An Approximate problem.

Let us consider the sequence of the approximation problem

$$\begin{cases} u_n \in K_\phi, \Theta(x, u_n, \nabla u_n) \in L^1(\Omega, \nu_2), \\ \langle A(u_n), T_k(v - u_n) \rangle + \int_\Omega \nu_2 \Theta(x, u_n, \nabla u_n) T_k(v - u_n) \, dx \geq \langle f_n, T_k(v - u_n) \rangle \\ \text{for all } v \in K_\phi \cap L^\infty(\Omega) \text{ and all } k > 0, \end{cases} \quad (3.9)$$

where (f_n) is a sequence of $C_0^\infty(\Omega)$ which converges strongly to $f \in L^1(\Omega)$ such that

$$\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}.$$

Thanks to Theorem 3.1 in [3], we can deduce that the previous problem (3.9) has at least one solution $u_n \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

Step 2: A priori estimates

Let $v = \phi^+$ be the test function in (3.9), then we get

$$\langle A(u_n), T_k(\phi^+ - u_n) \rangle + \int_\Omega \nu_2 \Theta(x, u_n, \nabla u_n) T_k(\phi^+ - u_n) \, dx \geq \int_\Omega f_n T_k(\phi^+ - u_n) \, dx. \quad (3.10)$$

Since $u_n - \phi^+$ and u_n have the same sign, by the sign condition (3.5) and since

$$\|f_n\|_{L^1(\Omega)} \leq c_1, \text{ we have}$$

$$\int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla(u_n - \phi^+) \, dx \leq \int_\Omega f_n T_k(u_n - \phi^+) \, dx \leq kc_1, \quad (3.11)$$

thus,

$$\int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla u_n \, dx \leq kc_1 + \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 |a(x, u_n, \nabla u_n)| |\nabla \phi^+| \, dx.$$

Using Young’s inequality, we get

$$\begin{aligned} & \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla u_n \, dx \\ & \leq kc_1 + \frac{\gamma^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 |a(x, u_n, \nabla u_n)|^{p'} \, dx + \frac{1}{\gamma^p} \frac{1}{p} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla \phi^+|^p \nu_1 \, dx, \end{aligned}$$

where γ is a positive constant.

From (3.2), we have

$$\begin{aligned} & \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla u_n \, dx \\ & \leq kc_1 + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} (h^{p'}(x) + |u_n|^p + |\nabla u_n|^p) \nu_1 \, dx + \frac{1}{p} \frac{1}{\gamma^p} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla \phi^+|^p \nu_1 \, dx \\ & \leq c_2 + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |u_n|^p \nu_1 \, dx + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 \, dx \\ & \leq c_2 + \frac{\gamma^{p'} \beta^{p'}}{p'} (\|\phi^+\|_\infty + k)^p \int_\Omega \nu_1 \, dx + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 \, dx \\ & \leq c_3 + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 \, dx. \end{aligned}$$

By (3.4), we have

$$\alpha \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 \, dx \leq c_3 + \frac{\gamma^{p'} \beta^{p'}}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 \, dx.$$

Choosing $0 < \gamma < \frac{(\alpha p')^{1/p'}}{\beta}$, this implies that

$$\int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 dx \leq c_4. \tag{3.12}$$

Moreover, by (3.9) and since $a(x, \tau, \zeta)\zeta \geq 0$, we obtain

$$\begin{aligned} \int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(u_n - \phi^+) dx &\leq k \int_{\Omega} |f_n| dx - \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla(u_n - \phi^+) dx \\ &\leq kc_1 + \int_{\{|u_n - \phi^+| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) |\nabla \phi^+| dx. \end{aligned}$$

Some reasoning as in the proof of (3.12) can be applied to get that

$$\int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(u_n - \phi^+) dx \leq c_3 + \gamma p' \frac{\beta p'}{p'} \int_{\{|u_n - \phi^+| \leq k\}} |\nabla u_n|^p \nu_1 dx.$$

When $|u_n - \phi^+| > k$, due to the fact that $T_k(u_n - \phi^+)$, $u_n - \phi^+$, u_n and $\Theta(x, u_n, \nabla u_n)$ have the same sign, we obtain

$$\int_{\{|u_n - \phi^+| \leq k\}} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(u_n - \phi^+) dx + \int_{\{|u_n - \phi^+| > k\}} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(u_n - \phi^+) dx \leq c_3,$$

thus,

$$k \int_{\{|u_n - \phi^+| > k\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \leq c_5.$$

Using (3.7), we get

$$|\Theta(x, u_n, \nabla u_n)| \geq \rho_2 \frac{\nu_1}{\nu_2} |\nabla u_n|^p \text{ for } |u_n| > \rho_1.$$

Taking $k > \rho_1 + \|\phi^+\|_{\infty}$, then $|u_n - \phi^+| > k$ implies $|u_n| > \rho_1$, which gives that

$$\int_{\{|u_n - \phi^+| > k\}} \nu_1 |\nabla u_n|^p dx \leq c_6. \tag{3.13}$$

By applying Theorem 2.10 ($\eta_1 = 1$) with $\nu_2 \leq \nu_1$, we have

$$\begin{aligned} \|u_n\|_{W_0^{1,p}(\Omega, \nu_1, \nu_2)}^p &= \int_{\Omega} |u_n|^p \nu_2 dx + \int_{\Omega} |\nabla u_n|^p \nu_1 dx \\ &\leq (\theta_1 + 1) \int_{\Omega} |\nabla u_n|^p \nu_1 dx. \end{aligned}$$

Finally, using (3.12) and (3.13), we conclude that u_n is bounded in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$, i.e

$$\|u_n\|_{W_0^{1,p}(\Omega, \nu_1, \nu_2)} \leq c_7, \tag{3.14}$$

here $c_i, 1 \leq i \leq 7$ are various positive constants independents of n .

Consequently, there exist a subsequence (still denoted by (u_n)) and $u \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$ such that

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega, \nu_1, \nu_2), \tag{3.15}$$

$$u_n \rightarrow u \text{ } \mu_2 - a.e. \tag{3.16}$$

Moreover, by Proposition 2.3 and Proposition 2.5 we deduce that $u_n \rightarrow u$ μ_1 -a.e. and a.e.

Step 3: We will show that $\nabla T_k(u_n) \rightarrow \nabla T_k(u)$ strongly in $(L^p(\Omega, \nu_1))^N$ which implies that

$\nabla u_n \rightarrow \nabla u$ a.e. in Ω .

Let $k \geq \|\phi^+\|_\infty$ and $\xi = (\frac{d(k)}{2\alpha})^2$. Let $\rho(\tau) = \tau e^{\xi\tau^2}$, $\theta_n = T_k(u_n) - T_k(u)$, $\lambda = e^{-4\xi k^2}$. By the choice of k , the function $v_n = u_n - \lambda\rho(\theta_n)$ is admissible for (3.9) and we obtain for $h > 0$,

$$\langle A(u_n), T_h(\lambda\rho(\theta_n)) \rangle + \int_\Omega \nu_2 \Theta(x, u_n, \nabla u_n) T_h(\lambda\rho(\theta_n)) dx \leq \int_\Omega f_n T_h(\lambda\rho(\theta_n)) dx. \tag{3.17}$$

Taking $h > 2k$, we get

$$|\lambda\rho(\theta_n)| \leq |\theta_n| \leq 2k < h,$$

and

$$\langle A(u_n), \rho(\theta_n) \rangle + \int_\Omega \nu_2 \Theta(x, u_n, \nabla u_n) \rho(\theta_n) dx \leq \int_\Omega f_n \rho(\theta_n) dx.$$

As $n \rightarrow \infty$, we have $f_n \rightarrow f$ strongly in $L^1(\Omega)$ and $\rho(\theta_n) \rightharpoonup 0$ weakly in $L^\infty(\Omega)$. Then, we get

$$\int_\Omega f_n \rho(\theta_n) dx \rightarrow 0.$$

As $\Theta(x, u_n, \nabla u_n) \rho(\theta_n) \geq 0$ on the subset $\{|u_n(x)| > k\}$, we deduce that

$$\langle A(u_n), \rho(\theta_n) \rangle + \int_{\{|u_n| \leq k\}} \nu_2 \Theta(x, u_n, \nabla u_n) \rho(\theta_n) dx \leq \varepsilon(n), \tag{3.18}$$

here $\varepsilon(n)$ is a real number that converges to 0 as n tends to $+\infty$.

Furthermore,

$$\begin{aligned} \langle A(u_n), \rho(\theta_n) \rangle &= \int_{\{|u_n| \leq k\}} \nu_1 a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx \\ &\quad + \int_{\{|u_n| > k\}} \nu_1 a(x, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx \\ &= \int_\Omega \nu_1 a(x, u_n, \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx \\ &\quad - \int_{\{|u_n| > k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla T_k(u) \rho'(\theta_n) dx \\ &= \int_\Omega \nu_1 (a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx \\ &\quad + \int_\Omega \nu_1 a(x, u_n, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx \\ &\quad - \int_{\{|u_n| > k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla T_k(u) \rho'(\theta_n) dx. \end{aligned}$$

From (3.2) and (3.14), we deduce that $(a(x, u_n, \nabla u_n) \rho'(\theta_n))_n$ is bounded in $(L^p(\Omega, \nu_1))^N$.

We have $\nabla T_k(u) \chi_{\{|u_n| > k\}} \rightarrow 0$ strongly in $(L^p(\Omega, \nu_1))^N$.

Consequently,

$$- \int_{\{|u_n| > k\}} \nu_1 a(x, u_n, \nabla u_n) \nabla T_k(u) \rho'(\theta_n) dx = \varepsilon(n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

By Lemma 2.16 and since $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$, we have

$T_k(u_n) \rightharpoonup T_k(u)$ in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

Then

$$\nabla T_k(u_n) \rightharpoonup \nabla T_k(u) \text{ weakly in } (L^p(\Omega, \nu_1))^N.$$

Since the sequence $(a(x, u_n, \nabla T_k(u)) \rho'(\theta_n))_n$ converges strongly in $(L^p(\Omega, \nu_1))^N$, we obtain

$$\int_\Omega \nu_1 a(x, u_n, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) dx = \varepsilon(n) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and it follows that

$$\langle A(u_n), \rho(\theta_n) \rangle = \int_{\Omega} \nu_1 \left(a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u)) \right) \left((\nabla T_k(u_n) - \nabla T_k(u)) \rho'(\theta_n) \right) dx + \varepsilon(n). \tag{3.19}$$

On the other hand,

$$\begin{aligned} & \left| \int_{\{|u_n| \leq k\}} \nu_2 \Theta(x, u_n, \nabla u_n) \rho(\theta_n) dx \right| \\ & \leq \int_{\{|u_n| \leq k\}} \nu_2 d(k) (h_1(x) + |\nabla u_n|^p) |\rho(\theta_n)| dx \\ & \leq \int_{\{|u_n| \leq k\}} \nu_2 d(k) h_1(x) |\rho(\theta_n)| dx + \int_{\{|u_n| \leq k\}} \nu_2 |\nabla u_n|^p d(k) |\rho(\theta_n)| dx \\ & \leq \int_{\{|u_n| \leq k\}} \nu_2 h_1(x) d(k) |\rho(\theta_n)| dx + \int_{\{|u_n| \leq k\}} \nu_2 \frac{a(x, u_n, \nabla u_n)}{\alpha} \nabla u_n d(k) |\rho(\theta_n)| dx \\ & \leq \int_{\{|u_n| \leq k\}} \nu_2 h_1(x) d(k) |\rho(\theta_n)| dx \\ & \quad + \int_{\{|u_n| \leq k\}} \nu_2 (a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u))) (\nabla T_k(u_n) - \nabla T_k(u)) |\rho(\theta_n)| \frac{d(k)}{\alpha} dx \\ & \quad + \int_{\{|u_n| \leq k\}} \nu_2 a(x, u_n, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) |\rho(\theta_n)| \frac{d(k)}{\alpha} dx \\ & \quad + \int_{\{|u_n| \leq k\}} \nu_2 a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u) |\rho(\theta_n)| \frac{d(k)}{\alpha} dx. \end{aligned}$$

Since $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ in $(L^p(\Omega, \nu_1))^N$ and

$$a(x, u_n, \nabla T_k(u)) |\rho(\theta_n)| \rightarrow 0 \text{ strongly in } (L^p(\Omega, \nu_1))^N,$$

we have

$$\int_{\{|u_n| \leq k\}} \nu_2 a(x, u_n, \nabla T_k(u)) (\nabla T_k(u_n) - \nabla T_k(u)) |\rho(\theta_n)| \frac{d(k)}{\alpha} dx = \varepsilon(n).$$

We conclude from (3.2) and (3.14) that $(a(x, u_n, \nabla T_k(u_n)))_n$ converges weakly in $(L^p(\Omega, \nu_1))^N$. Using the fact that $\nabla T_k(u) |\rho(\theta_n)| \frac{d(k)}{\alpha} \rightarrow 0$ strongly in $(L^p(\Omega, \nu_1))^N$, we obtain

$$\int_{\Omega} \nu_1 a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u) |\rho(\theta_n)| \frac{d(k)}{\alpha} = \varepsilon(n).$$

Furthermore,

$$\int_{\{|u_n| \leq k\}} \nu_2 d(k) h_1(x) |\rho(\theta_n)| dx = \varepsilon(n),$$

then

$$\begin{aligned} & \left| \int_{\{|u_n| \leq k\}} \nu_2 \Theta(x, u_n, \nabla u_n) \rho(\theta_n) dx \right| \\ & \leq \int_{\{|u_n| \leq k\}} \nu_1 \left(a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u)) \right) \left(\nabla T_k(u_n) - \nabla T_k(u) \right) |\rho(\theta_n)| \frac{d(k)}{\alpha} dx \\ & \quad + \varepsilon(n). \end{aligned}$$

Using (3.18) and (3.19), we have

$$\int_{\Omega} \nu_1 \left(a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u)) \right) \left(\nabla T_k(u_n) - \nabla T_k(u) \right) \left(\rho'(\theta_n) - \frac{d(k)}{\alpha} |\rho(\theta_n)| \right) dx \leq \varepsilon(n).$$

By choosing $\xi \geq \frac{d(k)}{(2\alpha)^2}$, we can deduce that for all $\tau \in \mathbb{R}$,

$$\left(\rho'(\tau) - \frac{d(k)}{\alpha} |\rho(\tau)| \right) \geq \frac{1}{2},$$

hence,

$$\frac{1}{2} \int_{\Omega} \nu_1 \left(a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u)) \right) \left(\nabla T_k(u_n) - \nabla T_k(u) \right) \leq \varepsilon(n).$$

Therefore,

$$\int_{\Omega} \nu_1 \left(a(x, u_n, \nabla T_k(u_n)) - a(x, u_n, \nabla T_k(u)) \right) \left(\nabla T_k(u_n) - \nabla T_k(u) \right) dx \rightarrow 0$$

as $n \rightarrow \infty$.

Moreover,

$$T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega, \nu_1, \nu_2).$$

By applying Lemma 3.1, we have $T_k(u_n) \rightarrow T_k(u)$ as $n \rightarrow \infty$ strongly in $W_0^{1,p}(\Omega, \nu_1, \nu_2)$, hence

$$\nabla T_k(u_n) \rightarrow \nabla T_k(u) \text{ strongly in } (L^p(\Omega, \nu_1))^N. \tag{3.20}$$

Therefore, there exists a subsequence still denoted by $(u_n)_n$ such that

$$\nabla u_n \rightarrow \nabla u \text{ a.e. in } \Omega. \tag{3.21}$$

Step 4: $\Theta(x, u_n, \nabla u_n) \rightarrow \Theta(x, u, \nabla u)$ strongly in $L^1(\Omega, \nu_2)$.

In fact, by (3.21), we get

$$\Theta(x, u_n, \nabla u_n) \rightarrow \Theta(x, u, \nabla u) \text{ a.e. in } \Omega. \tag{3.22}$$

Let $F \subset \Omega$ and for any $m > 0$, we have

$$\begin{aligned} & \int_F \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \\ & \leq \int_{F \cap \{|u_n| \leq m\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx + \int_{F \cap \{|u_n| > m\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx. \end{aligned} \tag{3.23}$$

Combining (3.6), (3.22) and using Vitali's theorem, we get that for $\varepsilon > 0$ there exists $\rho(\varepsilon, m)$ such that $\rho(\varepsilon, m) > |F|$ and then

$$\int_{F \cap \{|u_n| \leq m\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \leq \frac{\varepsilon}{2} \quad \forall n. \tag{3.24}$$

Let $v_n = u_n - W_m(u_n)$ where for $m > 1$, W_m is given by :

$$W_m(\tau) = \begin{cases} 0, & |\tau| \leq m - 1 \\ W'_m(\tau) = 1, & m - 1 \leq |\tau| \leq m. \\ \frac{\tau}{|\tau|}, & |\tau| \geq m. \end{cases}$$

If $u_n \leq m - 1$, then $W_m(u_n) \leq 0$ and $v_n \geq u_n \geq \phi$.

If $u_n \geq m - 1$, we have $0 \leq W_m(u_n) \leq 1$ and

$$u_n - W_m(u_n) \geq u_n - 1 \geq m - 2 \geq \phi \text{ for } m \geq 2 + \|\phi\|_{\infty}.$$

Then, v_n is admissible for (3.9) and we obtain

$$\langle A(u_n), T_k(W_m(u_n)) \rangle + \int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(W_m(u_n)) dx \leq \int_{\Omega} f_n T_k(W_m(u_n)) dx.$$

By choosing $k \geq 1$, we have

$$\int_{\Omega} \nu_1 a(x, u_n, \nabla u_n) \nabla u_n W'_m(u_n) dx + \int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) W_m(u_n) dx \leq \int_{\Omega} f_n T_k(W_m(u_n)) dx,$$

$$\int_{\{m-1 \leq |u_n| \leq m\}} \nu_1 a(x, u_n, \nabla u_n) \nabla u_n dx + \int_{\{|u_n| > m-1\}} \nu_2 \Theta(x, u_n, \nabla u_n) W_m(u_n) dx \leq \int_{\{|u_n| > m-1\}} |f_n| dx,$$

Since $a(x, u_n, \nabla u_n) \nabla u_n \geq 0$, we get

$$\int_{\{|u_n| > m-1\}} \nu_2 \Theta(x, u_n, \nabla u_n) W_m(u_n) dx \leq \int_{\{|u_n| > m-1\}} |f_n| dx.$$

As $W_m(u_n)$ and u_n have the same sign, we obtain

$$\int_{\{|u_n| > m-1\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| |W_m(u_n)| dx \leq \int_{\{|u_n| > m-1\}} |f_n| dx,$$

and then

$$\int_{\{|u_n| > m\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \leq \int_{\{|u_n| > m-1\}} |f_n| dx.$$

Since $f_n \rightarrow f$ strongly in $L^1(\Omega)$ and $|\{|u_n| > m-1\}| \rightarrow 0$ uniformly in n when $m \rightarrow \infty$, there exists $m(\varepsilon) > 1$ such that

$$\int_{\{|u_n| > m-1\}} |f_n| dx \leq \frac{\varepsilon}{2} \quad \forall n.$$

Then,

$$\int_{\{|u_n| > m\}} \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \leq \frac{\varepsilon}{2} \quad \forall n. \tag{3.25}$$

As a consequence of (3.23), (3.24) and (3.25), we obtain

$$\int_F \nu_2 |\Theta(x, u_n, \nabla u_n)| dx \leq \varepsilon \quad \forall n. \tag{3.26}$$

Then,

$\Theta(x, u_n, \nabla u_n)$ is equi-integrable.

By (3.22), (3.26) and Vitali's theorem, we deduce that

$$\Theta(x, u_n, \nabla u_n) \rightarrow \Theta(x, u, \nabla u) \text{ strongly in } L^1(\Omega, \nu_2).$$

Step 5: Passing to the limit

In the following, we pass to the limit in the approximate problem (3.9).

For all $v \in K_\phi \cap L^\infty(\Omega)$ and $k > 0$, we obtain

$$\int_{\Omega} \nu_1 a(x, u_n, \nabla u_n) \nabla T_k(v - u_n) dx + \int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(v - u_n) dx \geq \int_{\Omega} f_n T_k(v - u_n) dx.$$

Using (3.2) and (3.14), we get $a(x, u_n, \nabla u_n)$ is bounded in $(L^{p'}(\Omega, \nu_1))^N$.

Moreover, by (3.21) and by using Lemma 2.17, we obtain

$$a(x, u_n, \nabla u_n) \rightharpoonup a(x, u, \nabla u) \text{ weakly in } (L^{p'}(\Omega, \nu_1))^N. \tag{3.27}$$

Let $v \in L^\infty(\Omega)$ and set $h = k + \|v\|_\infty$

$$\begin{aligned} \left| \frac{\partial T_k(v - u_n)}{\partial x_i} \right| \nu_1^{1/p} &= \left(\chi_{|v - u_n| \leq k} \left| \frac{\partial(v - u_n)}{\partial x_i} \right| \right) \nu_1^{1/p} \\ &\leq \chi_{|u_n| \leq k + \|v\|_\infty} \left| \left(\frac{\partial(v - u_n)}{\partial x_i} \right) \right| \nu_1^{1/p} \\ &\leq \left| \frac{\partial v}{\partial x_i} \right| \nu_1^{1/p} + \left| \frac{\partial T_h(u_n)}{\partial x_i} \right| \nu_1^{1/p}, \quad i = 1, \dots, N. \end{aligned}$$

Combing the Vitali’s theorem with (3.20) and (3.21), we get

$$\nabla T_k(v - u_n) \rightarrow \nabla T_k(v - u) \text{ strongly in } (L^p(\Omega, \nu_1))^N, \tag{3.28}$$

for any $v \in W_0^{1,p}(\Omega, \nu_1, \nu_2) \cap L^\infty(\Omega)$.

By (3.27) and (3.28), we can pass to the limit in the first term of (3.9).

Since $\Theta(x, u_n, \nabla u_n) \rightarrow \Theta(x, u, \nabla u)$ strongly in $L^1(\Omega, \nu_2)$ and $f_n \rightarrow f$ strongly in $L^1(\Omega)$, then we can pass to the limit in

$$\langle A(u_n), T_k(v - u_n) \rangle + \int_{\Omega} \nu_2 \Theta(x, u_n, \nabla u_n) T_k(v - u_n) dx \geq \int_{\Omega} f_n T_k(v - u_n) dx,$$

and we consider the problem

$$\langle A(u), T_k(v - u) \rangle + \int_{\Omega} \nu_2 \Theta(x, u, \nabla u) T_k(v - u) dx \geq \langle f, T_k(v - u) \rangle$$

for all $v \in K_\phi \cap L^\infty(\Omega)$ and all $k \geq 0$

$u \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$, $u \geq \phi$ a.e. in Ω

$\Theta(x, u, \nabla u) \in L^1(\Omega, \nu_2)$.

Choose $\psi = T_\gamma(v)$ as a test function, when $\gamma \geq \|\phi^+\|_\infty$ and $v \in K_\phi$, then $\psi \in K_\phi \cap L^\infty(\Omega)$ and we have

$$\langle A(u), T_k(T_\gamma(v) - u) \rangle + \int_{\Omega} \nu_2 \Theta(x, u, \nabla u) T_k(T_\gamma(v) - u) dx \geq \int_{\Omega} f T_k(T_\gamma(v) - u) dx. \tag{3.29}$$

By Lemma 2.18 and Vitali’s theorem, we obtain

$$\nabla T_k(T_\gamma(v) - u) \rightarrow \nabla T_k(v - u) \text{ strongly in } (L^p(\Omega, \nu_1))^N.$$

Finally, passing to the limit in (3.29) as γ tends to infinity, we obtain

$$\langle Au, T_k(v - u) \rangle + \int_{\Omega} \nu_2 \Theta(x, u, \nabla u) T_k(v - u) dx \geq \int_{\Omega} f T_k(v - u) dx,$$

for any $v \in W_0^{1,p}(\Omega, \nu_1, \nu_2)$, $v \geq \phi$ a.e. in Ω .

Example 3.3.

Let $\Omega = \{v_1 = (a_1, b_1) \in \mathbb{R}^2 / a_1^2 + b_1^2 < 1\}$ and we consider the weight functions $\nu_1 = (a_1^2 + b_1^2)^{-1/2}$, $\nu_2 = (a_1^2 + b_1^2)^{1/2}$ belonging to \mathcal{A}_2 -weights (in this case $p = N = 2$). Furthermore, we define the following Carathéodory functions $a : \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $\Theta : \Omega \times \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$a((a_1, b_1), t, \xi) = e^{(a_1^2 + b_1^2)} \zeta,$$

$$\Theta((a_1, b_1), t, \zeta) = \frac{1}{a_1^2 + b_1^2} \arctan(t) (\cos^2(a_1 b_1) + 1) |\zeta|^2.$$

Note that it is easy to prove that $a((a_1, b_1), t, \zeta)$ verifies the growth condition (3.2) ($\beta = e$), the monotonicity condition (3.3) and the coercivity condition (3.4) ($\alpha = 1$).

Clearly, $\Theta((a_1, b_1), t, \zeta)$ satisfies conditions (3.5) and (3.6) ($d(t) = \frac{2}{a_1^2 + b_1^2} |\arctan(t)|$) and (3.7) with $\rho_1 = \frac{\pi}{4}$ and $\rho_2 = 1$.

On the other hand, we choose

$$f(a_1, b_1) = (a_1^2 + b_1^2)^{-1/3} \cos(a_1 b_1) \in L^1(\Omega).$$

Indeed, using Theorem 3.2, we can conclude for $\phi = 1$ and $A(u) = -\text{div}(\nu_1 a(x, u, \nabla u))$ that the unilateral problem

$$\begin{cases} u \in K_\phi, \Theta(x, u, \nabla u) \in L^1(\Omega, \nu_2), \\ \langle A(u), T_k(v - u) \rangle + \int_{\Omega} \nu_2 \Theta(x, u, \nabla u) T_k(v - u) dx \geq \langle f, T_k(v - u) \rangle \\ \text{for all } v \in K_\phi \text{ and all } k > 0, \end{cases}$$

admits at least one solution in the space $W_0^{1,p}(\Omega, \nu_1, \nu_2)$.

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