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ON FURTHER GENERALIZATIONS OF KNUTH'S OLD SUMS VIA HYPERGEOMETRIC FUNCTIONS

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Abstract We present strong generalizations and variations of the classical Knuth's old sums, using hypergeometric functions. By comparing the new identities with known results, numerous elementary representations of hypergeometric functions emerge as by-products.

1 Introduction

In many areas of mathematics occur algebraic identities, e.g. identities related to Ramanujan's continued fractions [2], identities of generalized Fibonacci and Lucas numbers [3], and identities involving Gaussian Pell polynomials [1]. In the present article we study the so-called Knuth's old sums, also known as Reed Dawson's identities, see Rathie et al. [4, p.1] which are given as

$$\sum_{k=0}^{2v} \left(\frac{-1}{2}\right)^k \binom{2v}{k}\binom{2k}{k} = 2^{-2v}\binom{2v}{v} \tag{1.1}$$

and

$$\sum_{k=0}^{2v+1} \left(\frac{-1}{2}\right)^k \binom{2v+1}{k}\binom{2k}{k} = 0. \tag{1.2}$$

These equations are closely related to Riordan's identities

$$\sum_{k=0}^{2v} \left(\frac{-1}{2}\right)^k \binom{2v+1}{k+1}\binom{2k}{k} = 2^{-2v}(2v+1)\binom{2v}{v} \tag{1.3}$$

and

$$\sum_{k=0}^{2v+1} \left(\frac{-1}{2}\right)^k \binom{2v+2}{k+1}\binom{2k}{k} = 2^{-2v-1}(v+1)\binom{2v}{v}, \tag{1.4}$$

see [4, p.2].

These relations have been generalized in the aforementioned article, introducing additive parameters into the binomial coefficients [4, p.3]:

$$\sum_{k=0}^{2v} \left(\frac{-1}{2}\right)^k \binom{2v+i}{k+i}\binom{2k}{k} = \pi(2v+1)_i \frac{2^{2i} i!}{(2i)!} \sum_{r=0}^{i} \frac{2^{-r}\binom{i}{r}\left(\frac{1+i-r}{2}\right)_v}{(i-r)!\Gamma^2\left(\frac{1+r-i}{2}\right)\left(\frac{2+i-r}{2}\right)_v}, \tag{1.5}$$

$$\sum_{k=0}^{2v+1} \left(\frac{-1}{2}\right)^k \binom{2v+1+i}{k+i}\binom{2k}{k} = 2\pi(2v+2)_i \frac{2^{2i} i!}{(2i)!} \sum_{r=0}^{i} \frac{2^{-r}\binom{i}{r}\left(\frac{2+i-r}{2}\right)_v}{(i-r+1)!\Gamma^2\left(\frac{r-i}{2}\right)\left(\frac{3+i-r}{2}\right)_v}. \tag{1.6}$$

Obviously, (1.1)-(1.4) arise as special cases of the relations (1.5) and (1.6). Furthermore, Rathie and Campbell [5, p.2] presented the following generalizations of (1.1) and (1.2):

$$\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^k\binom{n+i}{k+i}\binom{2k+2i}{k}=\begin{cases}\frac{\binom{n+i}{n/2}}{2^n}, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd,}\end{cases} \tag{1.7}$$

$$\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^k\frac{1}{\binom{k+i}{k}}\binom{n}{k}\binom{2k+2i}{k}=\begin{cases}\frac{2^{-n}\binom{n}{n/2}}{\binom{\frac{n}{2}+i}{n/2}}, & \text{if } n \text{ is even,}\\ 0, & \text{if } n \text{ is odd.}\end{cases} \tag{1.8}$$

We will show that the left sides of (1.5)-(1.8) and generalizations thereof can be expressed in a very compact form, using generalized hypergeometric functions. Such a function is defined by

$${}_pF_q(a_1,\dots,a_p;b_1,\dots,b_q;z)=\sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_k\dots(b_q)_k},$$

where $(n)_k=\Gamma(n+k)/\Gamma(n)$ is the Pochhammer symbol, satisfying $(n)_k=n(n+1)\dots(n+k-1)$ when k is a positive integer.

Knuth's old sums have been intensively studied in the literature [4, 5, 6]. In particular, different proofs have been given. In the present paper we focus on the development of strong generalizations of the above-mentioned identities by means of symbolic computation. Statement 1 gives a compact and generalized representation of the sums (1.5) and (1.6) via hypergeometric functions. Elementary function representations of hypergeometric functions arising from this are presented in Statement 2. Further variations of Knuth's sums are presented in Statement 3. Finally, a common generalization of the two identities (1.7) and (1.8) developed in [5] is given in Statement 4.

2 New results

Motivated by Rathie et al. [4] we present the following generalizations of Knuth's old sums, obtained by symbolic computation, which can be performed e.g. by software like Mathematica or Maple. Note that the following equations contain two additive parameters i and j introduced into the binomial coefficients.

Statement 1:

(i)

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v+i}{k+j}\binom{2k}{k}=\binom{2v+i}{j}\,{}_2F_1\left(\frac{1}{2},-2v-i+j;j+1;2\right)$$

$$+2^{-2v-1}\binom{2v+i}{2v+1+j}\binom{4v+2}{2v+1}\,{}_3F_2\left(1,1+j-i;2v+\frac{3}{2};2v+2,2v+2+j;2\right),$$

(ii)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1+i}{k+j}\binom{2k}{k}=\binom{2v+1+i}{j}\,{}_2F_1\left(\frac{1}{2},-2v-i+j-1;j+1;2\right)$$

$$-2^{-2v-2}\binom{2v+1+i}{2v+2+j}\binom{4v+4}{2v+2}\,{}_3F_2\left(1,1+j-i;2v+\frac{5}{2};2v+3,2v+3+j;2\right),$$

(iii)

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{2k+i}{k+j}=\binom{i}{j}\,{}_3F_2\left(-2v,\frac{i}{2}+1;\frac{i+1}{2};j+1,i-j+1;2\right),$$

(iv)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1}{k}\binom{2k+i}{k+j}=\binom{i}{j}\,{}_3F_2\left(-2v-1,\frac{i}{2}+1;\frac{i+1}{2};j+1,i-j+1;2\right),$$

(v)

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v+i}{k+i}\binom{2k+j}{k+j}=\binom{2v+i}{i}\,{}_3F_2\left(-2v,\frac{j}{2}+1;\frac{j+1}{2};i+1,j+1;2\right),$$

(vi)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1+i}{k+i}\binom{2k+j}{k+j}$$
$$=\binom{2v+i+1}{i}\,{}_3F_2\left(-2v-1,\frac{j}{2}+1;\frac{j+1}{2};i+1,j+1;2\right).$$

The identities (iii)-(vi) can also be obtained by rewriting the binomial coefficients in the left-hand summand in terms of Pochhammer symbols and converting the summand into a hypergeometric term.

Let us now consider special cases of (i) and (ii). Setting $i=j$ in (i) yields

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v+i}{k+i}\binom{2k}{k}=\binom{2v+i}{i}\,{}_2F_1\left(\frac{1}{2},-2v;i+1;2\right). \tag{2.1}$$

This is a compact alternative to the identity (1.5). Setting $i=0$ in (2.1) yields

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{2k}{k}={}_2F_1\left(\frac{1}{2},-2v;1;2\right) \tag{2.2}$$

and comparing this with (5.1) yields the elementary expression

$${}_2F_1\left(\frac{1}{2},-2v;1;2\right)=2^{-2v}\binom{2v}{v} \tag{2.3}$$

of a hypergeometric function. Setting $i=j$ in item (ii) of the statement results in

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1+i}{k+i}\binom{2k}{k}=\binom{2v+1+i}{i}\,{}_2F_1\left(\frac{1}{2},-2v-1;i+1;2\right), \tag{2.4}$$

which is an alternative to the complex formula (1.6) due to Rathie et al. [4]. Setting $i=0$ in the latter identity and comparing with (1.2) yields

$${}_2F_1\left(\frac{1}{2},-2v-1;1;2\right)=0, \tag{2.5}$$

for any nonnegative integer v. Further elementary representations of hypergeometric function values are obtained, comparing the right sides of (2.1) and (2.4) for $i=1,2,3$ with the special cases (15)-(20) in [4]:

Statement 2: The following identities apply:

(i)

$${}_2F_1\left(\frac{1}{2},-2v;2;2\right)=\frac{\Gamma(\frac{1}{2}+v)}{\sqrt{\pi}v!},$$

(ii)

$$ {}_2F_1\left(\frac{1}{2}, -2v; 3; 2\right) = \frac{\Gamma(\frac{3}{2}+v)(8v+6)}{\sqrt{\pi}v!(6v^2+9v+3)}, $$

(iii)

$$ {}_2F_1\left(\frac{1}{2}, -2v; 4; 2\right) = \frac{4}{15}\frac{\Gamma(\frac{5}{2}+v)(8v+15)}{\sqrt{\pi}v!\binom{2v+3}{3}}, $$

(iv)

$$ {}_2F_1\left(\frac{1}{2}, -2v-1; 2; 2\right) = \frac{\Gamma(\frac{3}{2}+v)}{\sqrt{\pi}(v+1)!}, $$

(v)

$$ {}_2F_1\left(\frac{1}{2}, -2v-1; 3; 2\right) = \frac{8}{3}\frac{\Gamma(\frac{5}{2}+v)}{\sqrt{\pi}v!\binom{2v+3}{2}}, $$

(vi)

$$ {}_2F_1\left(\frac{1}{2}, -2v-1; 4; 2\right) = \frac{4}{15}\frac{\Gamma(\frac{5}{2}+v)(8v+15)}{\sqrt{\pi}v!\binom{2v+4}{3}}. $$

We point out that none of the elementary representations of hypergeometric functions presented in this paper could be discovered by the software. In the following we present some variations of Knuth's old sums, where the number 2 in the second binomial coefficient is substituted for other positive integer values.

Statement 3:

(i)

$$ \sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{3k}{k} = {}_3F_2\left(\frac{1}{3}, \frac{2}{3}, -2v; \frac{1}{2}, 1; \frac{27}{8}\right). $$

(ii)

$$ \sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{4k}{k} = {}_4F_3\left(\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, -2v; \frac{1}{3}, \frac{2}{3}, 1; \frac{128}{27}\right). $$

(iii)

$$ \sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{5k}{k} = {}_5F_4\left(\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, -2v; \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1; \frac{3125}{512}\right). $$

(iv)

$$ \sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{6k}{k} = {}_6F_5\left(\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, -2v; \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1; \frac{23328}{3125}\right). $$

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(v)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1}{k}\binom{3k}{k} = {}_3F_2\left(\frac{1}{3},\frac{2}{3},-2v-1;\frac{1}{2},1;\frac{27}{8}\right).$$

(vi)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1}{k}\binom{4k}{k} = {}_4F_3\left(\frac{1}{4},\frac{2}{4},\frac{3}{4},-2v-1;\frac{1}{3},\frac{2}{3},1;\frac{128}{27}\right).$$

(vii)

$$\sum_{k=0}^{2v+1}\left(\frac{-1}{2}\right)^k\binom{2v+1}{k}\binom{5k}{k} = {}_5F_4\left(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5},-2v-1;\frac{1}{4},\frac{2}{4},\frac{3}{4},1;\frac{3125}{512}\right).$$

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Obviously, the general form of (i) ... (iv) is for any integer r:

$$\sum_{k=0}^{2v}\left(\frac{-1}{2}\right)^k\binom{2v}{k}\binom{rk}{k} = {}_rF_{r-1}\left(\frac{1}{r},\frac{2}{r},\dots,\frac{r-1}{r},-2v;\frac{1}{r-1},\frac{2}{r-1},\dots,\frac{r-2}{r-1},1;\frac{r^r}{2(r-1)^{r-1}}\right).$$

The generalization of (v) ... (vii) is completely analogous.

As a common generalization of (1.7) and (1.8) we finally obtain:

Statement 4:

$$\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^k\frac{\binom{n+l}{k+l}\binom{2k+2i}{k}}{\binom{k+j}{k}} = \binom{n+l}{l}\,{}_4F_3\left(1,-n,1+i,i+\frac{1}{2};2i+1,j+1,l+1;2\right).$$

For $i=l$ and $j=0$ we get

$$\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^k\binom{n+l}{k+l}\binom{2k+2l}{k} = \binom{n+l}{l}\,{}_4F_3\left(1,-n,1+l,l+\frac{1}{2};2l+1,1,l+1;2\right)$$

$$= \binom{n+l}{l}\,{}_2F_1\left(-n,l+\frac{1}{2};2l+1;2\right). \tag{2.6}$$

Note that identical factorials in numerator and denominator of (1.9) can be cancelled. Equation (2.6) is an alternative expression to Proposition 1 in [5]. For $i=j$ and $l=0$ the statement yields

$$\sum_{k=0}^{n}\left(\frac{-1}{2}\right)^k\frac{\binom{n}{k}\binom{2k+2i}{k}}{\binom{k+i}{k}} = {}_4F_3\left(1,-n,1+i,i+\frac{1}{2};2i+1,i+1,1;2\right)$$

$$= {}_2F_1\left(-n,i+\frac{1}{2};2i+1;2\right), \tag{2.7}$$

which is an alternative to Proposition 2 in Rathie and Campbell [5]. It is interesting to observe that the respective last hypergeometric functions in (2.6) and (2.7) are identical. If we compare (2.6) with Proposition 1 in [5] we obtain the following elementary representation of a hypergeometric function:

$${}_2F_1\left(-n,l+\frac{1}{2};2l+1;2\right) = \begin{cases}\dfrac{\binom{n+l}{n/2}}{2^n\binom{n+l}{l}}, & \text{if } n \text{ is even},\\ 0, & \text{if } n \text{ is odd}.\end{cases} \tag{2.8}$$

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