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On some Nonlinear elliptic unilateral problem with boundary conditions

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Abstract In this study, we investigate a nonlinear elliptic unilateral equation, described by the following model:

$$-\sum_{i=1}^{N} \partial^i \sigma_i(z, \vartheta, \nabla\vartheta) + N(z, \vartheta, \nabla\vartheta) = \mu - \operatorname{div} \phi(\vartheta) \quad \text{in } \Omega. \tag{0.1}$$

We prove the existence of entropy solutions for (0.1) in the anisotropic Sobolev space, under the hypotheses, $\mu = f - \operatorname{div} F$ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$, The nonlinear term $N(z, s, \nabla\vartheta)$ verifies only a growth condition. .

1 Introduction and Basic Assumptions

The study of anisotropic nonlinear elliptic problems is crucial for understanding nonlinear dynamics because these problems often model complex physical phenomena where properties vary in different directions. For example, in modeling Complex Materials and Media, anisotropic nonlinear elliptic problems arise in the study of materials and media with direction-dependent properties, such as composite materials, crystals, and biological tissues. These materials exhibit different behavior depending on the direction of the applied force or field, and nonlinear elliptic equations help capture these variations. On the other hand, for nonlinear Phenomena, there are many physical systems exhibit nonlinear behavior, where the response is not directly proportional to the input. Nonlinear elliptic problems help model such systems accurately, accounting for the complexities and interactions that linear models cannot capture. Finally we can also remark that These problems are relevant in various applications, including fluid dynamics, elasticity [18],image restoration [16], electromagnetism, and thermal conduction. For example, they can model heat flow in anisotropic materials or stress distribution in anisotropic elastic bodies, both of which are important for engineering design and analysis. Overall, the study of anisotropic nonlinear elliptic problems is essential for advancing our understanding of complex systems in nonlinear dynamics and for developing accurate models in various scientific and engineering disciplines.

On the other hand, anisotropic problems are a significant area of study in partial differential equations. A wealth of research has been devoted to developing anisotropic Sobolev spaces, which serve as the appropriate framework for analysing nonlinear problems that involve a growth exponent $\vec{p}$, where $\vec{p}$-laplacian is the differential operator formulated by

$$\Delta_{\vec{p}}(\vartheta) = \sum_{i=1}^{i=N} \partial_{z_i} \left(|\partial_{z_i}\vartheta|^{p_i-2} \partial_{z_i}\vartheta \right).$$

In the context of Lebesgue-Sobolev spaces, Benilan introduced an approach to entropy solutions adapted to the Boltzmann equation in [5]. Additionally, the works cited in [2] and [6] further investigate the existence of solutions for this class, specifically when $B(t) = |t|^p)$.

In [15], L. Boccardo and his collaborators explored the problem (0.1) involving the differential operator

$$A\vartheta = -\sum_{i=1}^{N} \frac{\partial}{\partial z_i}\left(\left|\frac{\partial \vartheta}{\partial z_i}\right|^{p_i-2}\frac{\partial \vartheta}{\partial z_i}\right).$$

In [14], Bendahmane et al. demonstrated the existence of solutions to the problem (0.1) for the case where

$$A\vartheta = -\sum_{i=1}^{N} \frac{\partial}{\partial z_i}\sigma_i\left(z, \frac{\partial \vartheta}{\partial z_i}\right),$$

and where the data $f = (f_1, \dots, f_m)^\top$ is vector-valued Radon measure on Ω.

In [7], Benboubker et al. established existence results for certain anisotropic elliptic problems of the form

$$-\operatorname{div} a(z, \vartheta, \nabla\vartheta) + N(z, \vartheta, \nabla\vartheta) + |\vartheta|^{p_0(z)-2}\vartheta = f - \operatorname{div} \phi(\vartheta).$$

Recently, there has been an extensive body of mathematical research on the existence of solutions to various parabolic and elliptic problems under diverse assumptions. For further reading, we refer the reader to [4, 8, 3, 9, 10, 11, 12, 15, 13].

The primary challenge in this study arises from the absence of the classical coercivity condition on the operator $A\vartheta$ in the space $W^{1,\vec{p}}(\Omega)$. To resolve this problem, we will add a penalization term, $\frac{1}{n}|\vartheta|^{p_0-2}\vartheta$ in the approximates equation (4.2) to facilitate the proof of our desired theorem.

The overview of this article is as follows: The next section provides definitions and key properties related to anisotropic Sobolev spaces. Section 3 presents the assumptions on $\sigma_i(z, s, \zeta)$, under which the problem (0.1) has a solution. The final section is dedicated to proving our main results.

2 Preliminaries

Let Ω be a bounded open subset of $I\!R^N$ $(N \geq 2)$.
Let $p_1, \dots, p_N$ be N real constants, with $\infty > p_i > 1$ for $i = 1, \dots, N$.
We set

$$\vec{p} = (1, p_1, \dots, p_N), \qquad D^0\vartheta = \vartheta \quad \text{and} \quad D^i\vartheta = \frac{\partial \vartheta}{\partial z_i} \quad \text{for} \quad i = 1, \dots, N,$$

and we set

$$\underline{p} = \min\{p_1, p_2, \dots, p_N\} \qquad \text{and} \qquad p_0 = \max\{p_1, p_2, \dots, p_N\}.$$

We introduce the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows:

$$W^{1,\vec{p}}(\Omega) = \left\{\vartheta \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i\vartheta \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \dots, N\right\},$$

endowed with the norm

$$\|\vartheta\|_{1,\vec{p}} = \|\vartheta\|_{1,1} + \sum_{i=1}^{N} \|D^i\vartheta\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\vartheta\|_{1,\vec{p}}\right)$ is a reflexive Banach (separable) space (cf [17]).

The closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}}(\Omega)$ is symbolized by $W_0^{1,\vec{p}}(\Omega)$.

Definition 2.1. For any positive value of κ, we introduce the following truncation function $T_\kappa(\cdot) : \mathbb{R} \to \mathbb{R}$, which will be relevant later on:

$$T_\kappa(s) = \begin{cases} s & \text{if } |s| \leq \kappa \\ \kappa\frac{s}{|s|} & \text{if } |s| > \kappa \end{cases}$$

We then define the space

$$\mathcal{T}_0^{1,\vec{p}}(\Omega) := \left\{ \vartheta : \Omega \to \mathbb{R} \mid \vartheta \text{ is measurable and } T_\kappa(\vartheta) \in W_0^{1,\vec{p}}(\Omega) \text{ for all } \kappa > 0 \right\}$$

3 Key assumptions

Let Ω be a bounded open subset of $I\!R^N\ ^N (N \geq 2)$, and $\vec{p} = (p_0, p_1, \ldots, p_N)$ such that $(\infty > p_i > 1)$ for $i = 0, 1, \ldots, N$. Choosing ψ as a measurable function on Ω with values in $\overline{\mathbb{R}}$, such that

$$\psi^+ \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega).$$

Consider the convex set K_ψ defined as follows:

$$K_\psi = \left\{ v \in W_0^{1,\vec{p}}(\Omega) \mid v \geq \psi \text{ almost everywhere in } \Omega \right\}$$

We introduce a Leray-Lions operator A mapping $W_0^{1,\vec{p}}(\Omega)$ to $W^{-1,\vec{p}}(\Omega)$, which is given by

$$A\vartheta = -\sum_{i=1}^{N} D^i \sigma_i(z, \vartheta, \nabla\vartheta)$$

where $\sigma_i : \Omega \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}$ are caratheodory functions, for $i = 1, \ldots, N$, which meet the following conditions:

$$|\sigma_i(z, s, \zeta)| \leq \beta \left(\gamma_i(z) + |s|^{\underline{p}-1} + |\zeta_i|^{p_i - 1} \right) \text{ for any } i = 1, \ldots, N, \tag{3.1}$$

$$\sigma_i(z, s, \zeta)\zeta_i \geq \alpha\, |\zeta_i|^{p_i} \qquad \text{for any } i = 1, \ldots, N, \tag{3.2}$$

for all $\zeta = (\zeta_1, \ldots, \zeta_N)$ and $\zeta' = (\zeta'_1, \ldots, \zeta'_N)$, we have

$$[\sigma_i(z, s, \zeta) - \sigma_i(z, s, \zeta')]\,(\zeta_i - \zeta'_i) > 0 \quad \text{for} \quad \zeta_i \neq \zeta'_i, \tag{3.3}$$

for a.e. $z \in \Omega$ and all $(s, \zeta) \in I\!R \times \mathbb{R}^{\mathbb{N}}$, where $\gamma_i(z) \in L^{p'_i}(\Omega)$ and $p_i - 1 > q_i > 0$ for $i = 1, \ldots, N$, where $\gamma_i(z), \alpha, \beta > 0$.
As a results of (3.2) and the continuity of $\sigma_i(z, s, \cdot)$ with respect to ζ, we obtain

$$\sigma_i(z, s, 0) = 0.$$

The nonlinear term $N(z, s, \zeta)$ is a Caratheodory function satisfying

$$|N(z, s, \zeta)| \leq f_0(z) + d(|s|) \sum_{i=1}^{N} |\zeta_i|^{p_i}\,, \tag{3.4}$$

with $d(\cdot) \in L^1(\mathbb{R}^{\mathbb{N}}) \cap L^\infty(\Omega)$, and $f_0(\cdot) \in L^1(\Omega)$.
We investigate the quasilinear elliptic problem outlined below

$$\begin{cases} A\vartheta + N(z, \vartheta, \nabla\vartheta) = f - \operatorname{div} F - \operatorname{div} \phi(\vartheta) & \text{in } \Omega \\ \vartheta = 0 & \text{on } \partial\Omega, \\ \vartheta \geq \psi, & \end{cases} \tag{3.5}$$

where

$$f \in L^1(\Omega), \quad F \in \prod_{i=1}^{N} L^{p'_i}(\Omega) \quad \text{and} \quad \phi(\cdot) \in C^0\left(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}\right). \tag{3.6}$$

Lemma 3.1. *[7] Given that (3.1)-(3.3) are satisfied, let us consider $(\vartheta_n)_{n\in N} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ be a sequence such that $\vartheta_n \rightharpoonup \vartheta \in W_0^{1,\vec{p}}(\Omega)$ and*

$$\int_\Omega \left(|\vartheta_n|^{p-2}\, \vartheta_n - |\vartheta|^{p-2} u \right) (\vartheta_n - \vartheta)\, dz + \sum_{i=1}^{N} \int_\Omega \left(\sigma_i\left(z, \vartheta_n, \nabla\vartheta_n\right) - \sigma_i\left(z, \vartheta_n, \nabla\vartheta\right)\right) \left(D^i\vartheta_n - D^i\vartheta\right) dz \to 0,$$

then $\vartheta_n \to \vartheta$ *in* $W_0^{1,\vec{p}}(\Omega)$.

4 The Existence results

Before state our main result let us introduce the notion of entropy solutions for the problem (3.5).

Definition 4.1. A function ϑ(measurable) is said entropy solution of the problem (3.5) with obstacle if

$$\begin{cases} T_\kappa(\vartheta) \in K_\psi, \\ \sum_{i=1}^{N} \int_\Omega \sigma_i(z,\vartheta,\nabla\vartheta) D^i T_\kappa(\vartheta - v) dz + \int_\Omega N(z,\vartheta,\nabla\vartheta) T_\kappa(\vartheta - v) dz \\ \leq \int_\Omega f T_\kappa(\vartheta - v) dz + \sum_{i=1}^{N} \int_\Omega F_i D^i T_\kappa(\vartheta - v) dz + \sum_{i=1}^{N} \int_\Omega \phi_i(\vartheta) D^i T_\kappa(\vartheta - v) dz \end{cases} \tag{4.1}$$

for any $v \in K_\psi \cap L^\infty(\Omega)$.

The purpose of this investigation is to present the principal result outlined below

Theorem 4.2. *Assuming that the conditions* (3.1)-(3.4) *and* (3.6) *are satisfied, the problem* (3.5) *has at least one entropy solution.*

4.1 Proof of Theorem 4.2

Step 1: Penalized approximate problem.

Consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $W^{-1,p'}(\Omega) \cap L^1(\Omega)$ for which $f_n \to f$ in $L^1(\Omega)$ and $|f_n| \leq |f|$ holds (for instance, $f_n = T_n(f)$). We then consider the problem.

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega \sigma_i\left(z, T_n\left(\vartheta_n\right), \nabla\vartheta_n\right) D^i\left(\vartheta_n - v\right) dz + \frac{1}{n} \int_\Omega \left|\vartheta_n\right|^{p_0-2} \vartheta_n\left(\vartheta_n - v\right) dz \\ &+ \int_\Omega N_n\left(z, \vartheta_n, \nabla\vartheta_n\right)\left(\vartheta_n - v\right) dz = \int_\Omega f_n\left(\vartheta_n - v\right) dz \\ &+ \sum_{i=1}^{N} \int_\Omega F_i D^i\left(\vartheta_n - v\right) dz + \sum_{i=1}^{N} \int_\Omega \phi_{n,i}\left(\vartheta_n\right) D^i\left(\vartheta_n - v\right) dz, \end{aligned} \tag{4.2}$$

for any $v \in K_\psi$, where

$$\phi_{n,i}(s) = \phi_i\left(T_n(s)\right) \text{ and } N_n(z,s,\zeta) = T_n(N(z,s,\zeta)).$$

Following the approach in [1], It can be confirmed that the problem (4.2) has at least one weak solution $\vartheta_n \in W_0^{1,\vec{p}}(\Omega)$.

Step 2: A priori estimates

Assume $\kappa \geq \max\left(1, \|\psi^+\|_\infty\right)$. Define v by

$$v = \vartheta_n - \eta T_\kappa\left(\vartheta_n - \psi^+\right)$$

where $v \in W_0^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$.For η sufficiently small, we have $v \in K_\psi$. Using v as an admissible test function in (4.2), we obtain

$$\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-\psi^+\right)dz+\frac{1}{n}\int_{\Omega}\left|\vartheta_n\right|^{p_0-2}\vartheta_nT_\kappa\left(\vartheta_n-\psi^+\right)dz$$
$$+\int_{\Omega}N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)T_\kappa\left(\vartheta_n-\psi^+\right)dz=\int_{\Omega}f_nT_\kappa\left(\vartheta_n-\psi^+\right)dz \tag{4.3}$$
$$+\sum_{i=1}^{N}\int_{\Omega}F_iD^iT_\kappa\left(\vartheta_n-\psi^+\right)dz+\sum_{i=1}^{N}\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-\psi^+\right)dz.$$

Concerning $\sum_{i=1}^{N}\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-\psi^+\right)dz$ we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-\psi^+\right)dz=&\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\phi_{n,i}\left(\vartheta_n\right)D^i\vartheta_ndz\\&-\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\phi_{n,i}\left(\vartheta_n\right)D^i\psi^+dz\end{aligned} \tag{4.4}$$

Taking $\Phi_{n,i}(t)=\int_0^t\phi_{n,i}(\tau)d\tau$, then $\Phi_{n,i}(0)=0$ and $\Phi_{n,i}(\cdot)\in C^1\mathbb{R}$, applying the Green formula, we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\phi_{n,i}\left(\vartheta_n\right)D^i\vartheta_ndz&=\sum_{i=1}^{N}\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^i\vartheta_n\cdot\chi_{\{|\vartheta_n-\psi^+|\le\kappa\}}dz\\&=\sum_{i=1}^{N}\int_{\Omega}D^i\Phi_{n,i}\left(\vartheta_n\cdot\chi_{\{|\vartheta_n-\psi^+|\le\kappa\}}\right)dz\\&=\sum_{i=1}^{N}\int_{\partial\Omega}\Phi_{n,i}\left(\vartheta_n\cdot\chi_{\{|\vartheta_n-\psi^+|\le\kappa\}}\right)\cdot n_id\sigma_i=0,\end{aligned} \tag{4.5}$$

Since $\vartheta_n=0$ on $\partial\Omega$, where $n=(n_1,n_2,\ldots,n_N)$ represents the normal vector on $\partial\Omega$, it follows that

$$\begin{aligned}\sum_{i=1}^{N}\left|\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\phi_{n,i}\left(\vartheta_n\right)D^i\psi^+dz\right|&\le\sum_{i=1}^{N}\int_{\{|\vartheta_n|\le\kappa+\|\psi^+\|_\infty\}}\left|\phi_{n,i}\left(\vartheta_n\right)\right|\left|D^i\psi^+\right|dz\\&\le\sum_{i=1}^{N}\sup_{|s|\le\kappa+\|\psi^+\|_\infty}\left|\phi_i(s)\right|\int_{\Omega}\left|D^i\psi^+\right|dz\le C_2.\end{aligned} \tag{4.6}$$

Using (3.4)), we have

$$\begin{aligned}\left|\int_{\Omega}N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)T_\kappa\left(\vartheta_n-\psi^+\right)dz\right|&\le\kappa\int_{\Omega}\left|f_0(z)\right|dz\\+\sum_{i=1}^{N}\int_{\Omega}d\left(\left|\vartheta_n\right|\right)\left|D^i\vartheta_n\right|^{p_i}&\left|T_\kappa\left(\vartheta_n-\psi^+\right)\right|dz.\end{aligned} \tag{4.7}$$

According to (4.4) – (4.7), we have

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^i\vartheta_ndz+\frac{1}{n}\int_{\Omega}\left|\vartheta_n\right|^{p_0-2}\vartheta_nT_\kappa\left(\vartheta_n-\psi^+\right)dz\\&\le\kappa\int_{\Omega}\left(\left|f(z)\right|+\left|f_0(z)\right|\right)dz+\sum_{i=1}^{N}\int_{\Omega}d\left(\left|\vartheta_n\right|\right)\left|D^i\vartheta_n\right|^{p_i}\left|T_\kappa\left(\vartheta_n-\psi^+\right)\right|dz\\&+\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}F_iD^i\left(\vartheta_n-\psi^+\right)dz+\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^i\psi^+dz+C_2.\end{aligned} \tag{4.8}$$

Since $\kappa \geq \|\psi^+\|_\infty$, then $T_\kappa(\vartheta_n - \psi^+)$ and ϑ_n have the same sign on $\{|\vartheta_n - \psi^+| > \kappa\}$, therefore

$$\begin{aligned}\frac{1}{n}\int_\Omega |\vartheta_n|^{p_0-2}\vartheta_n T_\kappa(\vartheta_n - \psi^+)\,dz &= \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}} |\vartheta_n|^{p_0-2}\vartheta_n(\vartheta_n - \psi^+)\,dz \\ &\quad + \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|>\kappa\}} |\vartheta_n|^{p_0-2}\vartheta_n T_\kappa(\vartheta_n - \psi^+)\,dz \\ &\geq \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}} |\vartheta_n|^{p_0}\,dz - \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\ \kappa\}} |\vartheta_n|^{p_0-1}|\psi^+|\,dz,\end{aligned}$$

taking into account (3.2), we get

$$\begin{aligned}&\alpha\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\vartheta_n|^{p_i}\,dz + \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0}\,dz \\ &\leq \sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}\sigma_i(z,T_n(\vartheta_n),\nabla\vartheta_n)D^i\vartheta_n dz + \frac{1}{n}\int_\Omega|\vartheta_n|^{p_0-2}\vartheta_n T_\kappa(\vartheta_n-\psi^+)\,dz \\ &\quad + \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0-1}|\psi^+|\,dz \leq \kappa\int_\Omega(|f(z)|+|f_0(z)|)\,dz \\ &+\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\sigma_i(z,T_n(\vartheta_n),\nabla\vartheta_n)|\,|D^i\psi^+|\,dz + \frac{1}{n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0-1}|\psi^+|\,dz\end{aligned} \tag{4.9}$$

By employing Young's inequality, we get

$$\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0-1}|\psi^+|\,dz \leq \frac{1}{2}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0}\,dz + C_1, \tag{4.10}$$

and

$$\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}F_i(D^i\vartheta_n - D^i\psi^+)\,dz \leq \frac{\alpha}{2}\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\vartheta_n|^{p_i}\,dz + C_4. \tag{4.11}$$

Moreover, considering (3.1) and using Young's inequality, we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\sigma_i(z,T_n(\vartheta_n),\nabla\vartheta_n)|\,|D^i\psi^+|\,dz \\ &\leq \beta\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}\gamma_i(z)|D^i\psi^+|\,dz + \beta\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{\underline{p}-1}|D^i\psi^+|\,dz \\ &+\beta\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\vartheta_n|^{p_i-1}|D^i\psi^+|\,dz \\ &\leq C_2 + \varepsilon\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{\underline{p}}\,dz + \frac{\alpha}{4}\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\vartheta_n|^{p_i}\,dz \\ &+\frac{1}{\varepsilon^{\underline{p}-1}}\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\psi^+|^{\underline{p}}\,dz + C_3\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\psi^+|^{p_i}\,dz.\end{aligned} \tag{4.12}$$

By integrating the results from (4.8) through (4.12), we can deduce that

$$\begin{aligned}&\frac{\alpha}{4}\sum_{i=1}^N\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|D^i\vartheta_n|^{p_i}\,dz + \frac{1}{2n}\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{p_0}\,dz \\ &\leq \kappa C + N\varepsilon\int_{\{|\vartheta_n-\psi^+|\leq\kappa\}}|\vartheta_n|^{\underline{p}}\,dz + C_4(\varepsilon)\end{aligned} \tag{4.13}$$

Consequently, a constant $C_5(\kappa,\varepsilon)$ exists such that

$$\frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{|\vartheta_n-\psi^+|\le\kappa\}}\left|D^i\vartheta_n\right|^{p_i}dz\le C_5(\kappa,\varepsilon). \tag{4.14}$$

And since

$$\{z\in\Omega,|\vartheta_n|\le\kappa\}\subset\left\{z\in\Omega,\left|\vartheta_n-\psi^+\right|\le\kappa+\left\|\psi^+\right\|_\infty\right\}$$

Thus

$$\begin{aligned}\sum_{i=0}^{N}\int_\Omega\left|D^iT_\kappa\left(\vartheta_n\right)\right|^{p_i}dz&=\sum_{i=1}^{N}\int_{\{|\vartheta_n|\le\kappa\}}\left|D^i\vartheta_n\right|^{p_i}dz+\int_\Omega\left|T_\kappa\left(\vartheta_n\right)\right|^{p_0}dz\\&\le\sum_{i=1}^{N}\int_{\left\{|\vartheta_n-\psi^+|\le\kappa+\|\psi^+\|_\infty\right\}}\left|D^i\vartheta_n\right|^{p_i}dz+\kappa^{p_0}|\Omega|\\&\le C_6\left(\kappa,\left\|\psi^+\right\|_\infty,\varepsilon\right)\end{aligned}$$

and we have

$$\left\|T_\kappa\left(\vartheta_n\right)\right\|_{1,\underline{p}}\le C_7\left(\kappa,\left\|\psi^+\right\|_\infty,\varepsilon\right),$$

where $C_7>0$ does not depend on n. Thus, the sequence $\left(T_\kappa\left(\vartheta_n\right)\right)_n$ is bounded in $W_0^{1,\vec{p}}(\Omega)$ uniformly in n, then there exists a subsequence still denoted $\left(T_\kappa\left(w_n\right)\right)_{n\in N}$ and a function $v_\kappa\in W_0^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases}T_\kappa\left(\vartheta_n\right)\rightharpoonup v_\kappa & \text{weakly in } W_0^{1,\vec{p}}(\Omega)\\ T_\kappa\left(\vartheta_n\right)\to v_\kappa & \text{strongly in } L^{\underline{p}}(\Omega) \text{ and a.e in } \Omega.\end{cases} \tag{4.15}$$

Now, applying (4.13) along with the Poincaré inequality, we have

$$\begin{aligned}\left\|\nabla T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}&=\sum_{i=1}^{N}\int_\Omega\left|D^iT_\kappa\left(\vartheta_n\right)\right|^{\underline{p}}dz\\&\le\sum_{i=1}^{N}\int_\Omega\left|D^iT_\kappa\left(\vartheta_n\right)\right|^{p_i}dz+N|\Omega|\\&\le\frac{4\kappa}{\alpha}C+\frac{4N\varepsilon}{\alpha}\left\|T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}+C_8(\varepsilon)\\&\le\frac{4\kappa}{\alpha}C+C'\frac{4N\varepsilon}{\alpha}\left\|\nabla T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}+C_8(\varepsilon).\end{aligned}$$

Now, we choose ε small enough $\left(C'\frac{4N\varepsilon}{\alpha}\le\frac{1}{2}\right)$, there exists a constant C_9 that does not depend on κ and n, such that

$$\left\|\nabla T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}\le C_9\kappa^{\frac{1}{\underline{p}}}\quad\text{for}\quad\kappa\ge1,$$

and we reach

$$\begin{aligned}\kappa\,\text{meas}\left\{|\vartheta_n|>\kappa\right\}&=\int_{\{|\vartheta_n|>\kappa\}}\left|T_\kappa\left(\vartheta_n\right)\right|dz\le\int_\Omega\left|T_\kappa\left(\vartheta_n\right)\right|dz\\&\le\|1\|_{p'}\left\|T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}\\&\le C\left\|\nabla T_\kappa\left(\vartheta_n\right)\right\|_{\underline{p}}\\&\le C_{10}\kappa^{\frac{1}{\underline{p}}},\end{aligned} \tag{4.16}$$

which yields.

$$\text{meas}\left\{|\vartheta_n|>\kappa\right\}\le C_{13}\frac{1}{\kappa^{1-\frac{1}{\underline{p}}}}\longrightarrow0\text{ as }\kappa\to\infty. \tag{4.17}$$

Finally, we show that $(\vartheta_n)_n$ is a Cauchy sequence in measure. Specifically, for every $\delta > 0$, we have

$$\begin{aligned}&\text{meas}\,\{|\vartheta_n - \upsilon_m| > \delta\} \le \text{meas}\,\{|\vartheta_n| > \kappa\} + \text{meas}\,\{|\upsilon_m| > \kappa\}\\ &+ \text{meas}\,\{|T_\kappa(\vartheta_n) - T_\kappa(\upsilon_m)| > \delta\}\end{aligned}$$

suppose that $\varepsilon > 0$, in view of (4.17) we can take $\kappa = \kappa(\varepsilon)$ large enough such that

$$\text{meas}\,\{|\vartheta_n| > \kappa\} \le \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\,\{|\upsilon_m| > \kappa\} \le \frac{\varepsilon}{3}. \tag{4.18}$$

Now, thanks to (4.15) we obtain

$$T_\kappa(\vartheta_n) \longrightarrow \eta_\kappa \text{ in } L^{\underline{p}}(\Omega) \text{ and a.e.in } \Omega.$$

Thus $(T_\kappa(\vartheta_n))_{n\in N}$ is a Cauchy sequence in measure, and for any $\kappa > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\kappa, \delta, \varepsilon)$ such that

$$\text{meas}\,\{|T_\kappa(\vartheta_n) - T_\kappa(\upsilon_m)| > \delta\} \le \frac{\varepsilon}{3} \quad \text{for all } m, n \ge n_0(\kappa, \delta, \varepsilon). \tag{4.19}$$

By combining (4.18) and (4.19), we can get : for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$

$$\text{meas}\,\{|\vartheta_n - \upsilon_m| > \delta\} \le \varepsilon \quad \text{for any } n, m \ge n_0.$$

Then, the sequence $(\vartheta_n)_n$ is a Cauchy sequence in measure and therefore converges almost everywhere, along a subsequence, to some measurable function w. By applying (4.15), we obtain

$$\begin{cases} T_\kappa(\vartheta_n) \rightharpoonup T_\kappa(\vartheta) & \text{in } W_0^{1,\vec{p}}(\Omega) \\ T_\kappa(\vartheta_n) \to T_\kappa(\vartheta) & \text{in } L^{\underline{p}}(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{4.20}$$

Step 3: Convergence of the gradient

In the rest of this note, we denote by $\varepsilon_i(n), i = 1, 2, \ldots$ various real-valued functions of real variables that converge to 0 as n tends to infinity. For $h > \kappa > 0$, we define

$$\theta := 4\kappa + h \quad \text{and} \quad \omega_n := T_{2\kappa}(\vartheta_n - T_h(\vartheta_n) + T_\kappa(\vartheta_n) - T_\kappa(\vartheta)).$$

choosing $v = \vartheta_n - \eta\omega_n$, we have $v \ge \psi$ for η small enough, thus by taking v as an admissible test in (4.2), we get

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega \sigma_i(z, T_n(\vartheta_n), \nabla\vartheta_n) D^i\omega_n dz + \frac{1}{n}\int_\Omega |\vartheta_n|^{p_0-2}\vartheta_n\omega_n dz\\ &+ \int_\Omega N_n(z, \vartheta_n, \nabla\vartheta_n)\omega_n dz \le \int_\Omega f_n\omega_n dz + \sum_{i=1}^{N}\int_\Omega \phi_{n,i}(\vartheta_n) D^i\omega_n dz\\ &\qquad + \sum_{i=1}^{N}\int_\Omega F_i D^i\omega_n dz.\end{aligned}$$

We can easily remark that ω_n have the same sign as ϑ_n on the set $\{|\vartheta_n| > \kappa\}$ and $\omega_n = T_\kappa(\vartheta_n) - T_\kappa(\vartheta)$ on the set $\{|\vartheta_n| \le \kappa\}$, also we have $D^i\omega_n = 0$ on $\{|\vartheta_n| > \theta\}$ then,

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|\vartheta_n|\le N\}} \sigma_i(z, T_\theta(\vartheta_n), \nabla T_\theta(\vartheta_n)) D^i\omega_n dz\\ &+ \frac{1}{n}\int_{\{|\vartheta_n|\le\kappa\}} |\vartheta_n|^{p_0-2}\vartheta_n(T_\kappa(\vartheta_n) - T_\kappa(\vartheta)) dz\\ &\le \int_\Omega (|f_n(z)| + |f_0(z)|)|\omega_n| dz + \sum_{i=1}^{N}\int_\Omega d(|\vartheta_n|)\left|D^i\vartheta_n\right|^{p_i}|\omega_n| dz\\ &+ \int_{\{|\vartheta_n|\le\theta\}} \phi_{n,i}(T_\theta(\vartheta_n)) D^i\omega_n dz + \sum_{i=1}^{N}\int_{\{|\vartheta_n|\le\theta\}} F_i D^i\omega_n dz.\end{aligned} \tag{4.21}$$

On the one hand, we have

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|\vartheta_n|\le\theta\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz\\
&=\sum_{i=1}^{N}\int_{\{|\vartheta_n|\le\kappa\}}\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\\
&\quad+\sum_{i=1}^{N}\int_{\{\kappa<|\vartheta_n|\le\theta\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz\\
&=\sum_{i=1}^{N}\int_\Omega\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\\
&\qquad\times\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\\
&\quad+\sum_{i=1}^{N}\int_\Omega\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\\
&\quad+\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)D^iT_\kappa(\vartheta)dz\\
&\quad+\sum_{i=1}^{N}\int_{\{\kappa<|\vartheta_n|\le\theta\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz.
\end{aligned}
\tag{4.22}
$$

For the second term on the right-hand side of (4.22), Lebesgue's dominated convergence theorem, implies taht $T_\kappa\left(\vartheta_n\right)\to T_\kappa(\vartheta)$ in $L^{p_i}(\Omega)$, then

$$\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\to\sigma_i\left(z,T_\kappa(\vartheta),\nabla T_\kappa(\vartheta)\right)\quad\text{in}\quad L^{p_i'}(\Omega),$$

since $D^iT_\kappa\left(\vartheta_n\right)$ converges to $D^iT_\kappa(\vartheta)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\varepsilon_1(n)=\sum_{i=1}^{N}\int_\Omega\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\longrightarrow 0 \text{ as } n\to\infty. \tag{4.23}$$

Considering the third term on the right-hand side of (4.22), and using (3.2) along with the continuity of $a(z,s,\zeta)$ in ζ, we obtain $a(z,s,0)=0$. Thus,

$$\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)D^iT_\kappa(\vartheta)dz=\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),0\right)D^iT_\kappa(\vartheta)dz=0. \tag{4.24}$$

For the final term on the right-hand side of (4.22), consider $z_n=\vartheta_n-T_h\left(\vartheta_n\right)+T_\kappa\left(\vartheta_n\right)-T_\kappa(\vartheta)$. Then,

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz\\
&=\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}\cap\{|z_n|\le 2\kappa\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\left(\vartheta_n-T_h\left(\vartheta_n\right)+T_\kappa\left(\vartheta_n\right)-T_\kappa(\vartheta)\right)dz\\
&\ge-\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}}\left|\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)\right|\left|D^iT_\kappa(\vartheta)\right|dz.
\end{aligned}
$$

We have $\left(\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)\right)_{n\in N}$ is bounded in $L^{p_i'}(\Omega)$, then there exists $\vartheta\in L^{p_i'}(\Omega)$ such that $\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)\rightharpoonup\vartheta_i$ in $L^{p_i'}(\Omega)$. Therefore,

$$
\begin{aligned}
\varepsilon_2(n)&=\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^iT_\kappa(\vartheta)dz\\
&\longrightarrow\sum_{i=1}^{N}\int_{\{|\vartheta|>\kappa\}}\vartheta_iD^iT_\kappa(\vartheta)dz=0,
\end{aligned}
$$

it follows that

$$\sum_{i=1}^{N}\int_{\{|\vartheta_n|>\kappa\}}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz\geq\varepsilon_2(n). \tag{4.25}$$

According to (4.22)-(4.25), we conclude that

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\\ \leq\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz+\varepsilon_3(n).\end{aligned} \tag{4.26}$$

Regarding the second term on the left-hand side of (4.21), taking into account (4.42), we obtain

$$\begin{aligned}\varepsilon_4(n)&=\frac{1}{n}\left|\int_{\{|\vartheta_n|\leq\kappa\}}\left|\vartheta_n\right|^{p_0-2}\vartheta_n\left(T_\kappa\left(\vartheta_n\right)-T_\kappa(\vartheta)\right)dz\right|\\&\leq\frac{2\kappa}{n}\int_{\{|\vartheta_n|\leq\kappa\}}|\vartheta_n|^{p_0-1}\,dz\longrightarrow 0 \text{ as } n\to\infty.\end{aligned} \tag{4.27}$$

Thanking to (4.21) and (4.26)-(4.27), we get

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\\ \leq\int_{\Omega}\left(|f_n(z)|+|f_0(z)|\right)|\omega_n|\,dz+\sum_{i=1}^{N}\int_{\{|\vartheta_n|\leq\theta\}}\phi_{n,i}\left(T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz\\ +\sum_{i=1}^{N}\int_{\Omega}F_iD^i\omega_n dz+\varepsilon_5(n).\end{aligned} \tag{4.28}$$

We have $f_n\to f$ in $L^1(\Omega)$ and $\omega_n\to T_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)$ weak-* in $L^\infty(\Omega)$, then

$$\int_{\Omega}(f_n+f_0)\omega_n dz=\int_{\Omega}(f+f_0)T_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz+\varepsilon_6(n). \tag{4.29}$$

Now, by choosing n large enough (for example $n\geq M$), we have

$$\int_{\Omega}\phi_{n,i}\left(T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz=\int_{\Omega}\phi_i\left(T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz, \tag{4.30}$$

since $D^i\omega_n\to D^iT_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)$ in $L^{p_i}(\Omega)$, then

$$\int_{\Omega}\phi_{n,i}\left(T_\theta\left(\vartheta_n\right)\right)D^i\omega_n dz=\int_{\Omega}\phi_i\left(T_\theta(\vartheta)\right)D^iT_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz+\varepsilon_7(n), \tag{4.31}$$

and

$$\sum_{i=1}^{N}\int_{\Omega}F_iD^i\omega_n dz=\sum_{i=1}^{N}\int_{\{h<|\vartheta|\leq h+2\kappa\}}F_iD^i\vartheta dz+\varepsilon_8(n). \tag{4.32}$$

Now thanks to (4.28)-(4.32), one has

$$\begin{aligned}
&\frac{b_\kappa}{\alpha}\sum_{i=1}^{N}\int_\Omega\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\\
&\times\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)|\omega_n|\,dz\\
&\geq\varepsilon_8(n)+\int_\Omega(f_n+f_0)T_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz\\
&+\sum_{i=1}^{N}\int_\Omega\phi_i\left(T_\theta(\vartheta)\right)D^iT_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz+\sum_{i=1}^{N}\int_{\{h<|\vartheta|\leq h+2\kappa\}}F_iD^i\vartheta dz+\varepsilon_9(n).
\end{aligned}\tag{4.33}$$

On the other hand, we can write

$$\int_\Omega(f_n+f_0)T_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz\longrightarrow0\quad\text{as}\quad h\to\infty.\tag{4.34}$$

For the term $\sum_{i=1}^{N}\int_\Omega\phi_i\left(T_\theta(\vartheta)\right)D^iT_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz$

We choose $\Psi_i(t)=\int_0^t\phi_i(\tau)d\tau$ then $\Psi_i(0)=0$ and $\Psi_i\in C^1(R)$. similarly as in (4.5), we show that

$$\begin{aligned}
&\int_\Omega\phi_i\left(T_\theta(\vartheta)\right)D^iT_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz\\
&=\int_\Omega\phi_i\left(T_{h+2\kappa}(\vartheta)\right)D^iT_{h+2\kappa}(\vartheta)dz-\int_\Omega\phi_i\left(T_h(\vartheta)\right)D^iT_h(\vartheta)dz=0.
\end{aligned}\tag{4.35}$$

For the convergence of the third term on the right-hand side of (4.33). Put $v=\vartheta_n-\eta T_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)$ for η small enough, therefore we take v as an admissible test in (3.5), we can write

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^iT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz+\frac{1}{n}\int_\Omega|\vartheta_n|^{p_0-2}\vartheta_nT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz\\
&\qquad+\int_\Omega N(z,\vartheta,\nabla\vartheta)T_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz\\
&=\int_\Omega f_nT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz+\sum_{i=1}^{N}\int_\Omega F_iD^iT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz\\
&\qquad+\sum_{i=1}^{N}\int_\Omega\phi_{n,i}\left(\vartheta_n\right)D^iT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz.
\end{aligned}\tag{4.36}$$

We have

$$\int_\Omega\phi_{n,i}\left(\vartheta_n\right)D^iT_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz=0\quad\text{for } i=1,\ldots,N.$$

According to (3.2),(3.4) and Young's inequality, we get

$$\begin{aligned}
\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\leq h+2\kappa\}}\left|D^i\vartheta_n\right|^{p_i}dz&\leq\int_\Omega(f_n+f_0)T_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz\\
&\quad+C_{15}\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\leq h+2\kappa\}}|F_i|^{p_i'}dz.
\end{aligned}\tag{4.37}$$

then

$$\frac{\alpha}{2}\sum_{i=1}^{N}\int_{\{h<|\vartheta|\le h+2\kappa]}\left|D^i\vartheta\right|^{p_i}dz \le \frac{\alpha}{2}\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{[h<|\vartheta_n|\le h+2\kappa|}\left|D^i\vartheta_n\right|^{p_i}dz$$
$$\le \lim_{n\to\infty}\int_\Omega (f_n+f_0)T_{2\kappa}\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz+\lim_{n\to\infty}C_{15}\sum_{i=1}^{N}\int_{\langle h<|\vartheta_n|\le h+2\kappa\}}|F_i|^{p_i'}dz \quad (4.38)$$
$$=\int_\Omega (f_n+f_0)T_{2\kappa}\left(\vartheta-T_h(\vartheta)\right)dz+C_{15}\sum_{i=1}^{N}\int_{[h<|\vartheta|\le h+2\kappa]}|F_i|^{p_i'}dz.$$

Now, let $h\longrightarrow\infty$ in (4.38), we can have

$$\limsup_{h\to\infty}\int_{\{h<|\vartheta|\le h+2\kappa\}}\left|D^i\vartheta\right|^{p_i}dz=0.$$

Therefore

$$\lim_{h\to\infty}\int_{[h<|\vartheta|\le h+2\kappa\}}|F_i|\left|D^i\vartheta\right|dz=0. \quad (4.39)$$

According (4.34)-(4.34) and (4.39), by letting $h\longrightarrow\infty$ in (4.33), we get

$$\sum_{i=1}^{N}\int_\Omega\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz\to 0,$$

and since $T_\kappa\left(\vartheta_n\right)\to T_\kappa(\vartheta)$ in $L^{p_0}(\Omega)$, then

$$\sum_{i=1}^{N}\int_\Omega\left(\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\kappa\left(\vartheta_n\right),\nabla T_\kappa(\vartheta)\right)\right)\left(D^iT_\kappa\left(\vartheta_n\right)-D^iT_\kappa(\vartheta)\right)dz$$
$$+\int_\Omega\left(\left|T_\kappa\left(\vartheta_n\right)\right|^{p_0-2}T_\kappa\left(\vartheta_n\right)-\left|T_\kappa(\vartheta)\right|^{p_0-2}T_\kappa(\vartheta)\right)\left(T_\kappa\left(\vartheta_n\right)-T_\kappa(\vartheta)\right)dz\longrightarrow 0 \text{ as } n\to\infty. \quad (4.40)$$

In view of Lemma 3.1 and (4.10), we conclude that

$$\begin{cases} T_\kappa\left(\vartheta_n\right)\to T_\kappa(\vartheta) & \text{strongly in } W_0^{1,\vec{p}}(\Omega),\\ D^i\vartheta_n\to D^i\vartheta & \text{a.e. in } \Omega \text{ for } i=1,\ldots,N.\end{cases} \quad (4.41)$$

Step 4: The equi-integrability

i) The equi-integrability of $\left(\frac{1}{n}\left|\vartheta_n\right|^{p_0-2}\vartheta_n\right)_n$.

We demonstrate that

$$\frac{1}{n}\left|\vartheta_n\right|^{p_0-2}\vartheta_n\longrightarrow 0 \text{ strongly in } L^1(\Omega). \quad (4.42)$$

By applying Vitali's Theorem, we demonstrate that the sequence $\left(\frac{1}{n}\left|\vartheta_n\right|^{p_0-2}\vartheta_n\right)_n$ is uniformly equiintegrable. Specifically, for η sufficiently small , we consider $v=\vartheta_n-\eta T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)$ as a test function in (4.2). Taking into account to (3.2)and applying Young's inequality, we can write

$$\alpha\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\le h+1\}}\left|D^i\vartheta_n\right|^{p_i}dz+\frac{1}{n}\int_{\{h<|\vartheta_n|\}}\left|\vartheta_n\right|^{p_0-2}\vartheta_nT_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz$$
$$\le\int_{\{h<|\vartheta_n|\}}|f_n|\,dz+\int_{\{h<|\vartheta_n|\}}|f_0(z)|\,dz,$$

Therefore

$$\frac{1}{n}\int_{\{h+1\leq|\vartheta_n|\}}|\vartheta_n|^{p_0-1}\,dz\leq\frac{1}{n}\int_{\{h<|\vartheta_n|\}}|\vartheta_n|^{p_0-2}\,\vartheta_n T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)dz$$
$$\leq\int_{\{h<|\vartheta_n|\}}|f_n|\,dz+\int_{\{h<|\vartheta_n|\}}|f_0(z)|\,dz,$$

Let $\eta>0$ be fixed, then there exists $h(\eta)\geq 1$ such that

$$\frac{1}{n}\int_{\{h(\eta)<|\vartheta_n|\}}|\vartheta_n|^{p_0-1}\,dz\leq\frac{\eta}{2}, \tag{4.43}$$

For any measurable subset $E\subset\Omega$, we obtain

$$\frac{1}{n}\int_E|\vartheta_n|^{p_0-1}\,dz\leq\frac{1}{n}\int_E\left|T_{h(\eta)}\left(\vartheta_n\right)\right|^{p_0-1}dz+\frac{1}{n}\int_{\{h(\eta)<|\vartheta_n|\}}|\vartheta_n|^{p_0-1}\,dz, \tag{4.44}$$

So, there exists $\beta(\eta)>0$ such that, for all $E\subseteq\Omega$ with meas $(E)\leq\beta(\eta)$, we have

$$\frac{1}{n}\int_E\left|T_{h(\eta)}\left(\vartheta_n\right)\right|^{p_0-1}dz\leq\frac{\eta}{2}. \tag{4.45}$$

Finally, by tanking to (4.43),(4.44) and (4.45), we have

$$\frac{1}{n}\int_E|\vartheta_n|^{p_0-1}\,dz\leq\eta \text{ for all } E \text{ such that } meas(E)\leq\beta(\eta). \tag{4.46}$$

which implies that $\left(\frac{1}{n}|\vartheta_n|^{p_0-2}\,\vartheta_n\right)_n$ is uniformly equi-integrable, then since

$$\frac{1}{n}|\vartheta_n|^{p_0-2}\,\vartheta_n\longrightarrow 0 \text{ a.e in } \Omega.$$

By applying Vitali's Theorem, the convergence (4.42) is concluded.

ii) The equi-integrability of $\left(N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)\right)_n$

To transition to the limit in the approximate problem, we need to prove that

$$N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)\to N(z,\vartheta,\nabla\vartheta) \text{ in } L^1(\Omega). \tag{4.47}$$

By applying Vitali's theorem, it is enough to show that $\left(N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)\right)_n$ is uniformly equi-integrable. We define

$$\bar{D}(s)=\frac{2}{\alpha}\int_0^s d(|\tau|)d\tau.$$

By taking $T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)e^{\bar{D}(|\vartheta_n|)}$ as a test function in (4.2), we have

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega\sigma_i\left(z,\vartheta_n,\nabla\vartheta_n\right)D^iT_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)e^{\bar{D}(|\vartheta_n|)}dz\\
&+\frac{2}{\alpha}\sum_{i=1}^N\int_\Omega\sigma_i\left(z,\vartheta_n,\nabla\vartheta_n\right)D^i\vartheta_n d\left(|\vartheta_n|\right)\left|T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)\right|e^{\bar{D}(|\vartheta_n|)}dz\\
&+\int_\Omega N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)e^{\bar{D}(|\vartheta_n|)}dz\\
&=\int_\Omega f_nT_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)e^{\bar{D}(|\vartheta_n|)}dz.
\end{aligned}$$

According to (3.2) and (3.4), we obtain

$$\begin{aligned}\alpha\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\le h+1\}}\sigma_i\left(z,\vartheta_n,\nabla\vartheta_n\right)D^i\vartheta_n e^{\bar{D}(|\vartheta_n|)}dz\\+2\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\}}\left|D^i\vartheta_n\right|^{p_i}d\left(|\vartheta_n|\right)\left|T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)\right|e^{\bar{D}(|\vartheta_n|)}dz\\\le\int_{\{h<|\vartheta_n|\}}\left(|f_n|+|f_0|\right)e^{\bar{D}(|\vartheta_n|)}dz\\+\sum_{i=1}^{N}\int_{\{h<|\vartheta_n|\}}\left|D^i\vartheta_n\right|^{p_i(z)}\left|T_1\left(\vartheta_n-T_h\left(\vartheta_n\right)\right)\right|d\left(|\vartheta_n|\right)e^{\bar{D}(|\vartheta_n|)}dz.\end{aligned}\tag{4.48}$$

it follows that

$$\sum_{i=1}^{N}\int_{\{h+1<|\vartheta_n|\}}d\left(|\vartheta_n|\right)\left|D^i\vartheta_n\right|^{p_i}dz\le e^{\bar{D}(\infty)}\int_{\{h<|\vartheta_n|\}}\left(|f|+|f_0|\right)dz.$$

Thus, for all $\eta>0$, there exists $h(\eta)\ge 1$ such that

$$\sum_{i=1}^{N}\int_{\{h(\eta)<|\vartheta_n|\}}d\left(|\vartheta_n|\right)\left|D^i\vartheta_n\right|^{p_i}dz\le\frac{\eta}{2}.\tag{4.49}$$

On the other hand, we set

$$d_{h(\eta)}:=\max\{d(s):|s|\le h(\eta)\},$$

for any measurable subset $E\subseteq\Omega$, we have

$$\begin{aligned}\sum_{i=1}^{N}\int_{E}d\left(|\vartheta_n|\right)\left|D^i\vartheta_n\right|^{p_i}dz\le d_{h(\eta)}\sum_{i=1}^{N}\int_{E}\left|D^iT_{h(\eta)}\left(\vartheta_n\right)\right|^{p_i}dz\\+\sum_{i=1}^{N}\int_{\{h(\eta)<|\vartheta_n|\}}d\left(|\vartheta_n|\right)\left|D^i\vartheta_n\right|^{p_i}dz.\end{aligned}\tag{4.50}$$

From (4.41), there exists $\tau(\eta)>0$ such that, for any meas $(E)\le\tau(\eta)$ we have

$$d_{h(\eta)}\sum_{i=1}^{N}\int_{E}\left|D^iT_{h(\eta)}\left(\vartheta_n\right)\right|^{p_i}dz\le\frac{\eta}{2}.\tag{4.51}$$

Based on (4.49), (4.50), and (4.51), we obtain

$$\sum_{i=1}^{N}\int_{E}d\left(|\vartheta_n|\right)\left|D^i\vartheta_n\right|^{p_i}dz\le\eta\quad\text{for all}\quad\text{meas}(E)\le\tau(\eta).\tag{4.52}$$

Thanks to (3.4), we deduce that $\left(N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)\right)_n$ is uniformly equi-integrable, and since

$$N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)\to N(z,\vartheta,\nabla\vartheta)\text{ a.e. in }\Omega.$$

By applying Vitali's theorem, we obtain the convergence result stated in (4.47).

Step 5: Passing to the limit

Let $v\in K_\psi\cap L^\infty(\Omega)$, by choosing $\vartheta_n-\eta T_\kappa\left(\vartheta_n-v\right)$ as a test in (4.2), with η smal enough, we obtain

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-v\right)dz+\frac{1}{n}\int_{\Omega}|\vartheta_n|^{p_0-2}\vartheta_nT_\kappa\left(\vartheta_n-v\right)dz\\\le\int_{\Omega}f_nT_\kappa\left(\vartheta_n-v\right)dz+\int_{\Omega}N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)T_\kappa\left(\vartheta_n-v\right)dz\\+\sum_{i=1}^{N}\int_{\Omega}F_iD^iT_\kappa\left(\vartheta_n-v\right)dz+\sum_{i=1}^{N}\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-v\right)dz.\end{aligned}\tag{4.53}$$

Let $\theta=\kappa+\|v\|_{\infty}$ then $\{|\vartheta_n-v|\leq\kappa\}\subseteq\{|\vartheta_n|\leq\theta\}$, and we obtain

$$\begin{aligned}
&\int_{\Omega}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-v\right)dz\\
&=\int_{\Omega}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)\left(D^iT_\theta\left(\vartheta_n\right)-D^iv\right)\chi_{\{|\vartheta_n-v|\leq\kappa\}}dz\\
&=\int_{\Omega}\left(\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla T_\theta\left(\vartheta_n\right)\right)-\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla v\right)\right)\left(D^iT_\theta\left(\vartheta_n\right)-D^iv\right)\chi_{\{|\vartheta_n-v|\leq\kappa\}}dz\\
&\quad+\int_{\Omega}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla v\right)\left(D^iT_\theta\left(\vartheta_n\right)-D^iv\right)\chi_{\{|\vartheta_n-v|\leq\kappa\}}dz.
\end{aligned}$$

It's clear that

$$\begin{aligned}
\lim_{n\to\infty}&\int_{\Omega}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla v\right)\left(D^iT_\theta\left(\vartheta_n\right)-D^iv\right)\chi_{\{|\vartheta_n-v|\leq\kappa\}}dz\\
&=\int_{\Omega}\sigma_i\left(z,T_\theta(\vartheta),\nabla v\right)\left(D^iT_\theta(\vartheta)-D^iv\right)\chi\{|\vartheta-v|\leq\kappa\}dz.
\end{aligned}$$

By applying the Fatou's Lemma, we get

$$\begin{aligned}
&\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_n\left(\vartheta_n\right),\nabla\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-v\right)dz\\
&\geq\sum_{i=1}^{N}\int_{\Omega}\left(\sigma_i\left(z,T_\theta(\vartheta),\nabla T_\theta(\vartheta)\right)-\sigma_i\left(z,T_\theta(\vartheta),\nabla v\right)\right)\left(D^iT_\theta(\vartheta)-D^iv\right)\chi_{\{|\vartheta-v|\leq\kappa\}}dz\\
&\quad+\lim_{n\to\infty}\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_\theta\left(\vartheta_n\right),\nabla v\right)\left(D^iT_\theta\left(\vartheta_n\right)-D^iv\right)\chi_{\{|\vartheta_n-v|\leq\kappa\}}dz\\
&=\sum_{i=1}^{N}\int_{\Omega}\sigma_i\left(z,T_\theta(\vartheta),\nabla T_\theta(\vartheta)\right)\left(D^iT_\theta(\vartheta)-D^iv\right)\chi_{\{|\vartheta-v|\leq\kappa\}}dz\\
&=\sum_{i=1}^{N}\int_{\Omega}\sigma_i(z,\vartheta\vartheta,\nabla\vartheta)D^iT_\kappa(\vartheta-v)dz.
\end{aligned}\tag{4.54}$$

Now, since $T_\kappa\left(\vartheta_n-v\right)\to T_\kappa(\vartheta-v)$ weak$-\star$ in $L^{\infty}(\Omega)$ and in view of (4.42) we conclude that

$$\frac{1}{n}\int_{\Omega}\left|\vartheta_n\right|^{p_0-2}\vartheta_nT_\kappa\left(\vartheta_n-v\right)dz\longrightarrow 0,\tag{4.55}$$

and

$$\int_{\Omega}N_n\left(z,\vartheta_n,\nabla\vartheta_n\right)T_\kappa\left(\vartheta_n-v\right)dz\longrightarrow\int_{\Omega}N(z,\vartheta,\nabla\vartheta)T_\kappa\left(\vartheta_n-v\right)dz,\tag{4.56}$$

also, since f_n tends to f in $L^1(\Omega)$ then

$$\int_{\Omega}f_nT_\kappa\left(\vartheta_n-v\right)dz\longrightarrow\int_{\Omega}fT_\kappa(\vartheta-v)dz.\tag{4.57}$$

Also, we have $\phi_{n,i}\left(\vartheta_n\right)=\phi_i\left(T_\theta\left(\vartheta_n\right)\right)$ in $\{|\vartheta_n-v|\leq\kappa\}$ for $n\geq M$, and since $T_\kappa\left(\vartheta_n-v\right)\rightharpoonup T_\kappa(\vartheta-v)$ in $W_0^{1,\vec{p}}(\Omega)$, then

$$\int_{\Omega}\phi_{n,i}\left(\vartheta_n\right)D^iT_\kappa\left(\vartheta_n-v\right)dz\longrightarrow\int_{\Omega}\phi_i(\vartheta)D^iT_\kappa(\vartheta-v)dz,\tag{4.58}$$

and

$$\int_{\Omega}F_iD^iT_\kappa\left(\vartheta_n-v\right)dz\longrightarrow\int_{\Omega}F_iD^iT_\kappa(\vartheta-v)dz.\tag{4.59}$$

In view of (4.53)-(4.59), the proof of Theorem 4.2 is achieved.

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