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Fuzzy Generalizations of Automorphism and Inner Automorphism of Groups

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Abstract: The fuzzification of various concepts from classical set theory emerged with the discovery of fuzzy sets as a generalization of crisp sets. Considering notions of classical group theory such as permutations on a set, homomorphisms, endomorphisms, and automorphisms of groups, this paper introduces terms such as fuzzy permutations on a set, fuzzy endomorphism, fuzzy automorphism, and fuzzy inner automorphisms of groups. These new concepts can be taken as fuzzy generalizations of their usual group theoretic counterparts. Notions of fuzzy permutation f_a^μ and fuzzy inner automorphism f_g^μ induced by a fuzzy normal subgroup μ are introduced. The set of all fuzzy permutations and the set of all fuzzy automorphisms induced by a fuzzy normal subgroup are denoted by $A_F^\mu(G)$ and $A_F(G,\mu)$ respectively. Fuzzy analogs of well-known theorems from classical group theory are presented. For the fuzzy analog of the Cayley theorem, we show that every group G is isomorphic to the fuzzy permutation group $A_F^\mu(G)$. We also establish the result that for a group G, $G/N_G(\mu)$ is isomorphic to a subgroup of $A_F(G,\mu)$.

1 Introduction

The study of fuzzy sets was initiated by Zadeh [1], since then a large number of mathematical structures like algebras, topological spaces, differential equations, etc. have been fuzzified by many mathematicians. Rosenfield [2] introduced the concept of fuzzy subgroups, assuming that subsets of a group are fuzzy. With this evolution of fuzzy group theory, various fuzzy counterparts of group theoretic concepts from classical group theory were introduced by many authors. To list a few in this context, Mukherjee and Bhattacharya [3] introduced the concept of fuzzy cosets and fuzzy normal subgroups and proved fuzzy analogs of various group theoretic concepts. Bhattacharya and Mukherjee [4] introduced the order of a fuzzy subgroup in a finite group, a fuzzy abelian group, and a fuzzy solvable group. Based on fuzzy binary operations, Yuan and Lee [5] proposed a new kind of fuzzy group. Nazmul [6] studied some properties of soft groups and fuzzy soft groups under soft mappings. Major developments in this field are compiled by Mordsen et al.[7]. Davaz and Cristea presented an extensive study of fuzzy algebraic hyperstructures with an interconnection between hypergroups and fuzzy sets in [8]. Mayerova et al.[9] studied fuzzy multi-hypergroups. Leoreanu et al. [10] presented a survey on the fuzzy degree of a hypergroup. Tahan et al. [11] initiated the study of fuzzy (m,n) filters of ordered semigroups. Fuzzy (m,n) filters based on fuzzy points in ordered semigroups have been studied by Mahboob et al. [12], followed by the characterization of an ordered semigroup in terms of fuzzy (m,n) structures [13]. Many authors contributed to fuzzification of various ring theroetic concepts. For some related study see [14, 15, 16, 17] The concept of homomorphism has a central place in the theory of algebraic structures. Apart from the classical abstract concept of homomorphism, fuzzy homomorphism may be proposed in several ways. Using fuzzy congruence on a fuzzy lattice, Korma [18] proposed the concept of homomorphism on the fuzzy lattice. The idea of a fuzzy kernel of a fuzzy homomorphism of rings has been studied by Addis et. al. [19]. Mahboob et al. [20] presented an extensive study of ordered Γ-semigroups and fuzzy Γ-ideals. Yao Bingxue [21] introduced the concept of fuzzy homomorphism. Group homomorphisms and isomorphisms have several applications in various fields including cryptographic protocols, the symmetry of objects, and artificial intelligence. Fuzzification of homomorphisms and automorphisms may lead to new tools and approaches that account for imprecision and uncertainty in these fields. In this paper, we introduced the concept of fuzzy automorphism, and fuzzy inner automorphism in groups, and obtained fuzzy analogs of some standard results from classical group theory. We introduced the idea of fuzzy permutation and inner automorphism induced by a fuzzy sub-group and a fuzzy normal subgroup of a group respectively. The preprint of the work is available at [22]. The contents of the paper are organized as follows: In section 2, some basic definitions, notations, and elementary results are presented. In Section 3, we introduced the notion of fuzzy automorphism and fuzzy inner automorphism and presented elementary group postulates. Section 4 & 5 concerns fuzzy permutation and inner automorphism induced by a fuzzy subgroup and a fuzzy normal subgroup of a group. Using these notions, we presented fuzzy analogs of Cayley's theorem and fuzzy analogs some well-known results from classical group theory.

2 Preliminaries

Definition 2.1. Let X be a non empty set. Then $\mu : X \to [0,1]$ is called a fuzzy subset of X.

We denote $\mathcal{FP}(X)$, the set of all fuzzy subsets of X also termed as the fuzzy power set of X.

Definition 2.2. Fuzzy map: Let X and Y be two non-empty sets. Then a fuzzy subset of $X \times Y$ given by $f : X \times Y \to [0,1]$ is called a fuzzy map if, for each $x \in X$, there exists a unique $y_x \in Y$ such that $f(x, y_x) = 1$.

In this case, we denote $f : X \cdots \to Y$, as a fuzzy map, and y_x is called the fuzzy image of x. We assert that two fuzzy maps f and g on a set are equal, i.e. $f \equiv g$, if and only if for each $x \in X$, both f and g have same fuzzy images.

Definition 2.3. Bijective fuzzy map: Let $f : X \cdots \to Y$ be a fuzzy map. Then f is said to be one-one, if for any $x_1, x_2 \in X$ and $y \in Y$, whenever $f(x_1, y) = f(x_2, y) = 1$, then $x_1 = x_2$. Further, f is called onto if, for each $y \in Y$, there exists $x \in X$ such that $f(x, y) = 1$. A fuzzy map is bijective if it is one-one and onto.

Definition 2.4. Composition of fuzzy maps: Let f and g be fuzzy subsets of the set $X \times Y$ and $Z \times X$ respectively where X, Y, and Z are non-empty sets. Then

$$f \circ g : Z \times Y \to [0,1] \text{ defined as } (f \circ g)(z, y) = \sup_{a \in X} \{ f(a, y) \mid g(z, a) = 1 \}.$$

If there is no such $a \in X$ such that $g(z, a) = 1$ for a given $z \in Z$, then using results ([7], Chapter 1), we assume that $(f \circ g)(z, y) = 0$.

Remark 2.5. If g is a fuzzy map on $Z \times X$, then

$$(f \circ g)(z, y) = f(a, y) \text{ where } g(z, a) = 1.$$

Definition 2.6. Fuzzy subgroup: Let G be a group and $\mu \in \mathcal{FP}(G)$. Then μ is called a fuzzy subgroup of G if for all $x, y \in G$, we have $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and $\mu(x^{-1}) \geq \mu(x)$. Further, the set of all fuzzy subgroups of G is denoted by $\mathcal{F}(G)$.

Definition 2.7. Normal fuzzy subgroup: Let $\mu \in \mathcal{F}(G)$. Then μ is a normal fuzzy subgroup of G if $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Definition 2.8. Fuzzy homomorphism: A fuzzy map $f : G \cdots \to G'$ is called a fuzzy homomorphism if for all $x_1, x_2 \in G$ and $y \in G'$, we have

$$f(x_1 x_2, y) = \sup_{y_1, y_2 \in G'} \{ f(x_1, y_1) \wedge f(x_2, y_2) \mid y = y_1 y_2 \}.$$

Definition 2.9. Kernel of a fuzzy homomorphism: Let $f : G \cdots \to G'$ be a fuzzy homomorphism. Then we define

$$K = \text{Ker} f = \{ x \in G \mid f(x, e') = 1 \}.$$

The next theorem is a basic result of fuzzy homomorphism. We refer to [21] for the proofs.

Theorem 2.10. *Let $f : G \cdots \to G'$ be a fuzzy homomorphism. Then*

(i) $y_{x_1 x_2} = y_{x_1} \cdot y_{x_2}$ *where* $y_{x_i} \in G'$ *denote the unique element corresponding to* $x_i \in G$ *for* $1 \leq i \leq 2$ *such that* $f(x_i, y_{x_i}) = 1$.

(ii) $f(e, e') = 1$ *where* e *and* e' *denote the respective identities of* G *and* G'.

(iii) $y_x^{-1} = y_{x^{-1}}$ *for any* $y_x \in G'$ *and* $x \in G$.

(iv) $f(x,y) = 1 \implies f(x^{-1}, y^{-1}) = 1$ *for any* $x \in G$ *and* $y \in G'$.

Now, we prove the following fuzzy analog of some results on a kernel of a homomorphism.

Theorem 2.11. *For a fuzzy homomorphism* $f : G \cdots \to G'$, *kernel* K *is a normal subgroup of* G. *Also,* f *is one-one if and only if* $K = \{e\}$.

Proof. By (ii) of Theorem (2.10), we have $f(e, e') = 1$ which shows that K is non empty. Let $x_1, x_2 \in K$. Then

$$\begin{aligned} f(x_1 x_2^{-1}, e') &= \sup_{y_1, y_2 \in G'} \{ f(x_1, y_1) \wedge f(x_2^{-1}, y_2) \mid e' = y_1 y_2 \} \\ &\geq f(x_1, e') \wedge f(x_2^{-1}, e') \\ &= f(x_1, e') \wedge f(x_2^{-1}, (e')^{-1}) = 1 \wedge 1 = 1 \end{aligned}$$

where above holds by (iv) of Theorem (2.10). Above shows that $x_1 x_2^{-1} \in K$, i.e., K is a subgroup of G. Now, let $g \in G$ and $x \in K$. Then $f(x, e') = 1$ and there exists a unique $y_g \in G'$ such that $f(g, y_g) = 1$. Now

$$f(g^{-1}xg, e') = \sup_{y_1, y_2 \in G'} \{ f(g^{-1}x, y_1) \wedge f(g, y_2) \mid e' = y_1 y_2 \} \geq f(g^{-1}x, y_g^{-1}) \wedge f(g, y_g)$$

$$= f(g^{-1}x, y_g^{-1}) \wedge 1 = f(g^{-1}x, y_g^{-1}) = \sup_{y_1, y_2 \in G'} \{ f(g^{-1}, y_1) \wedge f(x, y_2) \mid y_g^{-1} = y_1 y_2 \}$$

$$\geq f(g^{-1}, y_g^{-1}) \wedge f(x, e') = 1 \wedge 1 = 1.$$

Above shows that K is a normal subgroup of G. Remaining part of the theorem is self-evident. □

Example 2.12. Define a map $f : \mathbf{Z} \cdots \to \mathbf{Z_6}$ by

$$f(a,b) = \begin{cases} 1 & if \quad b = remainder\ when\ a\ is\ divided\ by\ 6 \\ 0 & if \quad otherwise \end{cases}.$$

Then f is a fuzzy epimorphisms of groups with kernel **6Z**.
Proof: Let $x_1, x_2 \in \mathbf{Z}$ and r_1, r_2 be respective remainders when x_1, x_2 are divided by 6. Using modular arithmetic, $r_1 r_2$ is the remainder when $x_1 x_2$ is divided by 6. Since $f(x_1, r_1) = 1 = f(x_2, r_2)$. Now for any $y \in Z_6$, $f(x_1 x_2, y) = 1$ if and only if $y = r_1 r_2$. Thus

$$f(x_1 x_2, y) = \sup_{y_1, y_2 \in Z_6} \{ f(x_1, y_1) \wedge f(x_2, y_2) \mid y = y_1 y_2 \}$$

This shows that f is a fuzzy homomorphism of groups. Now it is easy to see that f is a surjective fuzzy map with kernel **6Z**.

3 Fuzzy automorphism and fuzzy inner automorphism of a group

A fuzzy homomorphism $f : G \cdots \to G$ is said to be a fuzzy automorphism provided it is one-one and onto. Now, in this section, we discuss our main results on fuzzy automorphism and inner automorphism of a group G.

Lemma 3.1. *The composition of two fuzzy automorphisms of a group is again a fuzzy automorphism.*

Proof. Let f and g be two fuzzy automorphisms of G. Then for each $y \in G$ and $x_1, x_2 \in G$,

$$f(x_1 x_2, y) = \sup_{y_1, y_2 \in G} \{ f(x_1, y_1) \wedge f(x_2, y_2) \mid y = y_1 y_2 \}$$

and

$$g(x_1 x_2, y) = \sup_{y_1, y_2 \in G} \{ g(x_1, y_1) \wedge g(x_2, y_2) \mid y = y_1 y_2 \}.$$

First we show that $f \circ g$ is a homomorphism. Since f and g are fuzzy maps, there exist unique y_i, z_i, for $1 \leq i \leq 2$, such that

$$f(x_i, y_i) = 1 = g(x_i, z_i) \text{ for } 1 \leq i \leq 2.$$

Now we claim that $g(x_1x_2, z_1z_2) = 1$. Observe that

$$g(x_1x_2, z_1z_2) = \sup_{a,b\in G} \{g(x_1, a) \wedge g(x_2, b) \mid z_1z_2 = ab\}$$
$$\geq g(x_1, z_1) \wedge g(x_2, z_2) = 1.$$

Thus, claim holds. Now by definition 2.4, we have

$$(f \circ g)(x_1x_2, y) = f(z_1z_2, y) = \sup_{y_1,y_2\in G} \{f(z_1, y_1) \wedge f(z_2, y_2) \mid y = y_1y_2\}. \tag{3.1}$$

Now as

$$(f \circ g)(x_i, y_i) = \sup_{a\in G} \{f(a, y_i) \mid g(x_i, a) = 1\} = f(z_i, y_i),\ 1 \leq i \leq 2,$$

we have

$$\sup_{y_1,y_2\in G} \{(f \circ g)(x_1, y_1) \wedge (f \circ g)(x_2, y_2) \mid y = y_1y_2\}$$
$$= \sup_{y_1,y_2\in G} \{f(z_1, y_1) \wedge f(z_2, y_2) \mid y = y_1y_2\}$$
$$= (f \circ g)(x_1x_2, y)$$

where above holds by incorporating equation (3.1). Thus, $f \circ g$ is a fuzzy homomorphism. Now, we show that $f \circ g$ is a bijection. For any $y \in G$, take $x_1, x_2 \in G$ such that

$$(f \circ g)(x_1, y) = (f \circ g)(x_2, y) = 1$$

By definition, above can be written as

$$\sup_{a\in G} \{f(a, y) \mid g(x_1, a) = 1\} = \sup_{b\in G} \{f(b, y) \mid g(x_2, b) = 1\} = 1$$
$$\implies f(a, y) = f(b, y) = 1,$$

where a and b are the unique elements corresponding to x_1 and x_2 in G such that $g(x_1, a) = g(x_2, b) = 1$. The above implies that $a = b$ as f is one-one. This further implies that $g(x_1, a) = g(x_2, a) = 1$, and as g is one-one, this means $x_1 = x_2$ which completes the proof for $f \circ g$ to be one-one. Now we show that $f \circ g$ is onto. Take any $y \in G$. We have to prove the existence of some $x \in G$ such that $(f \circ g)(x, y) = 1$. Since f is onto, there exists some $a \in G$ such that $f(a, y) = 1$. Now as g is onto, so there exist some $x_a \in G$ such that $g(x_a, a) = 1$. Then observe that

$$(f \circ g)(x_a, y) = \sup_{b\in G} \{f(b, y) \mid g(x_a, b) = 1\}.$$

As g is a fuzzy map and $g(x_a, a) = 1 = g(x_a, b)$ implies that $a = b$. So above can be written as

$$(f \circ g)(x_a, y) = f(a, y) = 1.$$

Hence, the result. □

Lemma 3.2. *Associativity: Let $f, g,$ and h be fuzzy automorphisms of a group G. Then*

$$(f \circ g) \circ h \equiv f \circ (g \circ h). \tag{3.2}$$

Proof. Let $x, y \in G$. Then there exists $u \in G$ such that $h(x, u) = 1$. Again, $u \in G$ implies that there exists unique $z \in G$ such that $g(u, z) = 1$. Now,

$$((f \circ g) \circ h)(x, y) = \sup_{a\in G} \{(f \circ g)(a, y) \mid h(x, a) = 1\}$$
$$= (f \circ g)(u, y) = f(z, y).$$

Now consider the right side of equation (3.2). We have

$$(f \circ (g \circ h))(x, y) = \sup_{a\in G} \{f(a, y) \mid (g \circ h)(x, a) = 1\}. \tag{3.3}$$

Futher, observe that

$$(g \circ h)(x, z) = \sup_{b\in G} \{g(b, z) \mid h(x, b) = 1\}$$
$$= g(u, z) = 1.$$

So equation (3.3) yields

$$(f \circ (g \circ h))(x, y) = f(z, y).$$

In particular, fuzzy images of $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are same. Hence $(f \circ g) \circ h \equiv f \circ (g \circ h)$. □

Now, let us consider a fuzzy homomorphism $I : G \cdots \to G$ such that $I(x, y) = 1$ if and only if $x = y$. It is easy to verify that I is one-one and onto. The existence of such a map is shown in the next section. In the next lemma, we show that I is an identity map.

Lemma 3.3. *For any fuzzy automomorphism f, we have*

$$f \circ I \equiv f \equiv I \circ f.$$

Proof. Let $x \in G$ and y be the fuzzy image of x under f. Then

$$(f \circ I)(x, y) = \sup_{a \in G} \big\{ f(a, y) \mid I(x, a) = 1 \big\} = f(x, y) = 1.$$

Above implies that y is fuzzy image of x under $f \circ I$. Also

$$(I \circ f)(x, y) = \sup_{a \in G} \big\{ I(a, y) \mid f(x, a) = 1 \big\}.$$

But as y is the fuzzy image of x under f, above can be written as

$$(I \circ f)(x, y) = I(y, y) = 1$$

which means y is fuzzy image of x under $I \circ f$. Therefore, the result holds. □

Now we define the notion of an inverse fuzzy map. Let $f : G \cdots \to G$ be a one-one and onto fuzzy homomorphism. Define $g : G \times G \to [0, 1]$ by $g(y, x) = f(x, y)$.

Lemma 3.4. *g is a well defined bijective fuzzy map and*

$$g \circ f \equiv I \equiv f \circ g.$$

Proof. Suppose there exist $x_1, x_2 \in G$ corresponding to some $y \in G$ such that $g(y, x_1) = g(y, x_2) = 1$. By definition, this means $f(x_1, y) = 1 = f(x_2, y)$. But as f is one-one, we must have $x_1 = x_2$. This shows that g is a fuzzy map. It is easy to check that g is a bijective fuzzy map. Now take any $x \in G$ and y_1, y_2 to be its fuzzy images under $f \circ g$ and I respectively. Then, $(f \circ g)(x, y_1) = 1 = I(x, y_2)$. We need to show $y_1 = y_2$. $I(x, y_2) = 1$ implies that $x = y_2$. Now, by definition

$$(f \circ g)(x, y_1) = \sup_{a \in G} \big\{ f(a, y_1) \mid g(x, a) = 1 \big\} = 1.$$

As g is a fuzzy map, there exist a unique $a \in G$ such that

$$(f \circ g)(x, y_1) = f(a, y_1) = 1 \text{ and } g(x, a) = f(a, x) = 1.$$

But as f is one-one, above implies $x = y_1$ and hence $y_1 = y_2$. Therefore, $f \circ g \equiv I$. Similarly, we can show that $g \circ f \equiv I$. □

Lemma 3.5. *Map $g \circ f$ is a fuzzy homomorphism.*

Proof. We need to show that

$$(g \circ f)(x_1 x_2, z) = \sup_{z_1, z_2 \in G} \big\{ (g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2) \bigm| z = z_1 z_2 \big\}$$

for x_1, x_2 and $z \in G$. So, let $x_1, x_2 \in G$. Then there exist unique $y_{x_1}, y_{x_2} \in G$ such that

$$f(x_1, y_{x_1}) = 1 = f(x_2, y_{x_2}).$$

Since f is a fuzzy homomorphism, we have $f(x_1 x_2, y_{x_1} y_{x_2}) = 1$. Now, let $z_1, z_2 \in G$ be such that $z = z_1 z_2$. Then

$$\begin{aligned}
(g \circ f)(x_1 x_2, z) &= (g \circ f)(x_1 x_2, z_1 z_2) = g(y_{x_1} y_{x_2}, z_1 z_2) \\
&= f(z_1 z_2, y_{x_1} y_{x_2}) = \sup_{a, b \in G} \big\{ f(z_1, a) \wedge f(z_2, b) \bigm| y_{x_1} y_{x_2} = ab \big\} \\
&\geq f(z_1, y_{x_1}) \wedge f(z_2, y_{x_2}) = g(y_{x_1}, z_1) \wedge g(y_{x_2}, z_2) \\
&= (g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2).
\end{aligned}$$

Since z_1, z_2 are arbitrary, we have

$$(g \circ f)(x_1x_2, z) \geq \sup_{z_1,z_2 \in G} \{(g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2) \mid z = z_1z_2\}.$$

For the other way around, since f is a fuzzy homormorphism (Theorem 2.10(iv)),

$$f(x, y_x) = 1 \implies f(x^{-1}, y_{x^{-1}}) = 1.$$

Consider

$$\sup_{z_1,z_2 \in G} \{(g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2) \mid z = z_1z_2\} \geq (g \circ f)(x_1, x_1) \wedge (g \circ f)(x_2, x_1^{-1}z)$$

$$= 1 \wedge (g \circ f)(x_2, x_1^{-1}z) = g(y_{x_2}, x_1^{-1}z) = f(x_1^{-1}z, y_{x_2})$$
$$= \sup_{a,b \in G} \{f(x_1^{-1}, a) \wedge f(z, b) \mid y_{x_2} = ab\}$$
$$\geq f(x_1^{-1}, y_{x_1}^{-1}) \wedge f(z, y_{x_1}y_{x_2}) = f(z, y_{x_1}y_{x_2}) = g(y_{x_1}y_{x_2}, z)$$
$$= (g \circ f)(x_1x_2, z).$$

Thus, result. □

Lemma 3.6. *g is a fuzzy automorphism.*

Proof. We just need to show that g is a fuzzy homomorphism, i.e.

$$g(y_1y_2, z) = \sup_{z_1,z_2 \in G} \{g(y_1, z_1) \wedge g(y_2, z_2) \mid z = z_1z_2\}$$

for any y_1, y_2 and $z \in G$. Take $y_1, y_2, \in G$, this means there exist $x_1, x_2 \in G$ such that $f(x_1, y_1) = 1 = f(x_2, y_2)$ as f is onto. Since f is a fuzzy homomorphism, $f(x_1x_2, y_1y_2) = 1$. For $z = z_1z_2$, we have

$$(g \circ f)(x_1, z_1) = g(y_1, z_1) \text{ and } (g \circ f)(x_2, z_2) = g(y_2, z_2).$$

Now, consider

$$\sup_{z_1,z_2 \in G} \{g(y_1, z_1) \wedge g(y_2, z_2) \mid z = z_1z_2\} = \sup_{z_1,z_2 \in G} \{(g \circ f)(x_1, z_1) \wedge (g \circ f)(x_2, z_2) \mid z = z_1z_2\}$$

$$= (g \circ f)(x_1x_2, z) = g(y_1y_2, z)$$

where above holds because of Lemma 3.5. The above means that g is a fuzzy homomorphism. Lemmas 3.4 and 3.6 yields that $g = f^{-1}$, i.e. g is inverse of fuzzy automorphism f.

□

Let S be a non-empty set. By $A_F(S)$ we denote the set of all fuzzy bijective maps on the set S. From the above discussion, one can easily deduce that $A_F(S)$ is a group w.r.t. composition of fuzzy mappings. This group is termed a fuzzy permutation group on the set S. For a group G, Let $A_F(G)$ be the fuzzy permutation group on the group G and by $Aut_F(G)$ we denote the set of all fuzzy automorphisms on a group G. Then $A_F(G)$ and $Aut_F(G)$ form groups under the composition of fuzzy mappings.

4 Fuzzy permutation induced by a fuzzy subgroup and Cayley Theorem

Lemma 4.1. *Let $\mu \in F(G)$ such that $\mu(x) = 1$ if and only if $x = e$. Then for $a \in G$, the map $f_a^\mu : G \times G \to [0, 1]$ given by $f_a^\mu(x, y) = \mu(x^{-1}a^{-1}y)$ is a fuzzy permutation on G.*

Proof. Let $x_1, x_2 \in G$ such that for some $y \in G$, we have $f_a^\mu(x_1, y) = f_a^\mu(x_2, y) = 1$. This means $\mu(x_1^{-1}a^{-1}y) = \mu(x_2^{-1}a^{-1}y) = 1$. But this holds only if $x_1^{-1}a^{-1}y = x_2^{-1}a^{-1}y = e$ and hence $x_1 = x_2$. Thus, f_a^μ is one-one. For onto, let $y \in G$. Then $a^{-1}y$ is also in G. Observe that $f_a^\mu(a^{-1}y, y) = \mu(e) = 1$. Thus, f_a^μ is onto. □

Theorem 4.2. *Let $\mu \in F(G)$ such that $\mu(x) = 1$ if and only if $x = e$ and $A_F^\mu(G) = \{f_a^\mu : a \in G\}$. Then $A_F^\mu(G)$ is a group under the composition of fuzzy mappings.*

Proof. Let $A = A_F^\mu(G)$. Then for $f_a^\mu, f_b^\mu \in A$, we have

$$f_a^\mu \circ f_b^\mu(x,y) = f_a^\mu(bx,y) = \mu(x^{-1}b^{-1}a^{-1}y) = f_{ab}^\mu(x,y)$$

As for any f_a^μ, $f_a^\mu \circ I_e^\mu = I_e^\mu \circ f_a^\mu = f_a^\mu$, the fuzzy identity map $I = I_e^\mu$ is the identity element of $A_F^\mu(G)$. It is easy to see that $f_a^\mu \circ f_{a^{-1}}^\mu = I_e^\mu = f_{a^{-1}}^\mu \circ f_a^\mu$. □

Theorem 4.3. *(Cayley Theorem)* $G \cong A_F^\mu(G)$.

Proof. Define a map $\sigma : G \longrightarrow A_F^\mu(G) \subseteq A_F(G)$ by $\sigma(a) = f_a^\mu$. Since $f_a^\mu \circ f_b^\mu = f_{ab}^\mu$, σ is a homomorphism of groups. Let $a \in Ker\ \sigma$.Then $f_a^\mu = I_e^\mu$ i.e., $\mu(x^{-1}a^{-1}y) = \mu(x^{-1}y)\ \forall(x,y) \in G\times G$. In particular, for $y = x$, $\mu(x^{-1}a^{-1}x) = \mu(e) = 1$. This holds only when $x^{-1}a^{-1}x = e$. Eventually, $a = e$ and σ is a monomorphism of groups. For any $f_a^\mu \in A_F^\mu(G)$, $\sigma(a) = f_a^\mu$. Thus σ is an isomorphism and hence $G \cong A_F^\mu(G)$.

□

Definition 4.4. Fuzzy inner automorphism: A fuzzy map $f : G \cdots \to G$ is said to be a class preserving map if $f(x,y) = 1$ if and only if $y = a^{-1}xa$ for some $a \in G$. Let $\mathrm{Aut}_F(G)$ denote the group of all fuzzy automorphisms of G, then we say that $f \in \mathrm{Aut}_F(G)$ is an inner automorphism if and only if there exists $g \in G$ (fixed) such that $f(x,y) = 1$ if and only if $y = g^{-1}xg$. In this case, we denote f by f_g. Further, let $\mathrm{Inn}_F(G)$ denote the set of all fuzzy inner automorphisms of G.

Lemma 4.5. *Composition of fuzzy inner automorphisms is again a fuzzy inner automorphism.*

Proof. Let f_{g_1} and f_{g_2} be two fuzzy inner automorphisms of G. Then

$$(f_{g_1} \circ f_{g_2})(x,y) = \sup_{a\in G}\big\{f_{g_1}(a,y) \bigm| f_{g_2}(x,a) = 1\big\} = f_{g_1}(g_2^{-1}xg_2, y).$$

Now from above,

$$(f_{g_1} \circ f_{g_2})(x,y) = 1 \Longleftrightarrow y = g_1^{-1}g_2^{-1}xg_2g_1.$$

Similarly, by definition

$$f_{g_2g_1}(x,y) = 1 \Longleftrightarrow y = g_1^{-1}g_2^{-1}xg_2g_1.$$

Thus fuzzy images of $f_{g_1} \circ f_{g_2}$ and $f_{g_2g_1}$ are same and hence it follows that $f_{g_1} \circ f_{g_2} \equiv f_{g_2g_1}$. □

Lemma 4.6. *If $f_g \in Inn_F(G)$, then f_g^{-1} is also in $Inn_F(G)$ for any $g \in G$ and $f_g^{-1} \equiv f_{g^{-1}}$.*

Proof. Take x and y in G, then by definition of inverse fuzzy map, we have

$$f_g^{-1}(x,y) = 1 \Longleftrightarrow f_g(y,x) = 1 \Longleftrightarrow y = (g^{-1})^{-1}xg^{-1}.$$

Thus, f_g^{-1} is a fuzzy inner automorphism. Also note that

$$f_{g^{-1}}(x,y) = 1 \Longleftrightarrow y = (g^{-1})^{-1}xg^{-1}.$$

This shows that $f_g^{-1} \equiv f_{g^{-1}}$. □

Lemma 4.7. *$Inn_F(G)$ is a normal subgroup of $Aut_F(G)$.*

Proof. Lemmas 4.5 and 4.6 together show that $\mathrm{Inn}_F(G)$ is a subgroup of $\mathrm{Aut}_F(G)$. Now, let f be any element in $\mathrm{Aut}_F(G)$ and f_g be in $\mathrm{Inn}_F(G)$. Then we have

$$(f^{-1} \circ f_g \circ f)(x,y) = \sup_{a\in G}\big\{(f^{-1} \circ f_g)(a,y) \bigm| f(x,a) = 1\big\}.$$

Since $f \in \mathrm{Aut}_F(G)$, so for any $x \in G$, there exists a unique $z \in G$ such that $f(x,z) = 1$. Then

$$(f^{-1} \circ f_g \circ f)(x,y) = (f^{-1} \circ f_g)(z,y) = f^{-1}(g^{-1}zg, y).$$

As $g \in G$, there exists a unique $a \in G$ such that $f^{-1}(g,a) = 1$ which further implies that $f^{-1}(g^{-1},a^{-1}) = 1$. Now

$$\begin{aligned} f^{-1}(g^{-1}zg, a^{-1}xa) &= \sup_{y_1,y_2\in G}\big\{f^{-1}(g^{-1}z, y_1) \wedge f^{-1}(g, y_2) \bigm| a^{-1}xa = y_1y_2\big\} \\ &\geq f^{-1}(g^{-1}z, a^{-1}x) \wedge f^{-1}(g,a) = f^{-1}(g^{-1}z, a^{-1}x) \\ &= \sup_{y_1,y_2\in G}\big\{f^{-1}(g^{-1}, y_1) \wedge f^{-1}(z, y_2) \bigm| a^{-1}x = y_1y_2\big\} \\ &\geq f^{-1}(g^{-1}, a^{-1}) \wedge f^{-1}(z,x) = f^{-1}(g^{-1},a^{-1}) = 1. \end{aligned}$$

This implies that $(f^{-1} \circ f_g \circ f)(x, a^{-1}xa) = 1$ and hence $f^{-1} \circ f_g \circ f \in \mathrm{Inn}_F(G)$. Thus, $\mathrm{Inn}_F(G)$ is a normal subgroup of $\mathrm{Aut}_F(G)$. □

5 Fuzzy Inner Automorphisms Induced by Fuzzy Normal Subgroups

In this section, we define fuzzy inner automorphism induced by a fuzzy normal subgroup. These automorphisms can also be seen as typical examples of the inner automorphisms discussed in Section 4. Let G be a group and μ be a normal fuzzy subgroup of G and $\mu(x) = 1$ if and only if $x = e$. Let $g \in G$ and define $f_g^\mu : G \cdots \to G$ by $f_g^\mu(x, y) = \mu(x^{-1}gyg^{-1})$. We claim that f_g^μ is a fuzzy map. Observe that

$$f_g^\mu(x, g^{-1}xg) = \mu(x^{-1}gg^{-1}xgg^{-1}) = \mu(e) = 1.$$

Let if possible $f_g^\mu(x, z) = 1$ for some $z \in G$. Then this means

$$\mu(x^{-1}gzg^{-1}) = 1 \implies x^{-1}gzg^{-1} = e \implies z = g^{-1}xg.$$

Thus f_g^μ associates a unique $g^{-1}xg$ for each $x \in G$ such that $f_g^\mu(x, g^{-1}xg) = 1$. Thus, the claim holds.

Lemma 5.1. *For any $g \in G$, f_g^μ is a fuzzy homomorphism.*

Proof. Let x_1, x_2 and $y \in G$ such that $y = y_1y_2$ for $y_1, y_2 \in G$. Then

$$f_g^\mu(x_1x_2, y) = f_g^\mu(x_1x_2, y_1y_2) = \mu(x_2^{-1}x_1^{-1}gy_1y_2g^{-1}) = \mu(x_2^{-1}x_1^{-1}gy_1(g^{-1}g)y_2g^{-1})$$

$$= \mu(x_2^{-1}x_1^{-1}gy_1g^{-1}z) \quad (\text{where } z^{-1}gy_2g^{-1} = e)$$

$$= \mu(x_2^{-1}zz^{-1}x_1^{-1}gy_1g^{-1}z) \geq \mu(x_2^{-1}z) \wedge \mu(z^{-1}x_1^{-1}gy_1g^{-1}z).$$

Now as μ is normal, so above implies

$$f_g^\mu(x_1x_2, y) \geq \mu(x_2^{-1}z) \wedge \mu(x_1^{-1}gy_1g^{-1})$$

$$= \mu(x_2^{-1}gy_2g^{-1}) \wedge \mu(x_1^{-1}gy_1g^{-1}) = f_g^\mu(x_1, y_1) \wedge f_g^\mu(x_2, y_2).$$

This means that $f_g^\mu(x_1x_2, y) \geq f_g^\mu(x_1, y_1) \wedge f_g^\mu(x_2, y_2)$ for all $y_1, y_2 \in G$ such that $y = y_1y_2$ which further implies that

$$f_g^\mu(x_1x_2, y) \geq \sup_{y_1, y_2 \in G} \{f_g^\mu(x_1, y_1) \wedge f_g^\mu(x_2, y_2) \mid y = y_1y_2\}.$$

Now consider the other way around. Observe that

$$\sup_{y_1, y_2 \in G} \{f_g^\mu(x_1, y_1) \wedge f_g^\mu(x_2, y_2) \mid y = y_1y_2\} \geq f_g^\mu(x_1, g^{-1}x_1g) \wedge f_g^\mu(x_2, g^{-1}x_1^{-1}gy)$$

$$= \mu(x_1^{-1}g(g^{-1}x_1g)g^{-1}) \wedge \mu(x_2^{-1}g(g^{-1}x_1^{-1}gy)g^{-1})$$

$$= \mu(e) \wedge \mu(x_2^{-1}x_1^{-1}gyg^{-1}) = \mu(x_2^{-1}x_1^{-1}gyg^{-1})$$

$$= f_g^\mu(x_1x_2, y).$$

Thus, combining above results yields

$$f_g^\mu(x_1x_2, y) = \sup_{y_1, y_2 \in G} \{f_g^\mu(x_1, y_1) \wedge f_g^\mu(x_2, y_2) \mid y = y_1y_2\}.$$

Thus, f_g^μ is a fuzzy homomorphism.

□

Lemma 5.2. *For any $g \in G$, f_g^μ is a one-one and onto class preserving fuzzy homomorphism.*

Proof. First, let $f_g^\mu(x, y) = 1$. This means that $\mu(x^{-1}gyg^{-1}) = 1 \implies y = g^{-1}xg$. In the reverse direction, we have $f_g^\mu(x, g^{-1}xg) = \mu(x^{-1}g(g^{-1}xg)g^{-1}) = \mu(e) = 1$. This shows that f_g^μ is a class preserving map. Now, let $x_1, x_2 \in G$ such that for some $y \in G$, we have $f_g^\mu(x_1, y) = f_g^\mu(x_2, y) = 1$. This means $y = g^{-1}x_1g = g^{-1}x_2g$ and hence $x_1 = x_2$. Thus, f_g^μ is one-one. For onto, let $y \in G$. Then gyg^{-1} is also in G. Observe that $f_g^\mu(gyg^{-1}, y) = \mu(e) = 1$. Thus, f_g^μ is onto.

□

Lemma 5.3. *Let $f_{g_1}^\mu$ and $f_{g_2}^\mu$ be two fuzzy inner automorphisms of G induced by μ. Then $f_{g_1}^\mu \circ f_{g_2}^\mu$ is again a fuzzy inner automorphism of G induced by μ and $f_{g_1}^\mu \circ f_{g_2}^\mu = f_{g_2g_1}^\mu$ and in particular $f_{g_1}^\mu \circ f_{g_2}^\mu \equiv f_{g_2g_1}^\mu$.*

Proof. We have

$$(f^{\mu}_{g_1} \circ f^{\mu}_{g_2})(x,y) = \sup_{a\in G}\{f^{\mu}_{g_1}(a,y) \mid f^{\mu}_{g_2}(x,a)=1\} = f^{\mu}_{g_1}(g_2^{-1}xg_2,y)$$

$$= \mu(g_2^{-1}x^{-1}g_2g_1yg_1^{-1}) = \mu(x^{-1}(g_2g_1)y(g_2g_1)^{-1}) \quad (\mu \text{ is normal})$$
$$= f^{\mu}_{g_2g_1}(x,y).$$

since $f^{\mu}_{g_2g_1}(x,y)$ is a fuzzy inner automorphism induced by μ

$$\implies f^{\mu}_{g_1} \circ f^{\mu}_{g_2}$$

is a fuzzy inner automorphism induced by μ. Hence result. □

Lemma 5.4. *Let $f^{\mu}_{g_1}$, $f^{\mu}_{g_2}$ and $f^{\mu}_{g_3}$ be fuzzy inner automorphisms of G induced by μ. Then*

$$(f^{\mu}_{g_1} \circ f^{\mu}_{g_2}) \circ f^{\mu}_{g_3} \equiv f^{\mu}_{g_1} \circ (f^{\mu}_{g_2} \circ f^{\mu}_{g_3}) \equiv f^{\mu}_{g_3g_2g_1}.$$

Proof. The proof follows directly from the lemma's 5.3. □

Lemma 5.5. *$I^{\mu}_e(x,y) = \mu(x^{-1}y)$ is a fuzzy identity inner automorphism induced by μ. Here e denotes the identity of group G.*

Proof. Let f^{μ}_g be any fuzzy inner automorphism. Then note that $(f^{\mu}_g \circ I^{\mu}_e)(x,y) = f^{\mu}_g(x,y)$ which means $(f^{\mu}_g \circ I^{\mu}_e) \equiv f^{\mu}_g$. Also

$$(I^{\mu}_e \circ f^{\mu}_g)(x,y) = I^{\mu}_e(g^{-1}xg,y) = \mu(g^{-1}x^{-1}geye^{-1}) = \mu(g^{-1}x^{-1}gy) = \mu(x^{-1}gyg^{-1}) = f^{\mu}_g(x,y)$$

$$\implies I^{\mu}_e \circ f^{\mu}_g \equiv I^{\mu}_e.$$

Thus, I^{μ}_e is an identity map. Now, we show that I^{μ}_e is a fuzzy homomorphism. Let $x_1, x_2 \in G$ and $y \in G$ such that $y = y_1y_2$ for $y_1, y_2 \in G$. Then we have

$$I^{\mu}_e(x_1x_2,y) = I^{\mu}_e(x_1x_2,y_1y_2) = \mu(x_2^{-1}x_1^{-1}y_1y_2) = \mu(x_1^{-1}y_1y_2x_2^{-1})$$

$$\geq \mu(x_1^{-1}y_1) \wedge \mu(y_2x_2^{-1}) = I^{\mu}_e(x_1,y_1) \wedge I^{\mu}_e(x_2,y_2).$$

Since y_1 and y_2 are arbitrary, we have

$$I^{\mu}_e(x_1x_2,y) \geq \sup_{y_1,y_2\in G}\{I^{\mu}_e(x_1,y_1) \wedge I^{\mu}_e(x_2,y_2) \mid y = y_1y_2\}.$$

Further,

$$\sup_{y_1,y_2\in G}\{I^{\mu}_e(x_1,y_1) \wedge I^{\mu}_e(x_2,y_2) \mid y = y_1y_2\} \geq I^{\mu}_e(x_1,x_1) \wedge I^{\mu}_e(x_2,x_1^{-1}y)$$

$$= I^{\mu}_e(x_2,x_1^{-1}y) = \mu(x_2^{-1}x_1^{-1}y) = I^{\mu}_e(x_1x_2,y).$$

Therefore I^{μ}_e is a fuzzy homomorphism. It is easy to see that I^{μ}_e is one-one and onto. Hence result. □

Remark 5.6. Fuzzy map I^{μ}_e is a typical example of an identity map discussed in Lemma 3.3.

Lemma 5.7. *For any $g \in G$, $(f^{\mu}_g)^{-1} \equiv f^{\mu}_{g^{-1}}$.*

Proof. Observe that

$$(f^{\mu}_g \circ f^{\mu}_{g^{-1}})(x,y) = \sup_{a\in G}\{f^{\mu}_g(a,y) \mid f^{\mu}_{g^{-1}}(x,a)=1\} = f^{\mu}_g(gxg^{-1},y)$$

$$= \mu(gx^{-1}g^{-1}gyg^{-1}) = \mu(gx^{-1}yg^{-1}) = \mu(yg^{-1}gx^{-1})$$
$$= \mu(yx^{-1}) = I^{\mu}_e(x,y).$$

Similarly, we can show that $(f^{\mu}_{g^{-1}} \circ f^{\mu}_g)(x,y) = I^{\mu}_e(x,y)$. Thus, $(f^{\mu}_g)^{-1} = f^{\mu}_{g^{-1}}$ and hence $(f^{\mu}_g)^{-1} \equiv f^{\mu}_{g^{-1}}$. □

Theorem 5.8. *The set of all fuzzy inner automorphisms of G induced by a normal subgroup μ forms a group under composition.*

Proof. The result is straightforward. □

In group theory, Cayley's theorem states that a group G is isomorphic to a subgroup of some permutation group. In particular, $G/Z(G) \cong \text{Inn}(G)$. We have the following fuzzy analog of this result

Theorem 5.9. *For a group G, $G/N_G(\mu)$ is isomorphic to a subgroup of $A_F(G,\mu)$. Moreover, $G/Z(G) \cong$ $Inn_F(G,\mu)$ where $Z(G)$ denotes the center of G.*

Proof. Define a map
$$\zeta : G \to \text{Inn}_F(G,\mu) : g \mapsto f^{\mu}_{g^{-1}}.$$
First of all we show that ζ is a group homomorphism. Note that
$$\zeta(g_1g_2) = f^{\mu}_{g_2^{-1}g_1^{-1}} = f^{\mu}_{g_1^{-1}} \circ f^{\mu}_{g_2^{-1}} = \zeta(g_1) \circ \zeta(g_2)$$
which shows that ζ is a group homomorphism. Clearly, ζ is onto. Now we show that $\text{Ker}(\zeta) = Z(G)$ where $\text{Ker}(\zeta)$ denotes the kernel of ζ. Let $g \in \text{Ker}(\zeta)$. This means $\zeta(g) = f^{\mu}_{g^{-1}} = I^{\mu}_e$. This shows that $f^{\mu}_{g^{-1}} \equiv I^{\mu}_e$ and hence, fuzzy images of $f^{\mu}_{g^{-1}}$ and I^{μ}_e are same i.e., $gxg^{-1} = x$ for all $x \in G$. Therefore we deduce that $\text{Ker}(\zeta) \subseteq Z(G)$. Now to prove the reverse inclusion, take any $z \in Z(G)$. Then $\zeta(z) = f^{\mu}_{z^{-1}}$ but
$$f^{\mu}_{z^{-1}}(x,y) = \mu(x^{-1}z^{-1}yz) = \mu(x^{-1}y) = I^{\mu}_e(x,y).$$
In particluar $f^{\mu}_{z^{-1}} \equiv I^{\mu}_e$. This shows that $Z(G) \subseteq \text{Ker}(\zeta)$. Thus, $\text{Ker}(\zeta) = Z(G)$. So, by the fundamental theorem of homomorphism, we have $G/Z(G) \cong \text{Inn}_F(G,\mu)$.

□

The above result is analog to its counterpart from the classical group theory as the isomorphism involved there is a usual group isomorphism. On the other hand, if we go for the fuzzy isomorphism $\underset{F}{\cong}$ between G and $\text{Inn}_F(G,\mu)$, then we have the following interesting result.

Theorem 5.10. *For a group G, we have $G \underset{F}{\cong} Inn_F(G,\mu)$.*

Proof. Define a map
$$\theta : G \cdots \to \text{Inn}_F(G,\mu) \text{ by } (a, f^{\mu}_b) \mapsto \mu(a^{-1}b^{-1}).$$
Observe that $\theta(a, f^{\mu}_{a^{-1}}) = \mu(a^{-1}a) = 1$. This means θ is a fuzzy map which maps a to $f^{\mu}_{a^{-1}}$ uniquely. Now we show that θ is a fuzzy homomorphism. Let $a_1, a_2 \in G$ and $f^{\mu}_c \in \text{Inn}_F(G,\mu)$. So, we need to show that
$$\theta(a_1a_2, f^{\mu}_c) = \sup_{f^{\mu}_x, f^{\mu}_y \in \text{Inn}_F(G,\mu)} \left\{\theta(a_1, f^{\mu}_x) \wedge \theta(a_2, f^{\mu}_y) \mid f^{\mu}_c = f^{\mu}_x \circ f^{\mu}_y\right\}.$$
Now, for $f^{\mu}_x, f^{\mu}_y \in \text{Inn}_F(G,\mu)$ such that $f^{\mu}_x \circ f^{\mu}_y = f^{\mu}_c$ we have
$$\theta(a_1a_2, f^{\mu}_c) = \theta(a_1a_2, f^{\mu}_x \circ f^{\mu}_y) = \theta(a_1a_2, f^{\mu}_{yx}) = \mu(a_2^{-1}a_1^{-1}x^{-1}y^{-1}) = \mu(y^{-1}a_2^{-1}a_1^{-1}x^{-1})$$
$$\geq \mu(y^{-1}a_2^{-1}) \wedge \mu(a_1^{-1}x^{-1}) = \mu(a_2^{-1}y^{-1}) \wedge \mu(a_1^{-1}x^{-1}) = \theta(a_1, f^{\mu}_x) \wedge \theta(a_2, f^{\mu}_y).$$
Above implies that
$$\theta(a_1a_2, f^{\mu}_c) \geq \sup_{f^{\mu}_x, f^{\mu}_y \in \text{Inn}_F(G,\mu)} \left\{\theta(a_1, f^{\mu}_x) \wedge \theta(a_2, f^{\mu}_y) \mid f^{\mu}_c = f^{\mu}_x \circ f^{\mu}_y\right\}.$$
Now, for the other way around, observe that
$$\sup_{f^{\mu}_x, f^{\mu}_y \in \text{Inn}_F(G,\mu)} \left\{\theta(a_1, f^{\mu}_x) \wedge \theta(a_2, f^{\mu}_y) \mid f^{\mu}_c = f^{\mu}_x \circ f^{\mu}_y\right\}$$
$$= \sup_{f^{\mu}_x, f^{\mu}_y \in \text{Inn}_F(G,\mu)} \left\{\theta(a_1, f^{\mu}_x) \wedge \theta(a_2, f^{\mu}_y) \mid f^{\mu}_c = f^{\mu}_{yx}\right\} \geq \theta(a_1, f^{\mu}_{a_1^{-1}}) \wedge \theta(a_2, f^{\mu}_{ca_1})$$
$$= \mu(a_1^{-1}a_1) \wedge \mu(a_2^{-1}a_1^{-1}c^{-1}) = \mu((a_1a_2)^{-1}c^{-1}) = \theta(a_1a_2, f^{\mu}_c).$$
Thus, θ is a fuzzy homomorphism. Further,
$$\text{Ker}(\theta) = \{x \in G \ : \ \theta(x, I^{\mu}_e) = 1\} = \{x \in G \ : \ \mu(x^{-1}e^{-1}) = 1\} = \{e\}.$$
This implies that θ is one-one. Now take any $f^{\mu}_g \in \text{Inn}_F(G,\mu)$. Then as $g \in G$ implies that $\theta(g^{-1}, f^{\mu}_g) = \mu(gg^{-1}) = 1$ which means that θ is onto and hence θ is a fuzzy isomorphism. □

6 Conclusion

Homomorphisms and automorphisms of abstract algebraic structures have various applications in many real-world fields, including cryptography, symmetry of objects, database management, and graph theory. Fuzzification of algebraic concepts like homomorphisms and automorphisms may be used to develop tools and approaches that would account for imprecision and uncertainty in these fields. Keeping this in view, the present study introduces concepts like fuzzy permutation, fuzzy automorphism, and fuzzy inner automorphism in groups. It is shown that $A_F(G)$ and $Aut_F(G)$ form groups under the composition of fuzzy mappings. Finally, we introduce the idea of fuzzy permutation and fuzzy inner automorphism induced by fuzzy normal subgroups. This helps us to provide a foundation for concrete examples and deep insights into proposed notions. Using this we obtained fuzzy counterparts of some well-known results of group theory.

6.1 Future extension of the present work

The present study aims to propose the fuzzification of homomorphism and automorphism of groups. We proposed one way to obtain these using a fuzzy normal subgroup. This work can be extended to find other abstract ways to get fuzzy automorphisms. Moreover, some more fuzzy analogs of group theoretic concepts can be obtained from future prospectives.

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