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ON PILLAI-TYPE EQUATIONS INVOLVING PADOVAN NUMBERS AND S-UNITS

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Abstract In this paper, we explore the Padovan sequence $\{P_n\}_{n\geq 0}$, defined by the initial terms $P_0 = 0$, $P_1 = P_2 = 1$, and the recurrence relation $P_{n+3} = P_{n+1} + P_n$ for all $n \geq 0$. Given a positive integer c, we focus on analyzing the exponential Diophantine equation $P_n - 2^x 3^y = c$, where n, x and y are non-negative integers. Specifically, we prove that no positive integer c admits five or more distinct solutions $(n, x, y) \in \mathbb{Z}^3_{\geq 0}$.

1 Introduction

In recent decades, the resolution of longstanding problems in number theory, such as Fermat's Last Theorem, has inspired renewed interest in Diophantine equations, particularly those involving exponential terms. A classical example is the Pillai equation

$$a^x - b^y = c, \tag{1.1}$$

where $a > 1$, $b > 1$, and c are fixed integers. Originally studied by Pillai [16], this equation has been widely investigated under various modifications, often with the goal of determining the finiteness or uniqueness of its integer solutions (x, y). One notable result established that when a and b are coprime and positive, and $|c|$ exceeds a certain bound $c_0(a, b)$, equation (1.1) admits at most one solution.

Recent works have generalized the equation by replacing the powers of a or b with terms from integer sequences exhibiting exponential growth. Examples include Fibonacci numbers, Lucas numbers, Tribonacci numbers, and other linear recurrence sequences, where finiteness results analogous to Pillai's have been confirmed [4], [6], [10] and [9].

Let K be a number field with ring of integers $\mathcal{O}_K$, and let S be a finite set of prime ideals in $\mathcal{O}_K$. An element $x \in K$ is called an S-unit if its principal fractional ideal is a product of primes from S. For rational numbers, this means that both the numerator and denominator of x are composed solely of primes from S. Several investigations, including [3], [2] and [19], have considered variations of (1.1) where one of the exponential sequences is replaced by an S-unit.

In this work, we focus on a variant of the Pillai equation in which the powers of a are substituted with terms from the Padovan sequence $\{P_n\}_{n\geq 0}$, defined by

$$P_0 = 0,\ P_1 = P_2 = 1 \quad \text{and} \quad P_{n+3} = P_{n+1} + P_n \text{ for all } n \geq 0,$$

and the powers of b are replaced with products of S-units, where $S = \{2, 3\}$. More precisely, we study the Diophantine equation

$$P_n - 2^x 3^y = c, \tag{1.2}$$

for non-negative integers n, x, y and a fixed integer c.
The Padovan sequence is characterized by the polynomial $\psi(X) = X^3 - X - 1$. This polynomial is irreducible over $\mathbb{Q}[X]$ and has one positive real root α and two complex conjugate roots β and

γ ($\gamma = \overline{\beta}$), where

$$\alpha = \sqrt[3]{\frac{9+\sqrt{69}}{18}} + \sqrt[3]{\frac{9-\sqrt{69}}{18}} \text{ and } \beta = -\frac{\alpha}{2} + i\frac{\sqrt{3}}{2}\left(\sqrt[3]{\frac{9+\sqrt{69}}{18}} - \sqrt[3]{\frac{9-\sqrt{69}}{18}}\right).$$

Let

$$A_\alpha = \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)},\ A_\beta = \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} \text{ and } A_\gamma = \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)}.$$

We define

$$\chi_1 := \frac{\alpha}{\beta} \quad \text{and} \quad \chi_2 := -\frac{A_\alpha}{A_\beta}\left(\frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n}) - (1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})}\right),$$

with $n_2 < n_1 < n$ being nonnegative integers.
Since $P_1 = P_2 = P_3 = 1$ and $P_4 = P_5 = 2$, these repetitions simplify certain cases when analyzing (1.2).

We say that two algebraic numbers x and y are *multiplicatively independent* if there are no integers m and n, not both zero, such that $x^m y^n = 1$.

Theorem 1.1. *If χ_1 and χ_2 are multiplicatively independent, then there does not exist a $c \in \mathbb{N}^*$ such that the Diophantine equation (1.2) has at least five distinct solutions $(n, x, y) \in \mathbb{Z}^3_{\geq 0}$.*

If we take, for example, $n_1 - n = -2$ and $n_2 - n = -3$, it is easy to verify that χ_1 and χ_2 are multiplicatively independent. Indeed, if there existed integers u and v, not both zero, such that $\chi_1^u \chi_2^v = 1$, then taking logarithms would yield a linear relation over $\mathbb{Q}$, contradicting the known algebraic independence of the logarithms of χ_1 and χ_2. Therefore, the only solution is $u = v = 0$.

In the case where χ_1 and χ_2 are multiplicatively dependent, we encounter a difficulty in bounding the p-adic valuation (which will be defined later) of $\chi_1^n \chi_2 + 1$.

2 Methods

2.1 Preliminaries

The Padovan sequence can be expressed using the preceding coefficients through what is known as the Binet formula:

$$P_n = A_\alpha \alpha^n + A_\beta \beta^n + A_\gamma \gamma^n \quad \text{for all } n \geq 0. \tag{2.1}$$

The minimal polynomial $P(X)$ of A_α over $\mathbb{Z}$ is $23X^3 - 23X^2 + 6X - 1$, which has roots A_α, A_β and A_γ.
We have the following estimates from numerical approximations:

$$1.32 < \alpha < 1.33,$$
$$0.86 < |\beta| = |\gamma| < 0.87,$$
$$0.61 < |A_\alpha| < 0.62,$$
$$0.19 < |A_\beta| = |A_\gamma| < 0.20.$$

Also, it can be verified that $|\beta| = |\gamma| = \alpha^{-1/2}$.

Let $K := \mathbb{Q}(\alpha, \beta)$ be the splitting field of the polynomial $\psi(X)$ over $\mathbb{Q}$. The degree of the extension is $[K : \mathbb{Q}] = 6$. Moreover, the subfield $\mathbb{Q}(\alpha)$ has degree $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of K over $\mathbb{Q}$ is isomorphic to the symmetric group S_3, given by $G \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\}$. The elements of G correspond to the permutations of the roots of the polynomial $\psi(X)$.
We also recall a useful inequality related to logarithmic approximations, stated as [2, Lemma 1].

Lemma 2.1. *[2, Lemma 1] Let x be a real number such that $|x| < \frac{1}{2}$, then $|\log(1+x)| < \frac{3}{2}|x|$.*

To conclude this section, we provide an analytic argument corresponding to [19, Lemma 7].

Lemma 2.2. *Let $m \geq 1$. For any integer T such that $T > (4m^2)^m$ and $T > \frac{z}{(\log(z))^m}$, we have*

$$z < 2^m T(\log(T))^m.$$

We will need the following result, taken from [5, Lemma 2.6].

Lemma 2.3. *(Adapted from [5, Lemma 2.6]) The equation*

$$\frac{1-\gamma^s}{1-\gamma^r} = \frac{1-\alpha^s}{1-\alpha^r}, \tag{2.2}$$

has no integer solutions r, s satisfying $r > s \geq 1$.

2.2 Linear Forms in Logarithms

We frequently use Baker-type lower bounds for nonzero linear forms in two or three logarithms of algebraic numbers. Several such bounds are discussed in the literature, such as those by Baker and Wüstholz [1] and Matveev [14]. To state such inequalities, we first recall the definition of the height of an algebraic number.

Definition 2.4. Let δ be an algebraic number of degree d whose minimal polynomial over the integers is

$$a_0 x^d + a_1 x^{d-1} + \cdots + a_d = a_0 \prod_{i=1}^{d} (x - \delta^{(i)}),$$

where the leading coefficient a_0 is positive. The logarithmic height of δ is defined as

$$h(\delta) := \frac{1}{d}\left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\delta^{(i)}|, 1\}\right).$$

The following properties of the logarithmic height are presented in numerous works:

$$\begin{aligned} h(\delta \pm \gamma) &\leq h(\delta) + h(\gamma) + \log(2), \\ h(\delta\gamma^{\pm 1}) &\leq h(\delta) + h(\gamma), \\ h(\delta^s) &= |s| h(\delta) \quad (s \in \mathbb{Z}). \end{aligned} \tag{2.3}$$

A linear form in logarithms is an expression of the following form:

$$\Lambda = b_1 \log \delta_1 + \cdots + b_s \log \delta_s, \tag{2.4}$$

where $\delta_1, \ldots, \delta_s$ are positive real algebraic numbers, and $b_1, \ldots, b_s$ are nonzero integers. Let $\Gamma = e^{\Lambda} - 1$. With this notation, we begin by stating the main result of Matveev [14], which leads to the following estimate.

Theorem 2.5. *Let $\mathbb{Q}(\delta_1, \ldots, \delta_s)$ be a number field of degree D over $\mathbb{Q}$. Assume that $\Gamma \neq 0$. Then we have*

$$\log|\Gamma| > -1.4 \times 30^{s+3} \times s^{4.5} \times D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s,$$

where $B \geq \max\{|b_1|, \ldots, |b_s|\}$, and for each $i = 1, \ldots, s$, $A_i \geq \max\{Dh(\delta_i), |\log(\delta_i)|, 0.16\}$.

This result is the version of Bugeaud, Mignotte, and Siksek ([20, Theorem 9.4]).

Additionally, we use a p-adic version of Laurent's result, as developed by Bugeaud and Laurent in [7, Corollary 1]. Before presenting their result, we first define some necessary concepts.

Definition 2.6. Let p be a prime number. The p-adic valuation of an integer x, denoted by $v_p(x)$, is defined as follows:

$$v_p(x) := \begin{cases} \max\{k \in \mathbb{N} : p^k \mid x\}, & \text{if } x \neq 0, \\ \infty, & \text{if } x = 0. \end{cases}$$

Furthermore, for a rational number $x = \dfrac{a}{b}$, where a and b are integers, the p-adic valuation is defined as

$$v_p(x) = v_p(a) - v_p(b).$$

The formula for $v_p(x)$ in the case of rational numbers, as given in Definition 2.6, is independent of the specific choice of integers used to represent x as a fraction. This definition implies that for any rational number x, $v_p(x) = \text{ord}_p(x)$, where $\text{ord}_p(x)$ denotes the exponent of p in the prime factorization of x. For an algebraic number δ, its p-adic valuation is defined as $v_p(\delta) := \dfrac{v_p(a_d/a_0)}{d}$, where a_0 and a_d are integers associated with δ as in Definition 2.4, and d is the degree of δ. As an example, for a rational number $x = \dfrac{a_d}{a_0}$, with a_d and a_0 coprime integers and $a_0 \geq 1$, its minimal polynomial $f(X) = a_0 X - a_d$ has degree 1, and thus $v_p(x) = v_p\left(\dfrac{a_d}{a_0}\right)$, which aligns with Definition 2.6. The p-adic valuation naturally induces a corresponding absolute value.

In a similar manner to the previous context, let δ_1 and δ_2 be algebraic numbers over $\mathbb{Q}$, considered as elements of the local field $\mathbb{K}_p := \mathbb{Q}_p(\delta_1, \delta_2)$, where $D := [\mathbb{Q}_p(\delta_1, \delta_2) : \mathbb{Q}_p]$. As in Theorem 2.5, we need to use a modified height function. Specifically, we define the adjusted height of δ_i as follows:

$$h'(\delta_i) \geq \max\left\{h(\delta_i), \frac{\log(p)}{D}\right\}, \quad \text{for} \quad i = 1, 2.$$

Lemma 2.7. *(Bugeaud and Laurent, [7]) Let p be a prime number, b_1 and b_2 be positive integers, and let δ_1 and δ_2 be multiplicatively independent algebraic numbers such that $v_p(\delta_1) = v_p(\delta_2) = 0$. Define*

$$E := \frac{b_1}{h'(\delta_2)} + \frac{b_2}{h'(\delta_1)},$$

and

$$F := \max\{\log E + \log\log p + 0.4, 10, 10\log p\}.$$

Then, the following upper bound holds:

$$v_p\left(\delta_1^{b_1}\delta_2^{b_2} - 1\right) \leq \frac{24pg}{(p-1)(\log p)^4} F^2 D^4 h'(\delta_1) h'(\delta_2),$$

where $g > 0$ is the least positive integer such that $v_p(\delta_i^g - 1) > 0$.

Applying Theorem 2.5 and Lemma 2.7, we obtain upper bounds on the variables involved. However, these bounds are too large and thus require refinement. To achieve this, we use the following result from the theory of continued fractions (see Theorem 8.2.4 in [12]).

Lemma 2.8. *(Legendre). Let μ be an irrational number, and let the continued fraction expansion of μ be given by $[a_0, a_1, a_2, \ldots]$. Let $\dfrac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i]$ for all $i \geq 0$, denote the convergents of the continued fraction of μ. Let M be a positive integer and let N be the smallest index such that $q_N > M$. Define $a(M) := \max\{a_i,\ 0 \leq i \leq N\}$. Then, for all pairs $(u, v) \in \mathbb{Z}^2_{>0}$ with $0 < v < M$, we have*

$$\left|\mu - \frac{u}{v}\right| > \frac{1}{(a(M)+2)v^2}.$$

Lemma 2.9. *[5, Lemme 2.8] Let $k \in \mathbb{N}^*$. Then* *(i)* $\nu_2(z^k - 1) \le 1 + \nu_2(k)$,

(ii) $\nu_3(z^k - 1) \le 1 + \nu_3(k)$.

Since there is no known method based on continued fractions to derive a lower bound for linear forms in more than two variables with bounded integer coefficients, we instead use a method based on the LLL algorithm, which we describe below.

2.3 Reduced Bases for Lattices and LLL-Reduction Methods

Let k be a positive integer. A subset $\mathfrak{L}$ of the real vector space $\mathbb{R}^k$ is called a *lattice* if there exists a basis $\{b_1, b_2, \ldots, b_k\}$ of $\mathbb{R}^k$ such that

$$\mathfrak{L} = \left\{ \sum_{i=1}^{k} r_i b_i \mid r_i \in \mathbb{Z} \right\}.$$

We say that $b_1, b_2, \ldots, b_k$ form a basis for $\mathfrak{L}$ or that they span $\mathfrak{L}$. The integer k is called the rank of $\mathfrak{L}$. The determinant of the lattice, denoted $\det(\mathfrak{L})$, is defined as

$$\det(\mathfrak{L}) = |\det(b_1, b_2, \ldots, b_k)|,$$

where the b_i's are written as column vectors. This determinant is a positive real number and is independent of the choice of the basis (see [8, Sect. 1.2]).

Given a set of linearly independent vectors $b_1, b_2, \ldots, b_k$ in $\mathbb{R}^k$, we consider the Gram–Schmidt orthogonalization process. This technique is used to iteratively construct orthogonal vectors b_i^* (for $1 \le i \le k$) and corresponding coefficients $\mu_{i,j}$ (where $1 \le j \le i \le k$). Specifically, the definitions are given as follows:

$$b_i^* = b_i - \sum_{j=1}^{i-1} \mu_{i,j} b_j^*, \quad \mu_{i,j} = \frac{\langle b_i, b_j^* \rangle}{\langle b_j^*, b_j^* \rangle},$$

where $\langle \cdot, \cdot \rangle$ represents the standard inner product in $\mathbb{R}^k$. Here, b_i^* is the orthogonal projection of b_i onto the orthogonal complement of the span of $b_1, \ldots, b_{i-1}$. Consequently, b_i^* is orthogonal to $b_1^*, \ldots, b_{i-1}^*$ for $1 \le i \le k$. Thus, the sequence $b_1^*, b_2^*, \ldots, b_k^*$ forms an orthogonal basis of $\mathbb{R}^k$.

Definition 2.10. A basis $\{b_1, b_2, \ldots, b_n\}$ of a lattice $\mathfrak{L}$ is called *reduced* if it satisfies the following conditions:

$$|\mu_{i,j}| \le \frac{1}{2} \quad \text{for all } 1 \le j < i \le n, \text{ and } \|b_i^* + \mu_{i,i-1} b_{i-1}^*\|^2 \ge \frac{2}{3}\|b_{i-1}^*\|^2 \quad \text{for all } 1 < i \le n,$$

where $\|\cdot\|$ denotes the Euclidean norm. Note that the constant $\frac{2}{3}$ in the second condition is not fixed and can be replaced by any real number between $\frac{1}{4}$ and 1 (see [13, Section 1]).

Let $\mathfrak{L} \subseteq \mathbb{R}^k$ be a k-dimensional lattice with a reduced basis $\{b_1, \ldots, b_k\}$, and let B denote the matrix whose columns are $b_1, \ldots, b_k$. We define the function $l(\mathfrak{L}, y)$ as follows:

$$l(\mathfrak{L}, y) = \begin{cases} \min_{x \in \mathfrak{L}} \|x - y\|, & \text{if } y \notin \mathfrak{L}, \\ \min_{x \in \mathfrak{L}\setminus\{0\}} \|x\|, & \text{if } y \in \mathfrak{L}, \end{cases}$$

where $\|\cdot\|$ represents the Euclidean norm on $\mathbb{R}^k$.

Using the LLL-algorithm, it is possible to compute a lower bound for $l(\mathfrak{L}, y)$ in polynomial time. Specifically, there exists a positive constant c_1 such that $l(\mathfrak{L}, y) \ge c_1$ (see [18, Sect. V.4]).

Lemma 2.11. *Let $y \in \mathbb{R}^k$ and $z = B^{-1}y$, where $z = (z_1, \ldots, z_k)^T$ is the transpose of z. Define σ as follows:*

(i) If $y \notin \mathfrak{L}$, let i_0 denote the largest index such that $z_{i_0} \neq 0$, then $\sigma := \{z_{i_0}\}$, where $\{\cdot\}$ represents the fractional part or distance to the nearest integer.

(ii) If $y \in \mathfrak{L}$, then $\sigma := 1$.

Additionally, let $c_2 := \max_{1\le j\le k}\left\{\frac{\|b_1\|^2}{\|b_j^*\|^2}\right\}$.

Then

$$l(\mathfrak{L}, y)^2 \ge c_2^{-1}\sigma^2\|b_1\|^2 := c_1^2.$$

In our context, we work with real numbers $\eta_0, \eta_1, \ldots, \eta_k$ that are linearly independent over $\mathbb{Q}$. Additionally, we assume the existence of two positive constants c_3 and c_4 such that

$$|\eta_0 + a_1\eta_1 + \cdots + a_k\eta_k| \le c_3 \exp(-c_4 H), \tag{2.5}$$

where the integers a_i satisfy the constraints $|a_i| \le A_i$ for some given bounds A_i (with $1 \le i \le k$). For simplicity, we define $A_0 := \max_{1\le i\le k}\{A_i\}$.

The main approach, inspired by [11], involves approximating the linear form in (2.5) using a lattice construction. Specifically, we consider the lattice $\mathfrak{L}$ generated by the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ \lfloor C\eta_1 \rfloor & \lfloor C\eta_2 \rfloor & \cdots & \lfloor C\eta_{k-1} \rfloor & \lfloor C\eta_k \rfloor \end{pmatrix},$$

where C is a sufficiently large constant, typically chosen around the order of A_0^k. Let us assume that an LLL-reduced basis $b_1, \ldots, b_k$ of $\mathfrak{L}$ has been computed and that a lower bound $l(\mathfrak{L}, y) \ge c_1$ is established, where $y := (0, 0, \ldots, -\lfloor C\eta_0 \rfloor)$. The value of c_1 can be determined using the results of Lemma 2.11.

Under these conditions, the following statement corresponds to [18, Lemma VI.1].

Lemma 2.12. *(Adapted from [18, Lemma VI.1]) Define*

$$S := \sum_{i=1}^{k-1} A_i^2 \quad \text{and} \quad T := \frac{1 + \sum_{i=1}^{k} A_i}{2}.$$

If $c_1^2 \ge T^2 + S$, then the inequality (2.5) leads to one of the following conclusions:

(i) All coefficients satisfy $a_1 = a_2 = \cdots = a_{k-1} = 0$ and $a_k = -\frac{\lfloor C\eta_0 \rfloor}{\lfloor C\eta_k \rfloor}$.

(ii) The height H is bounded by $H \le \frac{1}{c_4}\left(\log(Cc_3) - \log\left(\sqrt{c_1^2 - S} - T\right)\right)$.

2.4 Bounds for Solutions to S-unit Equations

It is easy to verify that

$$\alpha^{n-3} \le P_n \le \alpha^{n-1} \quad \text{holds for all } n \ge 1. \tag{2.6}$$

Now, if $c \ge 0$ in (1.2), then $2^x 3^y = P_n - c \le P_n \le \alpha^{n-1}$. This implies that

$$x \log 2 + y \log 3 \le (n-1)\log\alpha < (n-1)\log 1.33 < n.$$

Hence,

$$x < n \text{ and } y < n.$$

Lemma 2.13. *([2, Proposition 1]). Let* $\Delta > 10^{80}$ *be a fixed integer and assume that*

$$2^x 3^y - 2^{x_1} 3^{y_1} = \Delta. \tag{2.7}$$

Then

$$2^x 3^y < \Delta(\log \Delta)^{60 \log\log \Delta}.$$

We now present and prove the following consequence of Lemma 2.13.

Lemma 2.14. *Assume that* (n, n_1, x, x_1, y, y_1) *is a solution to* $P_n - 2^x 3^y = P_{n_1} - 2^{x_1} 3^{y_1}$*, with* $n > 664$ *and* $n > n_1$*. Then, for* $X = x \log 2 + y \log 3$*, we have*

$$n \log \alpha - 3 < X < n \log \alpha + 60\big(\log(n \log \alpha)\big)^2.$$

Proof. If $n > 664$, then $\Delta = 2^x 3^y - 2^{x_1} 3^{y_1} = P_n - P_{n_1} \geq P_n - P_{n-1} = P_{n-5} > P_{659} > \alpha^{656} > 10^{80}$. So, we apply Lemma 2.13 with $\Delta = P_n - P_{n_1} \leq P_n \leq \alpha^{n-1}$, by (2.6). This yields

$$\begin{aligned} \exp(X) &< \Delta(\log \Delta)^{60 \log\log \Delta} \\ &\leq \alpha^{n-1}\big(\log \alpha^{n-1}\big)^{60 \log\log \alpha^{n-1}} \\ &< \alpha^n (\log \alpha^n)^{60 \log\log \alpha^n} \\ &= \alpha^n (n \log \alpha)^{60 \log(n \log \alpha)}. \end{aligned} \tag{2.8}$$

On the other hand,

$$0.13\alpha^n < \alpha^{n-7} < P_{n-5} = P_n - P_{n-1} \leq P_n - P_{n_1} = 2^x 3^y - 2^{x_1} 3^{y_1} < 2^x 3^y = \exp(X). \tag{2.9}$$

Combining (2.8) and (2.9), we get $0.13\alpha^n < \exp(X) < \alpha^n (n \log \alpha)^{60 \log(n \log \alpha)}$, and taking logarithms on both sides gives

$$n \log \alpha - 3 < X < n \log \alpha + 60\big(\log(n \log \alpha)\big)^2.$$

□

3 Proof of Theorem 1.1

For technical reasons, we assume that $n > 664$ and aim to establish an upper bound for n. Let (n, x, y), (n_1, x_1, y_1), (n_2, x_2, y_2) and (n_3, x_3, y_3) be elements of $\mathbb{Z}^3_{\geq 0}$ such that

$$P_n - 2^x 3^y = P_{n_1} - 2^{x_1} 3^{y_1} = P_{n_2} - 2^{x_2} 3^{y_2} = P_{n_3} - 2^{x_3} 3^{y_3}. \tag{3.1}$$

Without loss of generality, we may assume that $n > n_1 > n_2 > n_3$.

Lemma 3.1. *Let* $c \geq 1$ *be such that the Diophantine equation* (1.2) *has at least two representations as*

$$c = P_n - 2^x 3^y = P_{n_1} - 2^{x_1} 3^{y_1}.$$

Then, we have the following bound:

$$n - n_1 < 1.3 \cdot 10^{16} \log n.$$

Proof. Since $c \geq 1$, we have $P_n > 2^x 3^y$ and $P_{n_1} > 2^{x_1} 3^{y_1}$. We return to equation (1.2) and rewrite it as

$$\begin{aligned} P_n - 2^x 3^y &= P_{n_1} - 2^{x_1} 3^{y_1} \\ A_\alpha \alpha^n + A_\beta \beta^n + A_\gamma \gamma^n - 2^x 3^y &= A_\alpha \alpha^{n_1} + A_\beta \beta^{n_1} + A_\gamma \gamma^{n_1} - 2^{x_1} 3^{y_1} \\ A_\alpha \alpha^n - 2^x 3^y &= A_\alpha \alpha^{n_1} + A_\beta \beta^{n_1} + A_\gamma \gamma^{n_1} - A_\beta \beta^n - A_\gamma \gamma^n - 2^{x_1} 3^{y_1} \\ &< A_\alpha \alpha^{n_1} + A_\beta \beta^{n_1} + A_\gamma \gamma^{n_1} \\ &< A_\alpha \alpha^{n_1} + 2 \\ &< 4 A_\alpha \alpha^{n_1}, \end{aligned}$$

for every $n > 664$. Therefore, we deduce that

$$\left|2^x 3^y A_\alpha^{-1} \alpha^{-n} - 1\right| < 4\alpha^{-(n-n_1)}. \tag{3.2}$$

We proceed by applying Theorem 2.5 to the left-hand side of 3.2. Let

$$\Gamma_0 := 2^x 3^y A_\alpha^{-1} \alpha^{-n} - 1.$$

Notice that $\Gamma_0 \neq 0$. Indeed, if $\Gamma_0 = 0$, then it would correspond to $A_\beta \beta^n = 2^x 3^y$ by applying a Galois automorphism that maps α to β, which leads to a contradiction, since $|A_\beta \beta^n| < 1$ while $2^x 3^y \geq 1$ for all $x, y \geq 0$. Next, we consider the field $K := \mathbb{Q}(\alpha)$, which has degree $D = 3$. Here $s := 4$, and set

$$\delta_1 := 2, \quad \delta_2 := 3, \quad \delta_3 := A_\alpha, \quad \delta_4 := \alpha,$$

and

$$b_1 := x, \quad b_2 := y, \quad b_3 := -1, \quad b_4 := -n.$$

Next, we have

$$\max\{|b_1|, |b_2|, |b_3|, |b_4|\} = \max\{x, y, 1, n\} \leq n.$$

Since $|A_\alpha|, |A_\beta|, |A_\gamma| < 1$, it follows that $h(A_\alpha) = \dfrac{1}{3}\log 23$. Therefore, we can take

$$A_1 := Dh(\delta_1) = 3\log 2, \qquad A_2 := Dh(\delta_2) = 3\log 3,$$

$$A_3 := Dh(\delta_3) = \log 23, \qquad A_4 := Dh(\delta_4) = \log \alpha.$$

By applying Theorem 2.5, we get

$$\log|\Gamma_0| > -1.4 \cdot 30^7 \cdot 4^{4.5} \cdot 3^2 (1 + \log 3)(1 + \log n)(3\log 2)(3\log 3)(\log 23)(\log \alpha).$$

Simplifying this gives

$$\log|\Gamma_0| > -3.58 \cdot 10^{15} \log n. \tag{3.3}$$

This inequality holds for all $n > 664$. Comparing (3.2) with (3.3), we deduce

$$(n - n_1)\log \alpha - \log 4 < 3.58 \cdot 10^{15} \log n,$$

which implies

$$n - n_1 < 1.3 \cdot 10^{16} \log n.$$

□

Next, we state and prove the following result.

Lemma 3.2. *Let* $c \geq 1$, $X := x\log 2 + y\log 3$ *and* $X_1 := x_1 \log 2 + y_1 \log 3$. *Then*

$$X - X_1 < 4.4 \cdot 10^{30} (\log n)^2.$$

Proof. Returning to equation (1.2), we rewrite it in the form

$$A_\alpha \alpha^n - A_\alpha \alpha^{n_1} - 2^x 3^y = A_\beta \beta^{n_1} - A_\beta \beta^n + A_\gamma \gamma^{n_1} - A_\gamma \gamma^n - 2^{x_1} 3^{y_1}.$$

Factoring terms, we have

$$A_\alpha \alpha^{n_1}\left(\alpha^{n-n_1} - 1\right) 2^{-x} 3^{-y} - 1 = \frac{A_\beta(\beta^{n_1} - \beta^n)}{2^x 3^y} + \frac{A_\gamma(\gamma^{n_1} - \gamma^n)}{2^x 3^y} - \frac{2^{x_1} 3^{y_1}}{2^x 3^y}.$$

Taking the absolute values, we obtain

$$|\Gamma_1| = \left|\frac{A_\alpha \alpha^{n_1}(\alpha^{n-n_1} - 1)}{2^x 3^y} - 1\right| \leq \frac{1}{\exp(X)} + \frac{1}{\exp(X)} + \frac{1}{\exp(X - X_1)} \leq 3\exp(-(X - X_1)). \tag{3.4}$$

Notice that $\Gamma_1 \neq 0$, otherwise we would have $\dfrac{A_\alpha(\alpha^n - \alpha^{n_1})}{2^x 3^y} = 1$. Taking algebraic conjugates leads to $1 = \left|\frac{A_\beta(\beta^n - \beta^{n_1})}{2^x 3^y}\right| < 1$, which is a contradiction. Hence, we must have $\Gamma_1 \neq 0$. As before, we work in the field $\mathbb{Q}(\alpha)$, which has degree $D = 3$. Here $s := 4$,

$$\delta_1 := 2, \quad \delta_2 := 3, \quad \delta_3 := \alpha, \quad \delta_4 := A_\alpha(\alpha^{n-n_1} - 1),$$

$$b_1 := -x, \quad b_2 := -y, \quad b_3 := n_1, \quad b_4 := 1.$$

Next, $\max\{|b_1|, |b_2|, |b_3|, |b_4|\} = \max\{x, y, 1, n_1\} < n$, allowing us to set $B := n$. As previously, we can choose $A_1 := 3\log 2$, $A_2 := 3\log 3$ and $A_3 := \log\alpha$. Furthermore,

$$\begin{aligned} 3h(\delta_4) &= 3h(A_\alpha(\alpha^{n-n_1} - 1)) \\ &\leq 3h(A_\alpha) + 3h(\alpha^{n-n_1} - 1) \\ &\leq \log 23 + 3(n - n_1)h(\alpha) + 3\log 2 \\ &\leq \log 23 + 1.3 \cdot 10^{16} \log(n)\log(\alpha) + 3\log 2 \\ &< 3.7 \cdot 10^{15} \cdot \log(n), \end{aligned}$$

where we have used Lemma (3.1). Therefore, we take $A_4 := 3.7 \cdot 10^{15} \log n$. Then, by Theorem 2.5

$$\begin{aligned} \log|\Gamma_1| &> -1.4 \cdot 30^7 \cdot 4^{4.5} \cdot 3^2 (1 + \log 3)(1 + \log n)(3\log 2)(3\log 3)(\log\alpha)(3.7 \cdot 10^{15} \log n) \\ &> -4.3 \cdot 10^{30} (\log n)^2, \end{aligned} \tag{3.5}$$

for all $n > 664$. Comparing (3.4) and (3.5), we obtain

$$X - X_1 < 4.4 \cdot 10^{30} (\log n)^2. \tag{3.6}$$

□

To continue, we define

$$x_{\text{min}} := \min\{x, x_1, x_2\}, \quad y_{\text{min}} := \min\{y, y_1, y_2\}.$$

We state and prove the following result.

Lemma 3.3. *Assume that $c \geq 1$. Then, either*

$$x_{\text{min}}, y_{\text{min}} < 10^{22} (\log n)^3,$$

or

$$n < 22000.$$

Proof. Once again, we return to equation (1.2), assuming it admits two solutions (n, x, y) and (n_1, x_1, y_1). We rewrite it as

$$A_\alpha\alpha^n - A_\alpha\alpha^{n_1} + A_\beta\beta^n - A_\beta\beta^{n_1} + A_\gamma\gamma^n - A_\gamma\gamma^{n_1} = 2^x 3^y - 2^{x_1} 3^{y_1}.$$

Factoring terms, we get

$$A_\alpha\alpha^n \left(1 - \alpha^{n_1 - n}\right) + A_\beta\beta^n \left(1 - \beta^{n_1 - n}\right) + A_\gamma\gamma^n \left(1 - \gamma^{n_1 - n}\right) = 2^x 3^y - 2^{x_1} 3^{y_1}. \tag{3.7}$$

Suppose now that a third solution (n_2, x_2, y_2) to equation (1.2) exists. Then, we may derive a relation analogous to (3.7), which takes the form

$$A_\alpha\alpha^n \left(1 - \alpha^{n_2 - n}\right) + A_\beta\beta^n \left(1 - \beta^{n_2 - n}\right) + A_\gamma\gamma^n \left(1 - \gamma^{n_2 - n}\right) = 2^x 3^y - 2^{x_2} 3^{y_2}. \tag{3.8}$$

Next, we eliminate the term $A_\gamma \gamma^n$ from equations (3.7) and (3.8) to obtain

$$\begin{aligned} & A_\alpha \alpha^n \left[\left(1-\alpha^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right)-\left(1-\alpha^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)\right] \\ & + A_\beta \beta^n \left[\left(1-\beta^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right)-\left(1-\beta^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)\right] \\ & = \left(2^x 3^y - 2^{x_1} 3^{y_1}\right)\left(1-\gamma^{n_2-n}\right)-\left(2^x 3^y - 2^{x_2} 3^{y_2}\right)\left(1-\gamma^{n_1-n}\right). \end{aligned} \tag{3.9}$$

We begin by proving that the left-hand side of this expression is nonzero. Suppose, for contradiction, that it vanishes. Then we would have

$$\left(2^x 3^y - 2^{x_1} 3^{y_1}\right)\left(1-\gamma^{n_2-n}\right)-\left(2^x 3^y - 2^{x_2} 3^{y_2}\right)\left(1-\gamma^{n_1-n}\right)=0,$$

so that

$$\frac{1-\gamma^{n_1-n}}{1-\gamma^{n_2-n}}=\frac{2^x 3^y - 2^{x_1} 3^{y_1}}{2^x 3^y - 2^{x_2} 3^{y_2}}.$$

We begin by taking algebraic conjugates, which gives us the following relation:

$$\frac{1-\gamma^{n_1-n}}{1-\gamma^{n_2-n}}=\frac{1-\alpha^{n_1-n}}{1-\alpha^{n_2-n}}. \tag{3.10}$$

Moreover, since $n > n_1 > n_2$, it follows that $n - n_2 > n - n_1$, and by Lemma 2.3, tells us that (3.10) has no integer solutions for $n - n_1$ and $n - n_2$, with $n - n_2 > n - n_1$. Hence, the left-hand side of equation (3.9) is nonzero. Note also that the term $(1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})$ appearing on the right-hand side of equation (3.9) is nonzero. Indeed, if it were zero, we would have

$$\frac{1-\gamma^{n_1-n}}{1-\gamma^{n_2-n}}=\frac{1-\beta^{n_1-n}}{1-\beta^{n_2-n}},$$

a relation that is an algebraic conjugate of equation (3.10), which is known to have no integer solutions. Hence, the above expression cannot vanish. We now return to equation (3.9), which we rewrite as

$$\begin{aligned} & \frac{A_\alpha}{A_\beta}\left(\frac{\alpha}{\beta}\right)^n \left(\frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n})-(1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})}\right)+1 \\ & = \frac{(2^x 3^y - 2^{x_1} 3^{y_1})(1-\gamma^{n_2-n})-(2^x 3^y - 2^{x_2} 3^{y_2})(1-\gamma^{n_1-n})}{A_\beta \beta^n\left[(1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})\right]} \\ & = \frac{2^{x_{\min}} 3^{y_{\min}} \cdot A}{A_\beta \beta^n\left[(1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})\right]}, \end{aligned}$$

where

$$\begin{aligned} A \quad := \quad & \left(2^{x-x_{min}} 3^{y-y_{\min}} - 2^{x_1-x_{min}} 3^{y_1-y_{min}}\right)\left(1-\gamma^{n_2-n}\right) \\ & - \left(2^{x-x_{\min}} 3^{y-y_{\min}} - 2^{x_2-x_{min}} 3^{y_2-y_{min}}\right)\left(1-\gamma^{n_1-n}\right). \end{aligned}$$

Since $v_p(\beta) = 0$ and $v_p(A_\beta) = 0$ for $p = 2, 3$, we have

$$\begin{aligned} & v_2\left(\frac{A_\alpha}{A_\beta}\left(\frac{\alpha}{\beta}\right)^n \cdot \frac{\left(1-\alpha^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right)-\left(1-\alpha^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})}+1\right) \\ & = v_2\left(\frac{2^{x_{min}} 3^{y_{min}} \cdot A}{A_\beta \beta^n (1-\beta^{n_1-n})(1-\gamma^{n_1-n})-(1-\beta^{n_2-n})(1-\gamma^{n_2-n})}\right) \\ & = x_{\min} + \nu_2(A) - \nu_2((1-\beta^{n_1-n})(1-\gamma^{n_1-n})-(1-\beta^{n_2-n})(1-\gamma^{n_2-n})). \end{aligned}$$

Since A is an algebraic integer, it follows that $\nu_2(A) \geq 0$. Consequently, we obtain

$$\begin{aligned} x_{\min} \quad & \leq \nu_2\left(\frac{A_\alpha}{A_\beta}\left(\frac{\alpha}{\beta}\right)^n \frac{\left(1-\alpha^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right)-\left(1-\alpha^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)}{\left(1-\beta^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right)-\left(1-\beta^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)}+1\right) \\ & + \nu_2((1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})). \end{aligned} \tag{3.11}$$

Similarly,

$$\begin{aligned} y_{\min} &\leq \nu_3 \left(\frac{A_\alpha}{A_\beta} \left(\frac{\alpha}{\beta}\right)^n \frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n}) - (1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})} + 1 \right) \\ &+\nu_3((1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})). \end{aligned} \tag{3.12}$$

Next, we estimate

$$\nu_p\left(\left(1-\beta^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right) - \left(1-\beta^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)\right), \tag{3.13}$$

for $p = 2, 3$. Well, the number shown in (3.13) is an algebraic integer all whose conjugates are bounded in absolute value by

$$2\left(1+\alpha^{(n-n_1)/2}\right)\left(1+\alpha^{(n-n_2)/2}\right) < \alpha^{(n-n_1)/2} \cdot \alpha^{(n-n_2)/2+8} \leq \alpha^{n-n_2+8}.$$

Therefore, the constant term of the minimal polynomial of the number appearing inside the p-adic valuation in (3.13) is bounded above by $\alpha^{6(n-n_2)+48}$. It follows that

$$\nu_p\left(\left(1-\beta^{n_1-n}\right)\left(1-\gamma^{n_2-n}\right) - \left(1-\beta^{n_2-n}\right)\left(1-\gamma^{n_1-n}\right)\right) \leq \frac{(6(n-n_2)+48)\log\alpha}{\log p}.$$

Applying Lemma 3.1, we obtain that the above quantities are at most

$$3.2 \cdot 10^{16} \log n \quad (p=2) \quad \text{and} \quad 2 \cdot 10^{16} \log n \quad (p=3).$$

At this point, the relations (3.11) and (3.12) take the form

$$x_{\min} \leq \nu_2 \left(\frac{A_\alpha}{A_\beta} \cdot \left(\frac{\alpha}{\beta}\right)^n \cdot \frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n}) - (1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})} + 1 \right) + 3.2 \cdot 10^{16} \log n. \tag{3.14}$$

Similarly,

$$y_{\min} \leq \nu_3 \left(\frac{A_\alpha}{A_\beta} \cdot \left(\frac{\alpha}{\beta}\right)^n \cdot \frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n}) - (1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n}) - (1-\beta^{n_2-n})(1-\gamma^{n_1-n})} + 1 \right) + 2 \cdot 10^{16} \log n. \tag{3.15}$$

We apply Theorem 2.7 to (3.14) using the field $\mathbb{Q}(\alpha,\beta)$ of degree $D := 6$. Since $h(\delta_1) \leq 2h(\alpha) = \frac{2}{3}\log(\alpha)$ for all cases $p = 2, 3$, we can choose $h'(\delta_1) := \frac{2}{3}\log\alpha$ (note that $\frac{2}{3}\log\alpha > \frac{\log p}{D}$ for $p = 2, 3$). Furthermore,

$$\begin{aligned} h(\delta_2) &\leq 2h(A_\alpha) + h\left(1-\alpha^{n_1-n}\right) + h\left(1-\gamma^{n_2-n}\right) + h\left(1-\alpha^{n_2-n}\right) + h\left(1-\gamma^{n_1-n}\right) \\ &\quad + h\left(1-\beta^{n_1-n}\right) + h\left(1-\gamma^{n_2-n}\right) + h\left(1-\beta^{n_2-n}\right) + h\left(1-\gamma^{n_1-n}\right) + 2\log 2 \\ &\leq 2h(A_\alpha) + 4h\left(1-\alpha^{n_1-n}\right) + 4h\left(1-\alpha^{n_2-n}\right) + 2\log 2 \\ &\leq 4 + \tfrac{4}{3}\left(\log(\alpha^{(n-n_1)/2}+1) + \log(\alpha^{(n-n_2)/2}+1)\right) \\ &< \tfrac{2}{3}\log 23 + \tfrac{8}{3}(n-n_2)\log\alpha + 10\log 2 \\ &< 10^{16} \cdot \log\alpha \cdot \log n. \end{aligned}$$

This follows from Lemma 3.1 and the condition $n > 664$. Therefore,

$$E = \frac{b_1}{h'(\delta_2)} + \frac{b_2}{h'(\delta_1)} = \frac{n}{h'(\delta_1)} + \frac{1}{h'(\delta_2)} < n^2.$$

We now obtain

$$F = \max\{\log E + \log\log p + 0.4, 10, 10\log p\} \leq \max\left\{\log n^2 + \log\log p + 0.4, 10, 10\log p\right\}.$$

If $n > 22000$, then $F < 1 + \log n^2$ in both cases $p = 2$ and $p = 3$. Therefore, Lemma 2.7 gives

$$\begin{aligned}
\nu_p\left(\frac{A_\alpha}{A_\beta}\left(\frac{\alpha}{\beta}\right)^n \frac{(1-\alpha^{n_1-n})(1-\gamma^{n_2-n})-(1-\alpha^{n_2-n})(1-\gamma^{n_1-n})}{(1-\beta^{n_1-n})(1-\gamma^{n_2-n})-(1-\beta^{n_2-n})(1-\gamma^{n_1-n})}\right) &\leq \frac{24pg}{(p-1)(\log p)^4}F^2D^4h'(\lambda_1)h'(\lambda_2)\\
&< \frac{24pg}{(p-1)(\log p)^4}(1+2\log n)^2\cdot 6^4\cdot 10^{16}\cdot\log n\\
&< \frac{24pg}{(p-1)(\log p)^4}\cdot 4(\log n)^2\cdot\left(1+\frac{1}{2\log 22000}\right)\\
&\quad\cdot\log\alpha\cdot 10^{16}\log n\\
&= \frac{24pg}{(p-1)(\log p)^4}\cdot 4(\log n)^3\cdot\left(1+\frac{1}{2\log 22000}\right)\\
&\quad\cdot\log\alpha\cdot 10^{16}\\
&< \frac{pg\cdot 2.6\times 10^{20}}{(p-1)(\log p)^4}(\log n)^3.
\end{aligned}$$

Hence, inequalities (3.14) and (3.15) become

$$x_{\min} < \frac{2.6\times 10^{20}\times 2\times 3}{(2-1)(\log 2)^4}(\log n)^3 + 3.2\times 10^{16}\log n < 10^{22}(\log n)^3$$

and

$$y_{\min} < \frac{2.6\times 10^{20}\times 3\times 13}{(3-1)(\log 3)^4}(\log n)^3 + 2\times 10^{16}\log n < 6\times 10^{21}(\log n)^3.$$

In the above, we used $g = 3$ when $p = 2$ and $g = 13$ when $p = 3$. □

Finally, we consider a fourth and fifth solution (n_3, x_3, y_3) and (n_4, x_4, y_4), with $n > n_1 > n_2 > n_3 > n_4$, and determine an absolute bound for n. We state the following result.

Lemma 3.4. *If $c \geq 1$ and $n > 22000$, then*

$$n < 1.5\times 10^{38}, \quad x < 6.1\times 10^{37}, \quad y < 3.9\times 10^{37}.$$

Proof. Lemma 3.3 asserts that, for any set of three solutions, the smallest values of x and y are bounded above by $10^{22}(\log n)^3$. Therefore, in a set of five solutions, at most two solutions can have an x-value greater than $10^{22}(\log n)^3$ and similarly, at most two solutions can have a y-value exceeding this bound. As a result, at least one solution must have both x and y within this range. In particular, this means that the smallest solution satisfies these constraints. Thus

$$X_4 = x_4\log 2 + y_4\log 3 < (\log 2 + \log 3)\cdot 10^{22}(\log n)^3 < 1.8\cdot 10^{22}(\log n)^3.$$

With Lemmas 2.14 and 3.2, we get

$$\begin{aligned}
n\log\alpha - 3 &< X = X_4 + (X - X_4)\\
&< 1.8\cdot 10^{22}(\log n)^3 + 3\cdot 4.4\cdot 10^{30}(\log n)^2\\
&< 1.33\cdot 10^{31}(\log n)^3,
\end{aligned}$$

which implies

$$\frac{n}{(\log(n))^3} < 4.77\cdot 10^{31}. \tag{3.16}$$

We apply Lemma 2.2 to inequality (3.16) above with $z = n$, $m = 3$ and $T = 4.77\cdot 10^{31}$. Since $T > (4\cdot 3^2)^3$, we get

$$n < 2^3\cdot 4.77\cdot 10^{31}(\log 4.77\cdot 10^{31})^3 < 1.5\cdot 10^{38}.$$

Furthermore, by Lemma 2.14, we have the following inequality:

$$\begin{aligned} X &< n\log\alpha + 60(\log(n\log\alpha))^2, \\ x\log 2 + y\log 3 &< 1.5\cdot 10^{38}\log\alpha + 60(\log(1.5\cdot 10^{38}\log\alpha))^2, \\ &< 4.22\cdot 10^{37}. \end{aligned}$$

This gives

$$x < 6.1\cdot 10^{37} \text{ and } y < 3.85\cdot 10^{37}.$$

□

3.1 Reduction of the Upper Bound on n

In this subsection, we utilize the LLL-reduction method, continued fraction theory, and p-adic reduction techniques (as discussed in [15]) to derive a relatively small upper bound for n.
To start, we revisit equation (3.2). Assuming that $n - n_1 \geq 8$, we can express

$$|\Lambda_0| = \left|x\log 2 + y\log 3 - n\log\alpha - \log A_\alpha\right| \leq \frac{3}{2}|\Gamma_0| < 6\alpha^{-(n-n_1)},$$

where we applied Lemma 2.1 with $n - n_1 \geq 8$ and $x = \Gamma_0$. Consequently, we examine the approximation lattice

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ C\log 2 & C\log 3 & C\log\alpha \end{pmatrix},$$

with $C := 10^{115}$ and choose $v := (0, 0, \lfloor -C\log A_\alpha)\rfloor$. Now, by Lemma 2.11, we get

$$l(\mathfrak{L}, y)^2 \geq c_2^{-1}\sigma^2\|b_{1]}\|^2 := c_1^2 = 2.954\cdot 10^{76}.$$

Moreover, by Lemma 3.4, we have

$$x < A_1 := 6.1\cdot 10^{37}, \quad y < A_2 := 3.9\cdot 10^{37}, \quad n - 1 < A_3 := 1.5\cdot 10^{38}.$$

So, Lemma 2.12 gives $S = 5.242\cdot 10^{75}$ and $T = 1.25\cdot 10^{38}$. Since $c_1^2 \geq T^2 + S$, then choosing $c_3 := 6$ and $c_4 := \log\alpha$, we get

$$n - n_1 \leq \frac{1}{\log\alpha}\left(\log(10^{115}\cdot 6) - \log\left(\sqrt{2.954\cdot 10^{76} - 5.242\cdot 10^{75}} - 1.25\cdot 10^{38}\right)\right),$$

then

$$n - n_1 \leq 641.$$

We now return to equation (3.4) and suppose that $X - X_1 \geq 2$. Then we can express

$$|\Lambda_1| = \left|n_1\log\alpha + \log A_\alpha(\alpha^{n-n_1} - 1) - x\log 2 - y\log 3\right| < 4.5\exp(-(X - X_1)),$$

where we applied Lemma 2.1 in conjunction with the inequality $\exp(X - X_1) \geq \exp(2) > 6$. Consequently, we proceed with the same approximation lattice

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ C\log 2 & C\log 3 & C\log\alpha \end{pmatrix},$$

with $C := 10^{116}$, and we take $v := (0, 0, -\lfloor C\log A_\alpha(\alpha^{n-n_1} - 1)\rfloor)$. For all values $1 \leq n - n_1 \leq 641$, the constant C is sufficiently large, allowing us to apply 2.11. By Lemma 2.12, we obtain

$$l(\mathfrak{L}, y)^2 > c_1^2 = 1.6\cdot 10^{79}.$$

Moreover, applying Lemma 3.4, we also obtain

$$x < A_1 := 6.1 \cdot 10^{37}, \quad y < A_2 := 3.9 \cdot 10^{37}, \quad n < A_3 := 1.5 \cdot 10^{38}.$$

Thus, Lemma 2.12 still yields the same values for S and T as previously. Since $c_1^2 \geq T^2 + S$, we now choose $c_3 := 4.5$ and $c_4 := 1$, which leads to the bound $X - X_1 \leq 177$.
We now aim to derive improved bounds on $x_{\min}$ and $y_{\min}$ using p-adic reduction techniques, following the approach of [15]. Returning to equation (1.2), we suppose it admits two solutions (n, x, y) and (n_1, x_1, y_1). We rewrite this equation as

$$c = P_n - 2^x 3^y = P_{n_1} - 2^{x_1} 3^{y_1}.$$

To determine $\nu_p(P_n - P_{n_1})$, observe from the above identity that

$$\begin{aligned} \nu_p(P_n - P_{n_1}) &= \nu_p(2^x 3^y - 2^{x_1} 3^{y_1}) \\ &= \nu_p\left(2^{x_{\min}} 3^{y_{\min}} \left(2^{x - x_{\min}} 3^{y - y_{\min}} - 2^{x_1 - x_{\min}} 3^{y_1 - y_{\min}}\right)\right) \\ &= \nu_p\left(2^{x_{\min}} 3^{y_{\min}}\right) + \nu_p\left(2^{x - x_{\min}} 3^{y - y_{\min}} - 2^{x_1 - x_{\min}} 3^{y_1 - y_{\min}}\right). \end{aligned}$$

Thus, we have

$$\begin{aligned} \nu_p(2^{x_{\min}} 3^{y_{\min}}) &= \nu_p(P_n - P_{n_1}) - \nu_p\left(2^{x - x_{\min}} 3^{y - y_{\min}} - 2^{x_1 - x_{\min}} 3^{y_1 - y_{\min}}\right) \\ &\leq \nu_p(P_n - P_{n_1}). \end{aligned}$$

Next, we determine $\nu_p(P_n - P_{n_1})$ for all $n - n_1 \geq 1$, $n < 1.5 \cdot 10^{38}$ and $p = 2, 3$. We illustrate the method for $p = 2$, and subsequently automate the computation using MAPLE. Since $n < 1.5 \cdot 10^{38} < 2^{127}$, the binary representation of n consists of at most 127 digits. Let $d := n - n_1 \leq 641$, based on the results of the above reduction. Therefore, we need an upper bound for $\nu_2(P_n - P_{n_1}) = \nu_2(P_{n_1+d} - P_{n_1})$, but since $n_1 < n$, we instead bound

$$\nu_2(P_{n+d} - P_n), \quad d \in [1, 641], \quad n < 1.5 \cdot 10^{38}.$$

The Padovan sequence is periodic modulo 2^8 with period $7 \cdot 2^7 = 896 < 1000$. Using a simple program in MAPLE, we checked all $d \in [1, 641]$ for which there exists $n \leq 1000$ such that $2^8 \mid P_{n+d} - P_n$, and we see that all d in this interval satisfy this condition.
We illustrate the procedure with a specific example by choosing $d = 18$. Our aim is to determine a value $n_0(d) \in [1, 7 \cdot 2^7]$ such that for $n = n_0(d)$, we have $\nu_2(P_{n+d} - P_n) \geq 8$. This value is unique in this case, and it is $n_0(d) = 671$. Therefore, any $n < 1.5 \cdot 10^{38}$ satisfying $\nu_2(P_{n+d} - P_n) \geq 8$ must necessarily be of the form:

$$n = 671 + 7 \cdot 2^7 z \quad \text{for some } z \in \mathbb{Z}.$$

Our next objective is to determine a value of z for which $\nu_2(P_{n+18} - P_n)$ attains its maximum. To this end, we use the Binet formula for the Padovan sequence and obtain

$$\begin{aligned} P_{n+18} - P_n &= A_\alpha \alpha^{n+18} + A_\beta \beta^{n+18} + A_\gamma \gamma^{n+18} - (A_\alpha \alpha^n + A_\beta \beta^n + A_\gamma \gamma^n) \\ &= (\alpha^{18} - 1) A_\alpha \alpha^n + (\beta^{18} - 1) A_\beta \beta^n + (\gamma^{18} - 1) C_\gamma \gamma^n \\ &= (\alpha^{18} - 1) A_\alpha \alpha^{671 + 7 \cdot 2^7 z} + (\beta^{18} - 1) A_\beta \beta^{671 + 7 \cdot 2^7 z} + (\gamma^{18} - 1) A_\gamma \gamma^{671 + 7 \cdot 2^7 z} \\ &= (\alpha^{18} - 1) A_\alpha \alpha^{671} \exp_2(2^7 z \log(\alpha^7)) + (\beta^{18} - 1) A_\beta \beta^{671} \exp_2(2^7 z \log_2(\beta^7)) \\ &\quad + (\gamma^{18} - 1) A_\gamma \gamma^{671} \exp_2(2^7 z \log(\gamma^7)). \end{aligned}$$

In the above, we worked with $\log_2 \alpha^7$ (and similarly with α replaced by β and γ), which is well-defined, in contrast to $\log_2 \alpha$, which is not. Indeed, we have

$$\log_2 \alpha^7 = \log_2\left(1 - (1 - \alpha^7)\right) = -\sum_{n \geq 1} \frac{\left(1 - \alpha^7\right)^n}{n} = -\sum_{n \geq 1} \frac{\left(-2(\alpha^2 + \alpha)\right)^n}{n}. \tag{3.17}$$

On the right-hand side, we have

$$\begin{aligned}\left|\frac{\left(-2(\alpha^2+\alpha)\right)^n}{n}\right|_2 &= 2^{-(\nu_2(-2(\alpha^2+\alpha))^n-\nu_2(n))}\\ &= 2^{-n+\nu_2(n)}\\ &\leq 2^{-n+\frac{\log n}{\log 2}}.\end{aligned}$$

This shows that the expression on the right-hand side of (3.17) converges. Regarding the exponent in the exponential function, we have

$$\nu_2\left(2^7 z\log_2\alpha^7\right)\geq \nu_2(2^8 z)\geq 8.$$

Therefore,

$$\left|2^7 z\log_2\alpha^7\right|_2\leq 2^{-8}<2^{-1}.$$

Hence, the exponential in this input is convergent 2-adically. The same reasoning applies when α is replaced by β or γ. We now truncate the series in the logarithmic expression at $n=120$, and define

$$P:=-\sum_{n=1}^{120}\frac{(1-\alpha^7)^n}{n},$$

such that

$$\log_2\alpha^7 = P-\sum_{n\geq 121}\frac{(1-\alpha^7)^n}{n}.$$

Since $n-\nu_2(n)\geq n-\dfrac{\log n}{\log 2}$, and the function $n-\dfrac{\log n}{\log 2}$ exceeds 121 for all $n\geq 128$, it follows that $n-\nu_2(n)\geq 121$ whenever $n\geq 128$. Thus $\log_2\alpha^7=P+u$, where $\nu_2(u)\geq 121$. We therefore have $2^7 z\log_2\alpha^7 = 2^7 zP+2^7 zu$, so that

$$\exp_2(2^7 z\log_2\alpha^7)=\exp_2(2^7 z\cdot P+2^7 z\cdot u)=\exp_2(2^7 z\cdot P)\exp_2(2^7 z\cdot u).$$

For the exponential function, recall

$$\exp_2(y)=1+y+\frac{y^2}{2}+\cdots+\frac{y^n}{n!}+\cdots.$$

If $\nu_2(y)\geq 2$ and $n\geq 2$, we have

$$\nu_2\left(\frac{y^n}{n!}\right)=n\nu_2(y)-\nu_2(n!)\geq n\nu_2(y)-(n-\sigma_2(n)),$$

where $\sigma_2(n)$ is the sum of the binary digits of n. Since $\sigma_2(n)>0$, it follows that

$$\nu_2\left(\frac{y^n}{n!}\right)>n(\nu_2(y)-1)\geq\nu_2(y),$$

where the last inequality holds because it is equivalent to $\nu_2(y)\geq\dfrac{n}{n-1}$, which is true since $\nu_2(y)\geq 2\geq\dfrac{n}{n-1}$ for all $n\geq 2$.

In the sequence of equalities and inequalities above, $\sigma_2(n)$ denotes the sum of the binary digits of n. Consequently, we have

$$\exp_2(y)\equiv 1\pmod{2^{\nu_2(y)}},$$

provided $\nu_2(y)\geq 2$. Therefore

$$\exp_2(2^7 zu)\equiv 1\pmod{2^{\nu_2(2^7\cdot zu)}}\equiv 1\pmod{2^{7+\nu_2(u)}}\equiv 1\pmod{2^{128}}.$$

Thus

$$\exp_2(2^7 z \log_2 \alpha^7) \equiv \exp_2(2^7 zP) \pmod{2^{128}} \equiv \sum_{k\geq 0} \frac{(2^7 zP)^k}{k!} \pmod{2^{128}}.$$

Indeed,

$$\begin{aligned} \nu_2\left(\frac{(2^7 zP)^k}{k!}\right) &= k\nu_2(2^7 zP) - \nu_2(k!) \\ &\geq k(7 + \nu_2(z) + \nu_2(P)) - (k - \sigma_2(k)) \\ &\geq k(7 + \nu_2(P)) - (k - \sigma_2(k)) \\ &> 7k, \end{aligned}$$

since $\sigma_2(k) \geq 1$ and $\nu_2(P) \geq 1$, the resulting values are at least $7 \cdot 19 = 133$, which is greater than 128 for $k \geq 19$. Therefore, it is sufficient to truncate the series at $k = 18$, and we may write

$$\exp_2\left(2^7 z \log_2 \alpha^7\right) = \sum_{k=0}^{18} \frac{(2^7 zP)^k}{k!} \pmod{2^{128}}.$$

An analogous reasoning applies when α is replaced by β or γ, allowing us to write

$$Q := -\sum_{n=1}^{120} \frac{\left(1-\beta^7\right)^n}{n} \quad \text{and} \quad R := -\sum_{n=1}^{120} \frac{\left(1-\gamma^7\right)^n}{n}. \tag{3.18}$$

So that

$$\exp_2\left(2^7 z \log_2 \beta^7\right) = \sum_{k=0}^{18} \frac{(2^7 z \cdot Q)^k}{k!} \pmod{2^{128}},$$

and

$$\exp_2\left(2^7 z \log_2 \gamma^7\right) = \sum_{k=0}^{18} \frac{(2^7 z \cdot R)^k}{k!} \pmod{2^{128}}.$$

Thus

$P_{n+18} - P_n \equiv$

$$\sum_{k=0}^{18} \frac{\left(\alpha^{18}-1\right) A_\alpha \alpha^{671} (2^7 z \cdot P)^k + \left(\beta^{18}-1\right) A_\beta \beta^{671} (2^7 z \cdot Q)^k + \left(\gamma^{18}-1\right) A_\gamma \gamma^{671} (2^7 z \cdot R)^k}{k!} \pmod{2^{128}}.$$

The expression on the right-hand side is a polynomial of degree 18 in z, with rational coefficients that are in fact 2-adic integers—that is, their numerators are odd. We will demonstrate that, within our specified range, this polynomial does not vanish modulo 2^{128}. Consequently, this will imply that $\nu_2(P_{n+18} - P_n) < 128$ for all $n < 1.5 \times 10^{38}$.

Determining these quantities is not straightforward in MAPLE, due to the presence of large powers of α, β and γ in P, Q and R, respectively. However, we can compute

$$A := P + Q + R, \quad B := PQ + PR + QR, \quad C := PQR.$$

Next, the coefficients u_k are given by

$$u_k := \left(\alpha^{18}-1\right) A_\alpha \alpha^{671} P^k + \left(\beta^{18}-1\right) A_\beta \beta^{671} Q^k + \left(\gamma^{18}-1\right) A_\gamma \gamma^{671} R^k. \tag{3.19}$$

Form a linearly recurrence sequence of recurrence

$$u_{k+3} = Au_{k+2} - Bu_{k+1} + Cu_k, \quad \text{for } k \geq 0,$$

with u_0, u_1 and u_2 obtained from (3.19) for $k = 0, 1, 2$, respectively. Therefore, we can compute all the remaining coefficients iteratively and analyze the polynomial

$$f(z) := \sum_{k=0}^{18} (2^7 z)^k \frac{u_k}{k!} \pmod{2^{128}}.$$

All coefficients $\frac{u_k}{k!}$ are 2-adic integers and can be reduced modulo 2^{128}. Consequently, we obtain a polynomial in $\mathbb{Z}/(2^{128}\mathbb{Z})[z]$. Our objective is to determine a value of z such that this polynomial equals $0 \pmod{2^{128}}$. To this end, we proceed step by step, evaluating $f(z)$ modulo $2^{10}, 2^{11}, 2^{12}, 2^{13}$, and so on, we iteratively determine each binary digit of z that guarantees divisibility by increasingly higher powers of 2, following the principles of Hensel's lemma. This approach leads to

$$z = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + \cdots,$$

and after determining the binary digits up to 2^{101}, we express

$$z = 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 + 2^6 + 2^7 + 2^8 + 2^9 + \cdots + 2^{100} + 2^{101} + 2^{102}t,$$

and reduce $f(z)$ modulo 2^{127}, obtaining

$$2^{121}(8 + 9t) \pmod{2^{125}}.$$

Observe that z must be chosen as a multiple of 8, which gives

$$n \geq 2^{22} \cdot (\cdots + 2^{102} \cdot 8) = 2^{127} > 1.5 \cdot 10^{38}.$$

This analysis shows that, in practice, $\nu_2(P_{n+18} - P_n) < 124$.

A similar approach was carried out for other values of d and also for the case $p = 3$. When $p = 3$, the Padovan sequence $(P_n)_{n\geq 0}$ modulo 3^{k+1} has period $13 \cdot 3^k$, prompting us to work 3-adically with the logarithms $\log_3(\alpha^{13})$, $\log_3(\beta^{13})$ and $\log_3(\gamma^{13})$. In all instances, we found that $\nu_p(P_n - P_{n_1}) < 126$. Therefore, in all cases, it follows from (3.1) that

$$x_{\min}, y_{\min} < 126.$$

We now aim to refine the upper bound for n. Let c_X denote the bound on $X - X_1$. Then we have

$$X = X_2 + (X_1 - X_2) + (X - X_1) < x_{\min} \log 2 + y_{\min} \log 3 + 2c_X,$$

$$x \log 2 + y \log 3 < 126 \log 2 + 126 \log 3 + 2 \cdot 177 < 580.$$

This yields the bounds $x < 837$ and $y < 528$. Moreover, applying Lemma 2.14 gives

$$n \log \alpha - 3 < X < 580.$$

I obtain $n < 2100$.

4 Conclusion

To conclude the proof of Theorem 1.1, we observe that for $n > 664$, the parameters satisfy the bounds $n < 2100$, $x < 837$, and $y < 528$. To efficiently handle the large values of P_n, the MAPLE implementation employed batch processing to systematically check all triples (n, x, y) within these ranges. This method found no value of c admitting at least five representations of the form $P_n - 2^x 3^y$, in accordance with the statement of Theorem 1.1. The computation, performed on a 16 GB RAM laptop, was completed in about 2 hours.

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