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ON THE ASYMPTOTIC BEHAVIOR FOR $p(x)$-LAPLACIAN PROBLEM WITH STEKLOV NONLINEAR SUBREGION BOUNDARY CONDITION

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Abstract *In this work, we study the nonlinear Steklov boundary value for the p(x)-Laplacian problem. In fact, we have given the existence of a nontrivial weak solution on variable exponent Sobolev spaces. By introducing a preconditioner based on the norms of Luxembourg and a special kind of Sobolev space with variable exponent, the performance of the uniqueness is shown to be robust with respect to the new conditions of the p(x)-Laplacian problem with Steklov nonlinear subregion boundary condition. Our construction is mainly based on the epi-convergence method, which is used successfully in this new context, in order to establish the limit behavior by characterizing the weak limits of the energies for the solutions.*

1 Introduction

In this work, we are concerned with a generalized Steklov problem with $p(x)$- Laplacian operator. Recently, researches from different fields of nonlinear analysis have taken more interest in the asymptotic nonlinear analysis. In fact, the litterature contains a various and interesting studies of the nonlinear eigenvalue problems for the p-Laplacian subject to different kinds of boundary conditions on a bounded domain, we refer the reader to [20, 21, 22, 23]. Moreover, the study of asymptotic theory of the elliptic equations with variable exponent has been received considerable attention. Owing to their applications to a various range of phenomena including elastic mechanics[12], electro-rheological and thermo-rheological viscous flows of non-Newtonian fluids[26], image restoration[32] and mathematical biology[33]. Furthemore, many authors studied the case where $p(x)$-Laplacian with varied boundary conditions on a bounded domain (see for example, [39, 40, 25, 41, 42, 27, 28, 29, 17, 30, 31]. Accordingly, A. Zerouali et al [42] studied the following elliptic problem with nonlinear boundary conditions and variable exponent

$$\begin{aligned} -\operatorname{div}[a(x,\nabla u)] + |u|^{p(x)-2}u &= \lambda f(x,u) \quad \textit{in } \Omega, \\ a(x,\nabla u)\cdot v &= \mu g(x,u) \quad \textit{on } \partial\Omega. \end{aligned}$$

Indeed, the study of the p(x)-Laplace operator is also of interest both in nonlinear elasticity theory, viscoplastic, and in rheological theory of fluids (see [2, 7, 8, 11, 12, 13, 14]). Besides, many authors have studied the problem with Steklov eigenvalue problem see [15, 16, 18, 24, 19] and references therein.
Here, we are going to deal with the following Steklov boundary value problem involving the

$p(x)$-Laplacian ($p(x) \neq constant$) on a nanostructure which is a new topic.

$$\begin{cases} -\Delta_{\mathrm{p(x)}}\, u^{\varepsilon} + |u^{\varepsilon}|^{p(x)-2}u^{\varepsilon} = f \quad \text{in } \Omega, \\ |\nabla u^{\varepsilon}|^{p(x)-2}\dfrac{\partial u^{\varepsilon}}{\partial v} = -\lambda_{\varepsilon} g(x,u^{\varepsilon}) \quad \text{on } \Sigma_{\varepsilon}, \\ \left(|\nabla u^{\varepsilon}|^{p(x)-2}\right)\dfrac{\partial u^{\varepsilon}}{\partial v} = 0 \quad \text{on } \Gamma_{\varepsilon}. \end{cases} \tag{$\mathcal{P}^{\varepsilon}$}$$

where Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and $p \in C(\bar{\Omega})$, v is the outward unit normal vector on $\partial\Omega$. $g : \partial\Omega \times \mathbb{R} \mapsto \mathbb{R}$ are a Caratheodory function.

The operator Δ_p is p-Laplace operator with $p(x) > 1$: $\Delta_p u = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)$ with $|\nabla u| = \sqrt{\sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2}$, is called the $p(x)$-laplacian which is a natural generalization of the p-Laplacian (where $p > 1$ is a constant). When $p(x) \neq constant$, the $p(x)$-Laplacian possesses more complicated nonlinearity than the p-Laplacian. For this reason, some of the properties of the p-Laplacian problems may not hold for a general $p(x)$-Laplacian case. Reference [9, 10] represents one of the most important collections of results regarding the analysis of variable exponent Lebesgue and Sobolev spaces.

This operator is defined on Sobolev space $W^{1,p(x)}(\Omega)$ and $f \in L^{\infty}(\Omega)$.

We denote

$$\lambda_{\varepsilon} = \frac{1}{\varepsilon^{\alpha}}, \alpha \geq 0$$

,

$$\Sigma_{\varepsilon} = \left\{x \in \partial\Omega : |x_3| \leq \varepsilon^2\right\}, \quad \Gamma_{\varepsilon} = \partial\Omega \setminus \Sigma_{\varepsilon},$$

and

$$h^- = \min_{x\in\bar{\Omega}} h(x); \quad h^+ = \max_{x\in\bar{\Omega}} h(x); \quad \text{for all } h \in C(\bar{\Omega}),$$

$$p^*(x) = \begin{cases} \frac{3p(x)}{3-p(x)} & \text{if } p(x) < 3, \\ +\infty & \text{if } p(x) \geq 3, \end{cases}$$

$$p^{\partial}(x) = \begin{cases} \frac{2p(x)}{3-p(x)} & \text{if } p(x) < 3, \\ +\infty & \text{if } p(x) \geq 3, \end{cases}$$

$$C_+(\bar{\Omega}) = \left\{h \in C(\bar{\Omega}) : 1 < h^- < h^+ < +\infty\right\}.$$

We assume that variable exponents p is in $C_+(\bar{\Omega})$.

The energy functional corresponding to problem ($\mathcal{P}^{\varepsilon}$) is defined on $W^{1,p(x)}(\Omega)$ as

$$H(u^{\varepsilon}) \;=\; \Phi(u^{\varepsilon}) + \lambda_{\varepsilon} J(u^{\varepsilon}), \tag{1.1}$$

where

$$\Phi(u^{\varepsilon}) \;=\; \int_{\Omega}\left(\frac{|\nabla u^{\varepsilon}|^{p(x)}}{p(x)}\right)dx + \int_{\Omega}\left(\frac{|u^{\varepsilon}|^{p(x)}}{p(x)}\right)dx - \int_{\Omega} f u^{\varepsilon} dx, \tag{1.2}$$

$$J(u^{\varepsilon}) \;=\; \int_{\Sigma_{\varepsilon}} G(x,u^{\varepsilon})d\sigma \tag{1.3}$$

with $d\sigma$ is the 2-dimensional Hausdorff measure. For any $s \in \mathbb{R}$, we define

$$G(x,s) = \int_0^s g(x,t)dt.$$

Let us recall that $u^\varepsilon \in W^{1,p(x)}(\Omega)$ *is a weak solution of*($\mathcal{P}^\varepsilon$) *if*

$$\int_\Omega \left(|\nabla u^\varepsilon|^{p(x)-2}\right) \nabla u^\varepsilon \nabla v dx + \int_\Omega \left(|u^\varepsilon|^{p(x)-2}\right) u^\varepsilon v dx - \int_\Omega f v dx$$
$$= -\lambda_\varepsilon \int_{\partial\Sigma} g(x, u^\varepsilon) v d\sigma, \quad \text{for all } v \in W^{1,p(x)}(\Omega).$$

Now, we assume that g *satisfying to the following conditions*

(H_1) $g(x,.)$ *is a convex function for* $x \in \Omega$,

(H_2) $\exists\gamma > 0, \forall t \in \mathbb{R} |t|^{p-1} \leq g(x,t) \leq \gamma \cdot \left(1 + |t|^{p-1}\right)$, *for* $x \in \Omega$,

(H_3) $\begin{cases} \text{there exists a continuous } \omega : \mathbb{R}^+ \to \mathbb{R}^+ \\ \text{which is increasing and vanishing at the origin, such that} \\ \forall t \in \mathbb{R}, \forall x, y \in \Omega, |g(x,t) - g(y,t)| \leq \omega(|x-y|) \left(1 + |t|^{p-1}\right) \end{cases}$

The paper is structured as follows: Section 2 and 3 addresses the preliminary elements necessary to establish the context for the studied problem and to understand the rest of the article. In section 4, we recall some basic and elementary proporties concerning the notion of epiconvergence. Section 5 will be devoted to demonstrate the existence results of the problem in this case where the exponent is variable, and then we present the a priori estimates and the caracterization of the obtained solutions in order to solve the limit problem with interface conditions and consequently acquire a better understanding of the system's behavior near the nano interface boundary. Finally, Section 6 will mainly contains the main proof of the limit problem, the results obtained enriches the understanding of the behavior of the nanolayer in the considered structure, with potential practical applications, and provide an update on recent results on p(x)-Laplacian problem for nonlinear Steklov boundary conditions.

2 Notations and assumptions

In this section we shall fix some notation that will be frequently used along this paper.
We first recall some background facts concerning the generalized Lebesgue-Sobolev spaces. For $p \in C_+(\bar{\Omega})$, *we introduce the variable exponent Lebesgue space*

$$L^{p(x)}(\Omega) := \left\{ u : \Omega \subset \mathbb{R}^N \to \mathbb{R} \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < +\infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} := \inf \left\{ \alpha > 0; \int_\Omega \left| \frac{u(x)}{\alpha} \right|^{p(x)} dx \leq 1 \right\}$$

which is separable and reflexive Banach space (see, [37]). Let us define the space

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) / |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\| := \inf \left\{ \mu > 0; \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx + \int_\Omega \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}; \quad \forall u \in W^{1,p(x)}(\Omega).$$

Furthermore, let us define:
the operator m^ε *which transforms functions defined* u *on* Σ_ε *into functions defined on* $\partial\Sigma$, *by way of example, we cite [1]*

$$(m^\varepsilon u)(x') = \frac{1}{2\varepsilon^2} \int_{-\varepsilon^2}^{\varepsilon^2} u(x', x_3) dx_3$$

and

- *$d\sigma$: represents the surface measure on Σ_ε*
- $(x', x_3) \in \Sigma_\varepsilon$.
- $(x,t) = (x', x_3, t)$ *where* $x' = (x_1, x_2)$, $\nabla' = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$, *and* $\eta(\alpha) = \lim_{\varepsilon\to 0} \varepsilon^{2-\alpha}$ *with* $\alpha \geq 0$,
- *C will denote any constant with respect to ε.*

Proposition 2.1 (see [36, 38]).

(i) $W^{1,p(x)}(\Omega)$ *is separable reflexive Banach space;*

(ii) If $h \in C_+(\bar{\Omega})$ and $h(x) < p^*(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{h(x)}(\Omega)$ is compact and continuous.

(iii) If $h \in C_+(\bar{\Omega})$ and $h(x) < p^\partial(x)$ for any $x \in \bar{\Omega}$, then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{h(x)}(\partial\Omega)$ is compact and continuous.

Note that the following mapping ρ_p plays a crucial role in using and manipulating the generalized Lebesgue-Sobolev spaces. It's defined by

$$\rho_p(u) := \int_\Omega |\nabla u|^{p(x)} dx + \int_\Omega |u|^{p(x)} dx, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Proposition 2.2 ([35]). *For $u, u_k \in W^{1,p(x)}(\Omega)$; $\quad k = 1, 2, \ldots$, we have*

(i) $\|u\| \geq 1$ *implies* $\|u\|^{p^-} \leq \rho_p(u) \leq \|u\|^{p^+}$;

(ii) $\|u\| \leq 1$ *implies* $\|u\|^{p^+} \leq \rho_p(u) \leq \|u\|^{p^-}$;

(iii) $\|u_k\| \to 0$ *as* $k \to +\infty$ *if and only if* $\rho_p(u_k) \to 0$ *as* $k \to +\infty$;

(iv) $\|u_k\| \to +\infty$ *as* $k \to +\infty$ *if and only if* $\rho_p(u_k) \to +\infty$ *as* $k \to +\infty$.

3 Functionnal Setting

We introduce some notations used throughout this paper.
Let us consider the following functional spaces depending on the values of α:

$$W^{1,p(x)}_\partial(\Omega) = \left\{ u \in W^{1,p(x)}(\Omega) : u \in \partial\Omega/\partial\Sigma \right\}$$

For $\alpha = 2$

$$\mathbb{G}^\alpha = \left\{ u \in W^{1,p(x)}(\Omega) : u_{|\partial\Sigma} \in W^{1,p(x)}(\partial\Sigma) \right\} \text{ and } \quad \mathbb{D}^\alpha = D(\overline{\Omega}) \bigcap W^{1,p(x)}_\partial(\Omega)$$

and for $\alpha < 2$

$$\mathbb{G}^\alpha = W^{1,p(x)}_\partial(\Omega) \text{ and } \quad \mathbb{D}^\alpha = D(\overline{\Omega}) \bigcap W^{1,p(x)}_\partial(\Omega)$$

and for $\alpha > 2$

$$\mathbb{G}^\alpha = W^{1,p(x)}_0(\Omega) \text{ and } \quad \mathbb{D}^\alpha = D(\Omega)$$

In addition, the demonstration concerns the convergence of Σ_ε to $\partial\Sigma$ in the Hausdorff sense. To do this, we define a family of subvarieties

$$S_\varepsilon = \{x \in \partial\Omega | |x_3| \leq \varepsilon^2\}$$

and we evaluate the Hausdorff distance between Σ_ε and $\partial\Sigma$. We find that for all $x \in \Sigma_\varepsilon$, the distance to $\partial\Sigma$ is at most ε^2, and for all t in $\partial\Sigma$, the distance to Σ_ε is 0. Thus, the Hausdorff distance is equal to ε^2. Finally, we show that when ε tends to 0, the Hausdorff distance also tends to 0 and thus demonstrating the convergence of Σ_ε to $\partial\Sigma$ in the Hausdorff sense.

4 Epiconvergence notion

We will give an efficient notion of operator's sequence convergence, named epiconvergence, who is a special case of the $\Gamma-$convergence introduced by De Giorgi (1979) [6]. It is well suited to the asymptotic analysis of sequences of minimization problems.

Definition 4.1. [3, Definition 1.9] Let $(\mathbb{X}, \tau)$ be a metric space and $(F^\varepsilon)_\varepsilon$ and F be functionals defined on $\mathbb{X}$ and with value in $\mathbb{R} \cup \{+\infty\}$. F^ε epi-converges to F in $(\mathbb{X}, \tau)$, noted
$\tau - epilim_{\varepsilon \to 0} F^\varepsilon = F$,
if the following assertions are satisfied

- For all $x \in \mathbb{X}$, there exists x_ε^0, $x_\varepsilon^0 \xrightarrow{\tau} x$ such that $\limsup_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon^0) \leq F(x)$.
- For all $x \in \mathbb{X}$ and all x_ε with $x_\varepsilon \xrightarrow{\tau} x$, $\liminf_{\varepsilon \to 0} F^\varepsilon(x_\varepsilon) \geq F(x)$.

Note the following stability result of the epi-convergence.

Proposition 4.2. *[3, p. 40] Suppose that F^ε epi-converges to F in $(\mathbb{X}, \tau)$ and that $\Phi \colon \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$, is τ-continuous. Then $F^\varepsilon + \Phi$ epi-converges to $F + \Phi$ in $(\mathbb{X}, \tau)$.*

Theorem 4.3. *[3, theorem 1.10] Suppose that*

(i) F^ε admits a minimizer on $\mathbb{X}$,

(ii) The sequence $(\overline{u}^\varepsilon)$ is τ-relatively compact,

(iii) The sequence F^ε epi-converges to F in this topology τ.

Then every cluster point $\overline{u}$ of the sequence $(\overline{u}^\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon' \to 0} F^{\varepsilon'}(\overline{u}^{\varepsilon'}) = F(\overline{u}),$$

if $(\overline{u}^{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\overline{u}^\varepsilon)_\varepsilon$ which converges to $\overline{u}$.

5 Existence results

The operator Φ is well defined and of class C^1 (see [5]). The Fréchet derivative of Φ is the operator $\Phi' : W^{1,p(x)}(\Omega) \mapsto W^{1,p(x)}(\Omega)'$ defined as

$$\langle \Phi'(u^\varepsilon), v \rangle = \int_\Omega \left(|\nabla u^\varepsilon|^{p(x)-2} \right) \nabla u^\varepsilon \nabla v dx + \int_\Omega \left(|u^\varepsilon|^{p(x)-2} \right) u^\varepsilon v dx - \int_\Omega f v dx,$$

for any $u^\varepsilon, v \in W^{1,p(x)}(\Omega)$.
Firstly, we need to prove some properties of the operator Φ'.

Lemma 5.1. *The following statements holds.*
(1) Φ' is continuous and strictly monotone;
(2) Φ' is an homeomorphism.

Proof. (1) We have Φ' is the Fréchet derivative of Φ, one can show that Φ' is continuous. Basing on those elementary inequalities [34]

$$\begin{aligned} |x-y|^\gamma &\leq 2^\gamma \left(|x|^{\gamma-2}x - |y|^{\gamma-2}y \right)(x-y) && ,\text{if} \quad \gamma \geq 2, \\ |x-y|^2 &\leq \frac{1}{(\gamma-1)}(|x|+|y|)^{2-\gamma} \left(|x|^{\gamma-2}x - |y|^{\gamma-2}y \right)(x-y) && ,\text{if } 1 < \gamma < 2 \end{aligned}$$

for all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$.
Hence, we have for all $u^\varepsilon, v \in W^{1,p(x)}(\Omega)$ such that $u^\varepsilon \neq v$,

$$\begin{aligned} \langle \Phi'(u^\varepsilon) - \Phi'(v), u^\varepsilon - v \rangle = &\int_\Omega \left(|\nabla u^\varepsilon|^{p(x)-2} \nabla u^\varepsilon - |\nabla v|^{p(x)-2} \nabla v \right)(\nabla u^\varepsilon - \nabla v) dx \\ &+ \int_\Omega \left(|u^\varepsilon|^{p(x)-2} u^\varepsilon - |v|^{p(x)-2} v \right)(u^\varepsilon - v) dx > 0. \end{aligned}$$

which implies that Φ' is strictly monotone and consequently its injectivity.
(2) To prove that Φ' is a surjection, it suffices to verify that it is coercive. As a matter of fact, thanks to Proposition 2.2 and basing on $p^- - 1 > 0$. Thus, we deduce that for each $u^\varepsilon \in W^{1,p(x)}(\Omega)$ such that $\|u^\varepsilon\| \geq 1$

$$\begin{aligned}\frac{\langle \Phi'(u^\varepsilon), u^\varepsilon\rangle}{\|u^\varepsilon\|} &= \frac{\int_\Omega \left[|\nabla u^\varepsilon|^{p(x)} + |u^\varepsilon|^{p(x)}\right] dx - \int_\Omega f u^\varepsilon dx}{\|u^\varepsilon\|} \\ &\geq \frac{\rho_p(u^\varepsilon)}{\|u^\varepsilon\|} - C \\ &\geq \|u^\varepsilon\|^{p^- -1} - C \to \infty \quad \text{as} \quad \|u^\varepsilon\| \to \infty,\end{aligned}$$

where C is a real constant.

As a result, this concludes the surjectivity of the operator Φ', then Φ' admits an inverse mapping.
Now we are in position to state that Φ'^{-1} is continuous. To do so, let $(\varphi_n)_n$ be a sequence of $W^{1,p(x)}(\Omega)'$ such that $\varphi_n \to \varphi$ in $W^{1,p(x)}(\Omega)'$ as $n \to +\infty$. Also, let u_n^ε and u^ε in $W^{1,p(x)}(\Omega)$ such that $\Phi'^{-1}(\varphi_n) = u_n^\varepsilon$ and $\Phi'^{-1}(\varphi) = u^\varepsilon$. Using the coercivity of Φ', one conclude that the sequence (u_n^ε) is bounded in $W^{1,p(x)}(\Omega)$. Nevertheless $W^{1,p(x)}(\Omega)$ is reflexive (Proposition 2.1), for a subsequence still denoted (u_n^ε), we get $u_n^\varepsilon \rightharpoonup \hat{u^\varepsilon}$ weakly in $W^{1,p(x)}(\Omega)$ as $n \to +\infty$, which leads to

$$\lim_{n\to+\infty} \left\langle \Phi'(u_n^\varepsilon) - \Phi'(u), u_n^\varepsilon - \hat{u^\varepsilon}\right\rangle = \lim_{n\to+\infty} \left\langle \varphi_n - \varphi, u_n^\varepsilon - \hat{u^\varepsilon}\right\rangle = 0$$

In addition, one can show easily that Φ' has the property (S_+) (see for example [42]) and by the continuity of Φ' that $u_n^\varepsilon \mapsto \hat{u^\varepsilon}$ strongly in $W^{1,p(x)}(\Omega)$ and $\Phi'(u_n^\varepsilon) \mapsto \Phi'(\hat{u^\varepsilon}) = \Phi'(u^\varepsilon)$ in $W^{1,p(x)}(\Omega)'$ as $n \mapsto +\infty$. What's more, since Φ' is an injection, so we obtain that $\hat{u^\varepsilon} = u^\varepsilon$. This completes the proof.

Next, we will give the proof of the following lemma.

Lemma 5.2. *For the operator $J : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R}$ is sequentially weakly lower semi continuous, namely, $u_n \longrightarrow u_0$ in $W^{1,p(x)}(\Omega)$ implies $J(u_n) \longrightarrow J(u_0)$.*

Proof. *Let's consider $\Phi(u^\varepsilon)$ and $J(u^\varepsilon)$ as in (1.2) and (1.3) For each $u^\varepsilon, v \in W^{1,p(x)}(\Omega)$, it follows that*

$$\langle \Phi'(u^\varepsilon), v\rangle = \int_\Omega \left(|\nabla u^\varepsilon|^{p(x)-2}\right) \nabla u^\varepsilon \nabla v dx + \int_\Omega \left(|u^\varepsilon|^{p(x)-2}\right) u^\varepsilon v dx - \int_\Omega f v dx$$

$$\langle J'(u^\varepsilon), v\rangle = -\int_{\partial\Sigma} g(x, u^\varepsilon) v dx.$$

According to [31][Proposition 4.], the functional Φ is a continuous Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on $W^{1,p(x)}(\Omega)'$.
On the other hand, from (H1), we can claim that J is continuously Gâteaux differentiable functionals. Then, using the compacity of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ and the trace embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$ (Proposition 2.1). Therefore, those facts help us to show that J' is compact. Besides, Φ is bounded on each bounded subset of $W^{1,p(x)}(\Omega)$ under our assumptions. Hence, if $\|u^\varepsilon\| \geq 1$, we have

$$\begin{aligned}\Phi(u^\varepsilon) &= \int_\Omega \left(\frac{|\nabla u^\varepsilon|^{p(x)}}{p(x)}\right) dx + \int_\Omega \left(\frac{|u^\varepsilon|^{p(x)}}{p(x)}\right) dx \\ &\geq \frac{1}{p^+}\rho_p(u^\varepsilon) \\ &\geq \frac{1}{p^+}\|u^\varepsilon\|^{p^-}.\end{aligned}$$

This concludes the proof of the desired result. As a by-product we obtain the following result of existence and uniqueness

Proposition 5.3. *Under the hypothesis $H_1 - H_3$ and $f \in L^\infty(\Omega)$, the problem $(\mathcal{P}_\varepsilon)$ have a unique solution u^ε in $W^{1,p(.)}(\Omega)$.*

Lemma 5.4 (A priori estimate). *Under the hypothesis H1-H3 and assume that there exists a constant $C \geq 0$ such that*

$$H(v) \leq C, \forall v \in V.$$

Then we have the following assertions:

$$\int_\Omega \frac{1}{p(x)}|\nabla u_\varepsilon|^{p(x)} + \int_\Omega \frac{1}{p(x)}|u_\varepsilon|^{p(x)} \leq C \tag{5.1}$$

$$\int_{\Sigma_\varepsilon} G(x, u^\varepsilon) \leq C\varepsilon^\alpha \tag{5.2}$$

Indeed we do have u^ε is bounded in $W^{1,p(.)}_\partial(\Omega)$.

Proof. Since u^ε satisfies to

$$\int_\Omega \frac{1}{p(x)}|\nabla u^\varepsilon|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u^\varepsilon|^{p(x)}dx + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^\varepsilon)dx - \int_\Omega f u^\varepsilon dx \leq C.$$

So, we have

$$\begin{aligned}\int_\Omega \frac{1}{p(x)}|\nabla u^\varepsilon|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u^\varepsilon|^{p(x)}dx + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^\varepsilon)dx &\leq C + \int_\Omega f u^\varepsilon dx \\ &\leq C + C.\|u^\varepsilon\|_{W^{1,p(x)}(\Omega)} \\ &\leq C + \frac{1}{p^-}\|u^\varepsilon\|^{p^-}_{W^{1,p(x)}(\Omega)}\end{aligned}$$

If $\|u^\varepsilon\|_{W^{1,p(x)}(\Omega)} \leq 1$ then we have our results. Otherwise $\|u^\varepsilon\|_{W^{1,p(x)}(\Omega)} \geq 1$, then from proposition 2.2 we have

$$\begin{aligned}\int_\Omega \frac{1}{p(x)}|\nabla u^\varepsilon|^{p(x)}dx + \int_\Omega \frac{1}{p(x)}|u^\varepsilon|^{p(x)}dx + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^\varepsilon)dx &\leq C + \frac{1}{p^-}\|u^\varepsilon\|^{p^-}_{W^{1,p(x)}(\Omega)} \\ &\leq C + \frac{1}{p^-}\rho_p(u^\varepsilon)\end{aligned}$$

so, we obtain

$$\frac{1}{(p^-)'}\rho_p(u^\varepsilon) + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^\varepsilon)dx \leq C$$

Hence the assertions (5.1) and (5.2) is holds and from (5.1) the (u^ε) is bounded in $W^{1,p(x)}_\partial(\Omega)$. □

Proposition 5.5. *The solution of our problem $(u^\varepsilon)_\varepsilon$, possess a subsequence that converges weakly to an element u^* in $W^{1,p(x)}_\partial(\Omega)$ satisfy*

(i) If $\alpha = 2$

$$u^*{}_{|\partial\Sigma} \in W^{1,p(x)}_\partial(\partial\Sigma).$$

(ii) If $\alpha > 2$

$$u^*{}_{|\partial\Sigma} = 0.$$

Proof. From the lemma 5.4, the solution of our problem u^ε is borned in $W^{1,p}_\partial(\Omega)$, therefore, for any subsequence $(u^\varepsilon)_\varepsilon$, noted again u^ε, there exist $u^* \in W^{1,p(x)}_\partial(\Omega)$ such that $u^\varepsilon \rightharpoonup u^*$ in $W^{1,p(x)}_\partial(\Omega)$.

(i) $\alpha = 2$
We have

$$\begin{cases} \|m^\varepsilon u^\varepsilon - u^\varepsilon{}_{|\partial\Sigma}\|_{L^p(\partial\Sigma)} \leq C\varepsilon^{\frac{2(\alpha+p^- -1)}{p^-}}. \\ u^\varepsilon{}_{|\partial\Sigma} \rightharpoonup u^*{}_{|\partial\Sigma} \text{ in } L^p(\partial\Sigma) \end{cases}$$

We establish with the same way like in [1, Lemma 4 p.4] that

$$\int_{\partial\Sigma} |\nabla m^\varepsilon u^\varepsilon|^{p(x)} \leq \varepsilon^{\alpha-2}$$

so, we deduce that the sequence $\nabla' m^\varepsilon u^\varepsilon$ possess a subsequence noted again $\nabla' m^\varepsilon u^\varepsilon$ that converge weakly to an element u^2 in $L^p(\Sigma)^2$. So, $m^\varepsilon u^\varepsilon$ is borned in $W^{1,p(x)}(\Sigma)$, and since $m^\varepsilon u^\varepsilon \rightharpoonup u^*{}_{|\Sigma}$ in $L^p(\Sigma)$, one get, $m^\varepsilon u^\varepsilon \rightharpoonup u^*{}_{|\Sigma}$ in $W^{1,p(x)}(\Sigma)$;
Consequently $\nabla' u^*{}_{|\Sigma} = u^2$, we find $\nabla' u^*{}_{|\Sigma} \in L^p(\Sigma)$.
therefore we get, $u^*{}_{|\Sigma} \in W^{1,p}(\Sigma)$.

(ii) $\alpha > 2$
According to the following estimate

$$\int_{\partial\Sigma} |m^\varepsilon u^\varepsilon|^{p^-} \leq \varepsilon^{\alpha-2}$$

then the sequence $m^\varepsilon u^\varepsilon$ admits a subsequence noted $m^\varepsilon u^\varepsilon$ converges to 0 in $L^p(\partial\Sigma)^2$ and $m^\varepsilon u^\varepsilon \rightharpoonup u^*{}_{|\partial\Sigma}$ in $W^{1,p(x)}(\partial\Sigma)$, consequently we get $u^*{}_{|\partial\Sigma} = 0$.

This completes the proof of the proposition 5.5. □

6 Limit behavior of the problem

In this section we shall study the asymptotic behavior of the solution of the problem (Π^ε)*, when* ε *tend to* 0*, for that, we use the epiconvergence method. By the way, we have to determine the energy functionnal derivated from the minimisation problem* (Π^ε)*, moreover, we determine its epilimit* [1].
we set

$$\mathcal{H}^\varepsilon(v) = \begin{cases} \dfrac{1}{\varepsilon^\alpha}\displaystyle\int_{\Sigma_\varepsilon} G(x,v) & if\ v \in W^{1,p(x)}_\partial(\Omega), \\ +\infty & if\ v \in W^{1,p(x)}(\Omega) \setminus W^{1,p(x)}_\partial(\Omega). \end{cases} \tag{6.1}$$

$$\mathcal{S}(v) = \begin{cases} \displaystyle\int_\Omega \frac{1}{p}|\nabla v|^p + \int_\Omega \frac{1}{p}|v|^p - \int_{\partial\Sigma} f\ v & if\ v \in W^{1,p(x)}_\partial(\Omega), \\ +\infty & if\ v \in W^{1,p(x)}(\Omega) \setminus W^{1,p(x)}_\partial(\Omega). \end{cases} \tag{6.2}$$

We denote with τ_f *the weak topology in* $W^{1,p(x)}(\Omega)$*. Our goal now in the following theorem is to determine the epilimit of the fonctionnal* $\mathcal{H}^\varepsilon$*, when* ε *tend to* 0.

Theorem 6.1. *Depending to the values of* α*, there exist an energy functional* $\mathcal{H}^\alpha : W^{1,p(x)}(\Omega) \longrightarrow \mathbb{R} \cup \{+\infty\}$ *such that*

$$\tau_f - lim_e \mathcal{H}^\varepsilon = \mathcal{H}^\alpha \ in\ W^{1,p(x)}(\Omega), \tag{6.3}$$

where $\mathcal{H}^\alpha$ *is expressed by*

(i) If $\alpha \neq 2$.

$$\mathcal{H}^\alpha(v) = 0.$$

[1] voir par exemple [3].

(ii) If $\alpha = 2$.

$$\mathcal{H}^\alpha(v) = \begin{cases} 2\displaystyle\int_{\partial\Sigma} G(x, u_{|\partial\Sigma}) & \text{if } v \in W_\partial^{1,p(x)}(\Omega), \\ +\infty & \text{if } v \in W^{1,p(x)}(\Omega) \setminus W_\partial^{1,p(x)}(\Omega). \end{cases}$$

Proof. First, it remains to check the epilimit superior of $\mathcal{H}^\varepsilon$; then we deduce the epilimit inferior.

a) The upper epilimit.
Let $u \in \mathbb{G}^\alpha \subset W^{1,p(x)}(\Omega)$, since $\mathbb{D}^\alpha$ is dense in $\mathbb{G}^\alpha$, hence there exist a sequence (u^n) in $\mathbb{D}^\alpha$ such that $u^n \to u$ in $\mathbb{G}^\alpha$ quand $n \to +\infty$, therefore $u^n \to u$ in $W^{1,p(x)}(\Omega)$.
Now, Let θ be a regularised function satisfy

$$\theta(x_3) = 1 \text{ si } |x_3| \le 1,\ \theta(x_3) = 0 \text{ if } |x_3| \ge 2 \ \text{ et } |\theta'(x_3)| \le 2\ \forall x \in \mathbb{R},$$

and we put

$$\theta_\varepsilon(x) = \theta(\frac{x_3}{\varepsilon});$$

One can defines

$$u^{\varepsilon,n} = \theta_\varepsilon(x) {u^n}_{|\partial\Sigma} + (1 - \theta_\varepsilon(x)) u^n,$$

It follows that $u^{\varepsilon,n} \in W_0^{1,p(x)}(\Omega)$ so, it remains to show that clearly we have $u^{\varepsilon,n} \to u^n$ in $\mathbb{G}^\alpha$ when $\varepsilon \longrightarrow 0$. Moreover, in view of the formula

$$\mathcal{H}^\varepsilon(u^{\varepsilon,n}) = \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^{\varepsilon,n})dx,$$

we readily obtain

$$\mathcal{H}^\varepsilon(u^{\varepsilon,n}) = \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, (u^n)_{|\partial\Sigma})dx \tag{6.4}$$

$$= \varepsilon^{2-\alpha}\int_{\partial\Sigma}\int_{-1}^{1} G((x', t\varepsilon^2), (u^n)_{|\partial\Sigma})dtdx. \tag{6.5}$$

By passing to limit, we have

(i) If $\alpha = 2$.

$$\lim_{\varepsilon\to 0} \varepsilon^{2-\alpha}\int_{\partial\Sigma}\int_{-1}^{1} G((x', t\varepsilon^2), (u^n)_{|\partial\Sigma})dtdx = 2\int_{\partial\Sigma} G((x', 0), (u^n)_{|\partial\Sigma})dx'.$$

if $u \in W^{1,p(x)}(\Omega) \setminus W_\partial^{1,p(x)}(\Omega)$ then, one can conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{H}^\alpha(u^{\varepsilon,n}) \le +\infty$$

(ii) If $\alpha > 2$.
we have $(u^n)_{|\partial\Sigma} = 0$ and $G(.,0) = 0$, consequently we obtain

$$\varepsilon^{2-\alpha}\int_{\partial\Sigma}\int_{-1}^{1} G((x', t\varepsilon^2), (u^n)_{|\partial\Sigma})dtdx = 0$$

then

$$\limsup_{\varepsilon\to 0} \mathcal{H}^\varepsilon(u^{\varepsilon,n}) = 0$$

if $u \in W^{1,p(x)}(\Omega) \setminus W_\partial^{1,p(x)}(\Omega)$ then, one can conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{H}^\alpha(u^{\varepsilon,n}) \le +\infty$$

(iii) If $\alpha < 2$.
By Passing to upper limit, we get

$$\limsup_{\varepsilon\to 0} \varepsilon^{2-\alpha} \int_{\partial\Sigma}\int_{-1}^{1} G((x', t\varepsilon^2), (u^n)_{|\partial\Sigma}) dt dx = 0$$

then

$$\limsup_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon,n}) = 0$$

if $u \in W^{1,p(x)}(\Omega) \setminus W_{\partial}^{1,p(x)}(\Omega)$ then, one can conclude that

$$\limsup_{\varepsilon\to 0} \mathcal{H}^{\alpha}(u^{\varepsilon,n}) \leq +\infty$$

Since $u^n \to u$ in $\mathbb{G}^{\alpha}$, when $n \to +\infty$. Then according to the diagonalization lemma (see [3]), thus there exist a function $n(\varepsilon) : \mathbb{R}^+ \to \mathbb{N}$ increasing to $+\infty$ when $\varepsilon \to 0$, such that $u^{\varepsilon,n(\varepsilon)} \to u$ in $\mathbb{G}^{\alpha}$ when $\varepsilon \to 0$. Using straightforward computations yield for any n that approaches to $+\infty$, one may write,

(i) If $\alpha \neq 2$.

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon,n(\varepsilon)}) &\leq \limsup_{n\to+\infty} \limsup_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon,n}) \\ &\leq 0.\end{aligned}$$

(ii) If $\alpha = 2$.

$$\begin{aligned}\limsup_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon,n(\varepsilon)}) &\leq \limsup_{n\to+\infty} \limsup_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon,n}) \\ &\leq 2\int_{\partial\Sigma} G((x', 0), u_{|\partial\Sigma}).\end{aligned}$$

b) The lower epilimit.
We are going to determine the lower epi-limit. Let $u \in \mathbb{G}^{\alpha}$ and (u^{ε}) be a sequence in $W^{1,p(x)}(\Omega)$ such that $u^{\varepsilon} \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$, so that

$$\nabla u^{\varepsilon} \rightharpoonup \nabla u \quad \text{in } L^{p_-}(\Omega)^3 \tag{3.11}$$

(i) If $\alpha \neq 2$: Since

$$\mathcal{H}^{\varepsilon}(u^{\varepsilon}) \geq 0$$

According to (H_2) and by passage to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon}) \geq 0$$

(ii) If $\alpha = 2$: If $\liminf_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon}) = +\infty$, there is nothing to prove, because

$$\int_{\partial\Sigma} G(x', u_{|\partial\Sigma})\, dx' \leq +\infty$$

Otherwise, $\liminf_{\varepsilon\to 0} \mathcal{H}^{\varepsilon}(u^{\varepsilon}) < +\infty$, there exists a subsequence of $\mathcal{H}^{\varepsilon}(u^{\varepsilon})$ still denoted by $\mathcal{H}^{\varepsilon}(u^{\varepsilon})$ and a constant $C > 0$, such that $\mathcal{H}^{\varepsilon}(u^{\varepsilon}) \leq C$, which implies that

$$\frac{1}{\varepsilon^{\alpha}} \int_{\Sigma_{\varepsilon}} G(x, u^{\varepsilon})\, dx \leq C \tag{3.12}$$

and according to (H_2), we have

$$\frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_{\varepsilon}}|u^{\varepsilon}|^{p}\,dx\leq C \tag{3.13}$$

Recall that

$$\mathcal{H}^{\varepsilon}\left(u^{\varepsilon}\right)=\frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_{\varepsilon}}G\left(x,u^{\varepsilon}\right)$$

From the assumption (H1), $G(x,.)$ is a convex function for any $x\in\Omega$, so using the sub-differential inequality of

$$v\longmapsto\int_{\Sigma_{\varepsilon}}G(x,v)dx$$

we have

$$\int_{\Sigma_{\varepsilon}}G\left(x,u^{\varepsilon}\right)dx\geq\int_{\Sigma_{\varepsilon}}G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)dx+\int_{\Sigma_{\varepsilon}}g\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\left(u^{\varepsilon}-\,u^{\varepsilon}|_{\partial\Sigma}\right)dx$$

thus

$\mathcal{H}^{\varepsilon}\left(u^{\varepsilon}\right)\geq\frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_{\varepsilon}}G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)dx+\frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_{\varepsilon}}g\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\left(u^{\varepsilon}-\,u^{\varepsilon}|_{\partial\Sigma}\right)dx$ like in [4, Lemma 3.2] and $\alpha=2$, we show that

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon^{\alpha}}\int_{\Sigma_{\varepsilon}}g\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\left(u^{\varepsilon}-\,u^{\varepsilon}|_{\partial\Sigma}\right)dx=0 \tag{3.14}$$

So

$$\begin{aligned}\int_{\Sigma_{\varepsilon}}G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)dx&\leq\int_{\Sigma_{\varepsilon}}G\left(x,u^{\varepsilon}\right)dx-\int_{\Sigma_{\varepsilon}}g\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\left(u^{\varepsilon}-\,u^{\varepsilon}|_{\partial\Sigma}\right)dx\\&\leq\varrho_{\varepsilon}+\int_{\Sigma_{\varepsilon}}G\left(x,u^{\varepsilon}\right)dx\end{aligned}$$

where $\varrho_{\varepsilon}=-\int_{\Sigma_{\varepsilon}}g\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\left(u^{\varepsilon}-\,u^{\varepsilon}|_{\partial\Sigma}\right)dx$, according to (3.12), we obtain

$$\int_{\Sigma_{\varepsilon}}G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)dx\leq\varrho_{\varepsilon}+C\varepsilon^{\alpha}$$

From (3.12), let us define the following sequence

$$v_{\varepsilon}=m^{\varepsilon}\left(G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)\right)$$

we can show easily that

$$\int_{\partial\Sigma}v_{\varepsilon}dx'\leq\frac{C}{\varepsilon^{2}}\int_{\Sigma_{\varepsilon}}G\left(x,\,u^{\varepsilon}|_{\partial\Sigma}\right)dx$$

so we have

$$\int_{\partial\Sigma}|v_{\varepsilon}|\,dx'\leq C\left(\frac{\rho_{\varepsilon}}{\varepsilon^{2}}+\varepsilon^{\alpha-2}\right)$$

Let us show that

$$\int_{\partial\Sigma}|v_{\varepsilon}-G\left(\left(x',0\right),\,u^{\varepsilon}|_{\partial\Sigma}\right)|\,dx'=0$$

To simplify the writing, we put $G\left(x',\,u^{\varepsilon}|_{\partial\Sigma}\right)=G\left(\left(x',0\right),\,u^{\varepsilon}|_{\partial\Sigma}\right)$. We have

$$\begin{aligned}\int_{\partial\Sigma}|v_\varepsilon - G(x', u^\varepsilon|_{\partial\Sigma})|\,dx' &= \int_{\partial\Sigma}|v_\varepsilon - G(x', u^\varepsilon|_{\partial\Sigma})|\,dx' \\ &= \int_{\partial\Sigma}\left|\frac{1}{2\varepsilon^2}\int_{-\varepsilon^2}^{\varepsilon^2}(G(x, u^\varepsilon|_{\partial\Sigma}) - G(x', u^\varepsilon|_{\partial\Sigma}))\right| dx' \\ &\le \frac{C}{\varepsilon^2}\int_{\partial\Sigma}\int_{-\varepsilon^2}^{\varepsilon^2}\left|G(x, u^\varepsilon|_{\partial\Sigma}) - G\left(x', u^\varepsilon{}_{|\partial\Sigma}\right)\right| dx'\end{aligned}$$

according to (H3), we obtain

$$\begin{aligned}\int_{\partial\Sigma}|v_\varepsilon - G(x', u^\varepsilon|_{\partial\Sigma})|\,dx' &\le \frac{C}{\varepsilon^2}\int_{\partial\Sigma}\int_{-\varepsilon^2}^{\varepsilon^2} w(\varepsilon^2)\left(1 + |u^\varepsilon|_{\partial\Sigma}|^{p^-}\right) dx' \\ &\le Cw(\varepsilon^2)\int_{\partial\Sigma}\left(1 + \left|u^\varepsilon{}_{|\partial\Sigma}\right|^{p^-}\right) dx' \\ &\le Cw(\varepsilon^2)\end{aligned}$$

from (H3) and by passing to limit when ε close to 0 , hence

$$\lim_{\varepsilon\to 0}\int_{\partial\Sigma}|v_\varepsilon - G(x', u^\varepsilon|_{\partial\Sigma})|\,dx' = 0$$

let us $x' \mapsto \delta_\varepsilon(x')$ close to 0 in $L^1(\partial\Sigma)$ such that

$$v_\varepsilon = G(x', u^\varepsilon|_{\partial\Sigma}) + \delta_\varepsilon$$

we have

$$\begin{aligned}\mathcal{H}^\varepsilon(u^\varepsilon) &\ge \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} G(x, u^\varepsilon|_{\partial\Sigma})\,dx + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} g(x, u^\varepsilon|_{\partial\Sigma})(u^\varepsilon - u^\varepsilon|_{\partial\Sigma})\,dx \\ &\ge 2\varepsilon^{2-\alpha}\int_\Sigma v_\varepsilon dx' + \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} g(x, u^\varepsilon|_{\partial\Sigma})(u^\varepsilon - u^\varepsilon|_{\partial\Sigma})\,dx \\ &\ge 2\varepsilon^{2-\alpha}\int_{\partial\Sigma} G(x', u^\varepsilon|_{\partial\Sigma})\,dx' + \int_{\partial\Sigma}\delta_\varepsilon dx' \\ &+ \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} g(x, u^\varepsilon|_{\partial\Sigma})(u^\varepsilon - u^\varepsilon|_{\partial\Sigma})\,dx.\end{aligned}$$

From the sub-gradient inequality of $G(x', .)$, we obtain

$$\begin{aligned}\mathcal{H}^\varepsilon(u^\varepsilon) \ge& 2\varepsilon^{2-\alpha}\int_{\partial\Sigma} G\left(x', u_{|_{\partial\Sigma}}\right) dx' + \varepsilon^{2-\alpha}\int_{\partial\Sigma}\delta_\varepsilon dx' \\ &+ \varepsilon^{2-\alpha}\int_{\partial\Sigma} g\left(x', u_{|_{\partial\Sigma}}\right)\left(u^\varepsilon|_{\partial\Sigma} - u_{|_{\partial\Sigma}}\right) dx \\ &+ \frac{1}{\varepsilon^\alpha}\int_{\Sigma_\varepsilon} g\left(x, u^\varepsilon{}_{|\partial\Sigma}\right)(u^\varepsilon - u^\varepsilon|_{\partial\Sigma})\,dx\end{aligned}$$

Thanks to (3.14), $\delta_\varepsilon \rightharpoonup 0$ in $L^1(\partial\Sigma)$ and by passing to lower limit we obtain

$$\liminf_{\varepsilon\to 0}\mathcal{H}^\varepsilon(u^\varepsilon) \ge 2\varepsilon^{2-\alpha}\int_{\partial\Sigma} G\left(x', u_{|_{\partial\Sigma}}\right) dx'$$

(i) If $\alpha > 2$:

let $u \in W^{1,p(x)}(\Omega) \setminus \mathbb{G}^\alpha$ and $u^\varepsilon \in W_\partial^{1,p(x)}(\Omega)$ such that $u^\varepsilon \rightharpoonup u$ in $W^{1,p(x)}(\Omega)$ Assume that

$$\liminf_{\varepsilon \to 0} \mathcal{H}^\varepsilon(u^\varepsilon) < +\infty.$$

So there exists a constant $C > 0$ and a subsequence of $\mathcal{H}^\varepsilon(u^\varepsilon)$, still denoted by $\mathcal{H}^\varepsilon(u^\varepsilon)$, such that

$$\mathcal{H}^\varepsilon(u^\varepsilon) < C$$

According to (H2), so u^ε verify the assertions of lemma 4.3 and we can apply the proposition 4.4, thus $u_{|\partial\Sigma}$, so that $u \in \mathbb{G}^\alpha$ which contradicts the fact that $u \notin \mathbb{G}^\alpha$, as consequence

$$\liminf_{\varepsilon \to 0} \mathcal{H}^\varepsilon(u^\varepsilon) = +\infty.$$

and this completes the proof of the theorem 6.1.

□

In this subsection we shall discuss the determination of the limit problem related to the problem (5.1), when ε is considered to be converge to (approaches) zero. Basing on the results of the epiconvergence method ([3] theorem(thm1.10) and [3] proposition (prop: p.40). On other hand, in view of the τ_f-continuity of $\mathcal{G}$ in $W_\partial^{1,p}(\Omega)$, $\mathcal{H}^\varepsilon + \mathcal{G}$ τ_f-epiconverge to $\mathcal{H}^\alpha + \mathcal{G}$ in $W_\partial^{1,p}(\Omega)$.

Proposition 6.2. *For any $f \in L^{p'}(\Omega)$ and depending to the values of the parameter α, there exist $u^* \in W_\partial^{1,p(x)}(\Omega)$ satisfying*

$$u^\varepsilon \rightharpoonup u^* \text{ in } W_\partial^{1,p(x)}(\Omega),$$
$$\mathcal{H}^\alpha(u^*) + \mathcal{G}(u^*) = \inf_{v \in \mathbb{G}^\alpha} \left\{ \mathcal{H}^\alpha(v) + \mathcal{G}(v) \right\}.$$

Proof. According to the lemma 5.4, the family u^ε is bounded in $W_\partial^{1,p(x)}(\Omega)$, and therefore it posseses a τ_f- cluster value u^* in $W_\partial^{1,p(x)}(\Omega)$. At this stage we can use a classical result of epilimit (by applying theorem [3] theorem (thm1.10), one has u^* is a solution of the following limit problem

$$(\Pi^\alpha) \qquad \inf_{v \in W^{1,p(x)}(\Omega)} \Big\{ \mathcal{H}^\alpha(v) + \mathcal{G}(v) \Big\}.$$

Since $\mathcal{H}^\alpha$ equal $+\infty$ on $W^{1,p}(\Omega) \setminus \mathbb{G}^\alpha$, hence it follows that

$$(\Pi^\alpha) \qquad \inf_{v \in \mathbb{G}^\alpha} \{\mathcal{H}^\alpha(v) + \mathcal{G}(v)\}.$$

Now using the fact that the solution of the problem (Π^α) is unique, then u^ε admits a unique τ_f-adhesion value u^*, and consequently $u^\varepsilon \rightharpoonup u^*$ in $W_\partial^{1,p(x)}(\Omega)$. This proves the second part and therefore the proof of the point is now achieved. □

7 Conclusion

In this paper, we have established the existence of solutions and asymptotic behavior for Steklov problem with $p(x)$-Laplacian operator via theorem of epiconvergence. Also, some new technics of variational calculus have been discussed in this spacial and complex case. Therefore, our contribution is not only new in the given setting, but also generalizes special cases of previous works involving p and $p(x)$-Laplacian operators with Steklov conditions. The results of this work are variant, significant and so it is interesting and capable to develop its study in the future.

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