

MULTIPLICATION R_Γ -SEMIMODULES

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Abstract: This paper aims to introduce and investigate the properties of multiplication R_Γ -semimodules over Γ -semirings. This paper reviews the results related to multiplication modules, multiplication semimodules, and multiplication Γ -modules over Γ -semirings. In this paper, we study R_Γ -semimodules over Γ -semirings and examine in particular, when such R_Γ -semimodules are cyclic, finitely generated, essential or finitely co-generated.

1 Introduction

N. Nobusawa [13] first proposed the concept of Γ -ring in 1964 as a generalization of the ring. Berns [3] weakened slightly the conditions in the definition of Γ -ring in the sense of Nobusawa. After the Γ -ring was defined by Barnes and Nobusawa, many researchers studied the Γ -ring. Berns [3], Kyuno [8], and Luh [9] studied the structure of Γ -ring and obtained various generalizations analogous to corresponding parts in ring theory. In 1995, M.M.K. Rao [14] introduced the concept of a Γ -semiring as a generalization of Γ -rings and semirings. Later on Jagatap and Pawar[7], Rao and Venkateswarlu [15], studied various concepts of structure space of prime ideals and k - weakly primary ideals respectively and they enriched the theory of Γ -semirings. The concept of multiplication module was studied by Mehdi[2], Barnard[4], El-Bast and Smith [10] with the development of Γ -ring by Nobusawa [13] and Barnes [3]. The concept of multiplication Γ -modules was studied by Estaji et al. [5]. Further, Nazari and Ghalandarzadeh [11] investigate some properties of multiplication semimodules and generalize some results on multiplication modules to semimodules. In 2024, Swomin [17] introduced and investigated several properties of k -content semimodules as a generalization of content modules.

In this paper, we prove that the surjective Γ -homomorphic image of multiplication R_Γ -semimodule is multiplication R_Γ -semimodule. Also, we define $T_p(M)$, P -cyclic, and proved some results on this property. Further, we define essential and finitely co-generated R_Γ -semimodule and prove that if R is a Γ -semiring and M is a faithful multiplication R_Γ -semimodule, then an R_Γ -subsemimodule N of M is essential if and only if an essential ideal I of R exists such that $N = I\Gamma M$. Further, if R is a Γ -semiring and M a faithful multiplication R_Γ -semimodule, then M is finitely co-generated if and only if R is finitely co-generated.

2 Preliminaries

We recall from [6, 14, 16], some basic notations, definitions and examples of Γ -semirings and R_Γ -semimodules needed for this paper.

Let R and Γ be two additive commutative semigroups. Then R is called a Γ -semiring if there exists a map $R \times \Gamma \times R \rightarrow R$ ($(x, \alpha, y) \rightarrow x\alpha y$) such that for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ the following conditions hold: (i) $(x + y)\alpha z = x\alpha z + y\alpha z$, (ii) $x\alpha(y + z) = x\alpha y + x\alpha z$, (iii)

$x(\alpha + \beta)z = x\alpha z + x\beta z$, (iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Let A and B be semirings and let $R = Hom(A, B)$ and $\Gamma = Hom(B, A)$ denote the sets of homomorphisms from A to B and B to A respectively, then R is a Γ -semiring with operations of pointwise addition and composition of mappings.

Let M be a Γ -ring and let R be the set of ideals of M define addition in a natural way and if $A, B \in R$, $\gamma \in \Gamma$, let $A\gamma B$ denote the ideal generated by $\{x\gamma y \mid x, y \in M\}$, then R is a Γ -semiring.

A Γ -semiring R is said to have a zero element if $0\alpha x = 0 = x\alpha 0$ and $0 + x = x = x + 0$ for all $x \in R$ and $\alpha \in \Gamma$. R is said to have identity element 1 if for all $x \in R$ there exist $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$. R is said to be commutative if $x\alpha y = y\alpha x$ for all $x, y \in R$ and $\alpha \in \Gamma$. An element x of R is said to be Γ -multiplicative Γ -idempotent if there exists $\alpha \in \Gamma$ such that $x\alpha x = x$. If every element of R is Γ -multiplicative Γ -idempotent, then R is called multiplicative Γ -idempotent Γ -semiring. An element x of R is said to be strong Γ -multiplicative Γ -idempotent if for all $\alpha \in \Gamma$, $x\alpha x = x$. If every element of R is strong Γ -multiplicative Γ -idempotent, then R is called strong multiplicative Γ -idempotent Γ -semiring. A non-empty subset I of R is said to be the left (right) ideal of R if I is a sub semigroup of $(R, +)$ and $x\alpha y \in I$ ($y\alpha x \in I$) for all $y \in I$, $x \in R$ and $\alpha \in \Gamma$. If I is the left and right ideal of R , then I is known as an ideal of R . Let R be a Γ -semiring and $x_1, x_2, \dots, x_n \in R$, then finitely generated ideal $\langle x_1, x_2, \dots, x_n \rangle$ of R is the set of all linear combinations of the elements x_1, x_2, \dots, x_n . That is, $\langle x_1, x_2, \dots, x_n \rangle = \{\sum_{i=1}^n r_i \alpha_i x_i \mid r_i \in R, \alpha_i \in \Gamma\}$. An ideal in R is finitely generated if a finite set of elements can generate it. A proper ideal M of R is a maximal ideal if no other proper ideal of R containing M properly exists.

Let R be a Γ -semiring. An additive commutative monoid M is said to be right R_Γ -semimodule if there exist a mapping $M \times \Gamma \times R \rightarrow M$ (images to be denoted by $m\alpha r$ where $m \in M$, $\alpha \in \Gamma$, $r \in R$) satisfying the following conditions: (i) $(m+n)\alpha r = m\alpha r + n\alpha r$, (ii) $m\alpha(r+s) = m\alpha r + m\alpha s$, (iii) $m(\alpha + \beta)r = m\alpha r + m\beta r$, (iv) $(m\alpha r)\beta s = m\alpha(r\beta s)$, (v) $0_M \alpha r = 0_M = m\alpha 0_R$ for all $r, s \in R$, $\alpha, \beta \in \Gamma$ and $m, n \in M$. Similarly, a left R_Γ -semimodule is defined analogously.

Let M be an additive commutative monoid of all $m \times n$ matrices over the set of non-negative rational numbers and R be an additive commutative semigroup of all $m \times n$ matrices over the set of nonnegative integers and Γ be the additive commutative semigroup of all $n \times m$ matrices over the set of non-negative integers. Then with respect to usual matrix multiplication, R is a Γ -semiring and M is an R_Γ -semimodule.

An R_Γ -semimodule M is with unity or unitary if there exists an element $1 \in R$, $\alpha \in \Gamma$ satisfying $m\alpha 1 = m = 1\alpha m$ for all $m \in M$. A non empty subset N of M is a right R_Γ -subsemimodule of M if and only if (i) $m + n \in N$ (ii) $m\alpha r \in N$ for all $m, n \in N$, $r \in R$ and $\alpha \in \Gamma$. It is obvious that $0_M \in N$. The left R_Γ -subsemimodule is defined analogously. Similar, to the module theory over commutative semirings, it is easy to see that if $m_1, m_2, \dots, m_n \in M$, then the finitely generated R_Γ -subsemimodule $\langle m_1, m_2, \dots, m_n \rangle$ of M is the set of all linear combinations of the elements m_1, m_2, \dots, m_n . That is, $\langle m_1, m_2, \dots, m_n \rangle = \{\sum_{i=1}^n r_i \alpha_i m_i \mid r_i \in R, \alpha_i \in \Gamma\}$. Let M be a R_Γ -semimodule and I be an ideal of R . Let X be a nonempty subset of M . Then $I\Gamma X = \{\sum_{i=1}^n a_i \alpha_i m_i \mid a_i \in I, \alpha_i \in \Gamma, m_i \in X\}$ is a R_Γ -subsemimodule of M . If M, N are R_Γ -semimodules, then $f : M \rightarrow N$ is called a R_Γ -homomorphism if $f(m + n) = f(m) + f(n)$ and $f(m\alpha n) = f(m)\alpha f(n)$ for all $m, n \in M$ and $\alpha \in \Gamma$.

Remark: Throughout this paper, R will be a Γ -semiring with zero elements ‘0’ and identity element ‘1’, and all R_Γ -semimodules are assumed to be unitary unless otherwise stated.

3 Multiplication R_Γ -Semimodules

In this section, we investigate some results of multiplication R_Γ -semimodules which are generalization of the results in multiplication semimodules, modules and ideals [1, 11, 12, 18] and show in particular that they share some of the properties like finitely generated R_Γ -semimodule, cyclic R_Γ -semimodule, faithful R_Γ -semimodule, essential R_Γ -semimodule and finitely co-generated R_Γ -semimodule. Further, note that if M is a multiplication R_Γ -semimodule, then clearly so is every homomorphic image of M .

Let R be a commutative Γ -semiring and M be an R_Γ -semimodule. Let N and L be R_Γ -

subsemimodules of M . We define $(N : L) = \{r \in R \mid r\alpha L \subseteq N, \text{ for all } \alpha \in \Gamma\}$. Then $(N : L)$ is an ideal of R . $(\mathbf{0} : N)$ is an ideal of R called the annihilating ideal of N . It is denoted by $\text{ann}(N)$. Furthermore, N is said to be faithful if and only if $(\mathbf{0} : N) = \mathbf{0}$.

Before starting the results, we first define multiplication R_Γ -semimodule with an example.

Definition 3.1. Let R be a Γ -semiring. An R_Γ -semimodule M is a multiplication R_Γ -semimodule if for each R_Γ -subsemimodule N of M there exists an ideal I of R such that $N = I\Gamma M$.

Example 3.2. Let R be a strong multiplicative Γ -idempotent Γ -semiring. Then every ideal of R is a multiplication R_Γ -semimodule. For, let J be an ideal of R such that $I \subseteq J$. If $x \in I$, then $x = x\alpha x \in I\Gamma J$ for all $\alpha \in \Gamma$. Therefore, $I = I\Gamma J$ and hence J is a multiplication R_Γ -semimodule.

Proposition 3.3. Let R be a Γ -semiring and $\{I_\lambda\}_{\lambda \in \Lambda}$ be a family of ideals of R . If M is an R_Γ -semimodule then $(\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M = \sum_{\lambda \in \Lambda} (I_\lambda \Gamma M)$.

Proof. Let $x \in (\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M$. Then, there exist $x_1, x_2, \dots, x_k \in \sum_{\lambda \in \Lambda} I_\lambda$, $\alpha_1, \alpha_2, \dots, \alpha_k \in \Gamma$ and $m_1, m_2, \dots, m_k \in M$ such that $x = \sum_{t=1}^k x_t \alpha_t m_t$, it follows that for $1 \leq t \leq k$, $x_t = \sum_{j=1}^{k_t} i_{\lambda_{j_t}}$, with $i_{\lambda_{j_t}} \in I_{\lambda_{j_t}}$. Hence, $x = \sum_{t=1}^k (\sum_{j=1}^{k_t} i_{\lambda_{j_t}}) \alpha_t m_t = \sum_{t=1}^k \sum_{j=1}^{k_t} i_{\lambda_{j_t}} \alpha_t m_t \in \sum_{\lambda \in \Lambda} (I_\lambda \Gamma M)$. Therefore, $(\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M \subseteq \sum_{\lambda \in \Lambda} (I_\lambda \Gamma M)$. Also, since for every $\lambda \in \Lambda$, $I_\lambda \Gamma M \subseteq (\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M$, we conclude that $\sum_{\lambda \in \Lambda} (I_\lambda \Gamma M) \subseteq (\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M$. Hence, $(\sum_{\lambda \in \Lambda} I_\lambda)\Gamma M = \sum_{\lambda \in \Lambda} (I_\lambda \Gamma M)$. \square

Theorem 3.4. Let R be a Γ -semiring and M an R_Γ -semimodule, then M is a multiplication R_Γ -semimodule if and only if there exists an ideal I of R such that $R\Gamma m = I\Gamma M$ for all $m \in M$.

Proof. Let M be a multiplication R_Γ -semimodule and $m \in M$. Since $R\Gamma m$ is an R_Γ -subsemimodule of M and M is a multiplication R_Γ -semimodule, so there exists an ideal I of R such that $R\Gamma m = I\Gamma M$. Conversely, assume that an ideal I of R exists, such that $R\Gamma m = I\Gamma M$ for all $m \in M$. Let N be an R_Γ -subsemimodule of M . Therefore, for every $n \in N$ there exists an ideal I_n of R such that $R\Gamma n = I_n \Gamma M$. This implies that $N = \sum_{n \in N} R\Gamma n = \sum_{n \in N} I_n \Gamma M = (\sum_{n \in N} I_n)\Gamma M$. Let $I = \sum_{n \in N} I_n$. Then, $N = I\Gamma M$. Hence, M is a multiplication R_Γ -semimodule. \square

The following Theorem is a generalization of (cf. lemma 1.1, [18]).

Theorem 3.5. Let M be an R_Γ -semimodule over a Γ -semiring R . Then M is a multiplication R_Γ -semimodule if and only if for every cyclic R_Γ -subsemimodule N of the R_Γ -semimodule M there exists an ideal I of R , such that $N = I\Gamma M$.

Proof. Let N be an R_Γ -subsemimodule of an R_Γ -semimodule M and $\{N_i\}_{i \in \Lambda}$ be the set of all cyclic R_Γ -subsemimodule of N . Let $I = \sum_{i \in \Lambda} (N_i : M)$. Then clearly, I is an ideal. So, $N_i = (N_i : M)\Gamma M$ for all i . Thus, $N = \sum_{i \in \Lambda} N_i = \sum_{i \in \Lambda} (N_i : M)\Gamma M = I\Gamma M$. Converse is obvious. \square

Theorem 3.6. Let M be a multiplication R_Γ -semimodule over a Γ -semiring R . If N is a finitely generated R_Γ -subsemimodule of M , then there exists a finitely generated ideal I of R such that $N = I\Gamma M$.

Proof. Let $N = \langle x_1, x_2, \dots, x_n \rangle$. By assumption, we have $N = (N : M)\Gamma M$. Therefore, there exists $a_{ij} \in (N : M)$, $\alpha_{ij} \in \Gamma$ and $y_{ij} \in M$ such that $x_i = a_{i1}\alpha_{i1}y_{i1} + a_{i2}\alpha_{i2}y_{i2} + \dots + a_{im}\alpha_{im}y_{im}$ for all $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Let I be an ideal of R generated by $\langle a_{11}, a_{12}, \dots, a_{nm} \rangle$. it is easy to see that $I \subseteq (N : M)$ and $I\Gamma M \subseteq (N : M)\Gamma M$. On the other hand, since for each $x_i \in N$, $x_i \in I\Gamma M$. We must have $N \subseteq I\Gamma M$. Thus, $N \subseteq I\Gamma M \subseteq (N : M)\Gamma M \subseteq N$. Hence, $N = I\Gamma M$ and I is finitely generated. \square

Let M and N be R_Γ -semimodules and $f : M \rightarrow N$ be an R_Γ -homomorphism. If M' is an R_Γ -subsemimodule of M and I is an ideal of R then $f(I\Gamma M') = I\Gamma f(M')$. Now suppose that f is surjective and N' is an R_Γ -subsemimodule of N . Put $M' = \{m \in M \mid f(m) \in N'\}$. Then $f(M') = N'$.

It is well known that every homomorphic image of a multiplication semimodule is a multiplication semimodule [cf. [11], Theorem1]. A similar result holds for multiplication R_Γ -semimodules.

Theorem 3.7. *Let M, N be R_Γ -semimodules of a Γ -semiring R and $f : M \rightarrow N$ be a surjective R_Γ -homomorphism. If M is a multiplication R_Γ -semimodule, then N is a multiplication R_Γ -semimodule.*

Proof. Let N' be an R_Γ -subsemimodule of N , then there exists an R_Γ -subsemimodule M' of M such that $f(M') = N'$. Since M is a multiplication R_Γ -semimodule so there exists an ideal I of R such that $M' = I\Gamma M$. Now, $N' = f(M') = f(I\Gamma M) = I\Gamma f(M) = I\Gamma N$. Hence, N is a multiplication R_Γ -semimodule. □

Definition 3.8. Let M be an R_Γ -semimodule and P a maximal ideal of a commutative Γ -semiring R . Then following, [11] we define $T_P(M) = \{m \in M \mid \text{there exist } r \in R, \alpha \in \Gamma \text{ and } q \in P \text{ such that } r + q = 1 \text{ and } r\alpha m = 0\}$. If each elements of R and M commutes then clearly, $T_P(M)$ is an R_Γ -subsemimodule of M . Further, M is P -cyclic if there exist $r \in R, q \in P, \alpha \in \Gamma$ and $m \in M$ such that $r + q = 1$ and $r\alpha M \subseteq R\Gamma m$.

Theorem 3.9. *Let R be a commutative Γ -semiring and M be an R_Γ -semimodule then M is multiplication R_Γ -semimodule if either $T_P(M) = M$ or M is P -cyclic for all maximal ideal P of R .*

Proof. Let N be an R_Γ -subsemimodule of M and $I = (N : M)$ then $I\Gamma M \subseteq N$. Let $x \in N$ and $J = \{r \in R \mid r\alpha x \in I\Gamma M \text{ for all } \alpha \in \Gamma\}$. J is an ideal of R . If $J \neq R$, then there exists a maximal ideal P of R such that $J \subseteq P$. If $M = T_P(M)$ then there exists $r \in R, \alpha \in \Gamma$ and $q \in P$ such that $r + q = 1$ and $0 = r\alpha x \in I\Gamma M$. But if $r \in R$ and $r\alpha x \in I\Gamma M$ then $r \in J$ so $J = R$, a contradiction as P being a maximal ideal. Thus, M is P -cyclic. This implies that there exist $r \in R, q \in P, \alpha \in \Gamma$ and $m \in M$ and such that $r + q = 1$ and $r\alpha M \subseteq R\Gamma m$. Thus, $r\alpha N$ is an R_Γ -subsemimodule of $R\Gamma m$ and $r\alpha N = K\Gamma m$ where $K = \{r \in R \mid r\alpha m \in r\alpha N\}$. Moreover, $(r\Gamma K)\Gamma M \subseteq K\Gamma M \subseteq N$. Therefore, $r\Gamma K \subseteq I$. Thus, $r\alpha r\alpha x \in r\alpha r\alpha N = r\alpha(r\alpha N) \subseteq r\alpha K\Gamma m \subseteq I\Gamma M$. Hence, $r\alpha r\alpha x \in I\Gamma M$. So, $r\alpha r \in J$. Since $J \subseteq P$ so, P is a maximal ideal, which is a contradiction. Thus, $J = R$ and $x \in I\Gamma M$. This implies that $N \subseteq I\Gamma M$. Also, $I\Gamma M \subseteq N$. Hence, $N = I\Gamma M$. □

Theorem 3.10. *Let R be a commutative Γ -srming. If M is a multiplication R_Γ -semimodule, then for every maximal ideal P of R either $M = \{m \in M \mid m = q\alpha m \text{ for some } q \in P, \alpha \in \Gamma\}$ or M is P -cyclic.*

Proof. Let M be a multiplication R_Γ -semimodule and P be a maximal ideal of R .

Case-I

When $M = P\Gamma M$. If $m \in M$, then there exists an ideal I of R such that $R\Gamma m = I\Gamma M$. Hence, $R\Gamma m = I\Gamma P\Gamma M = P\Gamma I\Gamma M = P\Gamma m$. Therefore $m = q\alpha m$ for some $q \in P$ and $\alpha \in \Gamma$.

Case-II

Let $M \neq P\Gamma M$. Then there exists $x \in M$ such that $x \notin P\Gamma M$. So, for any ideal I of R , $R\Gamma x = I\Gamma M$. If $I \subseteq P$, then $x \in I\Gamma M \subseteq P\Gamma M$, which is a contradiction. Thus $I \not\subseteq P$. Since P is a maximal ideal of R , so $P + I = R$. Thus, there exists $r \in I, \alpha \in \Gamma$ and $q \in P$ such that $r + q = 1$. Moreover, $r\alpha M \subseteq I\Gamma M = R\Gamma x$. Hence, M is P -cyclic. □

Theorem 3.11. *Let R be a Γ -semiring and M a faithful R_Γ -semimodule. Then M is a multiplication R_Γ -semimodule if and only if*

- (i) *For any non-empty collection $\{I_\lambda : \lambda \in \Lambda\}$ of ideals in R , $(\bigcap_{\lambda \in \Lambda} I_\lambda)\Gamma M = \bigcap_{\lambda \in \Lambda} (I_\lambda\Gamma M)$.*
- (ii) *For any R_Γ -subsemimodule N of M and any ideal I in R with $N \subseteq I\Gamma M$, there exists an ideal J in R such that $J \subseteq I$ and $N \subseteq J\Gamma M$.*

Proof. Suppose conditions (i) and (ii) hold. Let N be an R_Γ -subsemimodule of M and $S = \{I : I \text{ is an ideal of } R \text{ and } N \subseteq I\Gamma M\}$. Clearly $R \in S$. Let I_λ be any non-empty collection of ideals in S . By (i) $\bigcap_{\lambda \in \Lambda} I_\lambda \in S$. So, by Zorn's lemma, S has a minimal element, say I . Then $N \subseteq I\Gamma M$. Suppose that $N \neq I\Gamma M$. By (ii), there exists an ideal J with $J \subseteq I$ and $N \subseteq J\Gamma M$. In this case, $J \in S$, contradicting the choice of I . Thus, $N = I\Gamma M$. It follows that M is a multiplication R_Γ -semimodule.

Conversely, Suppose M is a multiplication R_Γ -semimodule.

(i) Let $N = \bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma M)$.

Since each $I_\lambda \Gamma M$ is an R_Γ -subsemimodule of M , so $N = \bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma M)$ is an R_Γ -subsemimodule of M . But M is a multiplication R_Γ -semimodule so there exists an ideal J of R such that $N = J \Gamma M$. Thus, $J \subseteq I_\lambda$ for all $\lambda \in \Lambda$. Hence, $J \subseteq \bigcap_{\lambda \in \Lambda} I_\lambda$ implies that $J \Gamma M \subseteq (\bigcap_{\lambda \in \Lambda} I_\lambda) \Gamma M$. This implies that $\bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma M) \subseteq (\bigcap_{\lambda \in \Lambda} I_\lambda) \Gamma M$. Again, let $m \in (\bigcap_{\lambda \in \Lambda} I_\lambda) \Gamma M$. Then for every $\lambda \in \Lambda$, there exists $a_\lambda \in I_\lambda, \alpha \in \Gamma$ such that $m = a_\lambda \alpha m$. Hence, $m \in I_\lambda \Gamma M$ for all λ . Thus, $m \in \bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma M)$. This implies that $(\bigcap_{\lambda \in \Lambda} I_\lambda) \Gamma M \subseteq \bigcap_{\lambda \in \Lambda} (I_\lambda \Gamma M)$.

(ii) Let N be an R_Γ -subsemimodule of M and I an ideal of R such that $N \subseteq I \Gamma M$. Since, M is a multiplication R_Γ -semimodule then there exists an ideal K of R such that $N = K \Gamma M$. Let $J = I \cap K$. Clearly, $J \subseteq I$ and $N \subseteq I \Gamma M \cap K \Gamma M = (I \cap K) \Gamma M = J \Gamma M$. \square

Now, we have an immediate result of Theorem 3.11.

Corollary 3.12. *Let R be a Γ -semiring and M a faithful R_Γ -semimodule. Then M is a multiplication R_Γ -semimodule if and only if*

(i) *For any non-empty collection $\{I_\lambda : \lambda \in \Lambda\}$ of ideals in R , $(\bigcap_{\lambda \in \Lambda} I_\lambda) \Gamma M = \bigcap_{\lambda \in \Lambda} ((I_\lambda + \text{ann}(M)) \Gamma M)$.*

(ii) *For any R_Γ -subsemimodule N of M and any ideal I in R with $N \subseteq I \Gamma M$, there exists an ideal J in R such that $J \subseteq I$ and $N \subseteq J \Gamma M$.*

The following definition is analogous to the definition in [12].

Definition 3.13. An R_Γ -subsemimodule E of an R_Γ -semimodule M is called an essential R_Γ -subsemimodule provided $E \cap N \neq \mathbf{0}$ for every non-zero R_Γ -subsemimodule $N \subseteq M$. If E is essential in M , then we denote it by $E \subseteq_e M$. An essential ideal of R is an essential R_Γ -subsemimodule of the R_Γ -semimodule R .

We generalize the following results from [4, 12] to R_Γ -semimodules.

Theorem 3.14. *Let R be a Γ -semiring and M a faithful multiplication R_Γ -semimodule. Then an R_Γ -subsemimodule N of M is essential if and only if an essential ideal I of R exists such that $N = I \Gamma M$.*

Proof. Suppose that N is an essential R_Γ -subsemimodule of M . Since M is a multiplication R_Γ -semimodule so for R_Γ -subsemimodule N of M there exists an ideal I of R such that $N = I \Gamma M$. Suppose that $I \cap J = \mathbf{0}$ for some ideal J of R . By Theorem 3.11, we have $J \Gamma M = \mathbf{0}$. But M is faithful. This implies that $J = \mathbf{0}$. Hence, I is an essential ideal of R . Conversely, suppose that I is an essential ideal of R . Let K be an R_Γ -subsemimodule of M such that $(I \Gamma M) \cap K = \mathbf{0}$. Since, M is a multiplication R_Γ -semimodule, so there exists an ideal J of R with $K = J \Gamma M$. Thus, $(I \cap J) \Gamma M \subseteq I \Gamma M \cap J \Gamma M \subseteq I \Gamma M \cap K = \mathbf{0}$. Also, M is faithful, therefore it follows that $I \cap J = \mathbf{0}$ this implies $J = \mathbf{0}$. Hence, $I \Gamma M$ is an essential R_Γ -subsemimodule of M . \square

Let R be a Γ -semiring and M an R_Γ -semimodule. Then, we define,

(i) $Z(R) = \{r \in R \mid r \alpha s = 0 \text{ for some } 0 \neq s \in R \text{ and for all } 0 \neq \alpha \in \Gamma\}$. That is the set of all zero divisors of R .

(ii) $Z(M) = \{r \in R \mid r \alpha m = 0 \text{ for some } 0 \neq m \in M \text{ and for all } 0 \neq \alpha \in \Gamma\}$. That is the set of all zero divisors of M .

(iii) We define the socle of M by $\text{Soc}(M)$, to be $\text{Soc}(M) = \bigcap \{N \mid N \subseteq_e M\}$. (cf. [12])

Theorem 3.15. *Let M be a commutative and faithful multiplication R_Γ -semimodule, then*

(i) $\text{Soc}(M) = \text{Soc}(R) \Gamma M$

(ii) $Z(M) = Z(R)$.

Proof. (i) Let $E(M)$ denotes the collection of all essential R_Γ -subsemimodule of the R_Γ -semimodule M and $E(R)$ denotes the set of all essential ideal of R .

Now, $\text{Soc}(M) = \bigcap_{N \in E(M)} N = \bigcap_{E \in E(R)} (E \Gamma M) = (\bigcap_{E \in E(R)} E) \Gamma M = \text{Soc}(R) \Gamma M$.

- (ii) Let $r \in Z(M)$. Then for some $0 \neq m \in M$ and for all $0 \neq \alpha \in \Gamma$, $r\alpha m = 0$. Since $R\Gamma m$ is a R_Γ -subsemimodule of M and M being a multiplication R_Γ -semimodule so there exists an ideal I of R such that $R\Gamma m = I\Gamma M$. Now, $r\Gamma I\Gamma M = r\Gamma R\Gamma m = R\Gamma r\Gamma m = 0$ and hence $r\Gamma I = \mathbf{0}$, since M is faithful. Clearly, $I \neq \mathbf{0}$ and $r \in Z(R)$. Thus, $Z(M) \subseteq Z(R)$. Conversely, Let $a \in Z(R)$. Then for some $0 \neq s \in M$ and for all $0 \neq \alpha \in \Gamma$, $r\alpha s = 0$. Since M is faithful so $s\Gamma M \neq \mathbf{0}$ and $r\alpha(s\Gamma M) = \mathbf{0}$. This implies that $r \in Z(M)$. Hence, $Z(R) \subseteq Z(M)$. \square

Definition 3.16. An R_Γ -semimodule M is finitely co-generated if for every set A of an R_Γ -subsemimodule of M $\cap A = \mathbf{0}$ if and only if $\cap F = \mathbf{0}$ for some finite set $F \subseteq A$. The Γ -semiring R is called finitely co-generated if it is finitely co-generated as an R_Γ -semimodule.

Theorem 3.17. Let R be a Γ -semiring and M a faithful multiplication R_Γ -semimodule. Then M is finitely co-generated if and only if R is finitely co-generated.

Proof. Let M be finitely co-generated. Let $I_\lambda (\lambda \in \Lambda)$ be a non empty collection of ideals of R such that $\cap_{\lambda \in \Lambda} I_\lambda = \mathbf{0}$. So by Theorem 3.11, $\cap_{\lambda \in \Lambda} (I_\lambda \Gamma M) = \mathbf{0}$. Since M is finitely co-generated, so there exists a finite subset Λ' of Λ such that $\cap_{\lambda \in \Lambda'} (I_\lambda \Gamma M) = \mathbf{0}$. Thus, $(\cap_{\lambda \in \Lambda'} I_\lambda) \Gamma M = \mathbf{0}$. But M is faithful so $\cap_{\lambda \in \Lambda'} I_\lambda = \mathbf{0}$. It follows that R is finitely co-generated. Conversely, suppose R is finitely co-generated. Let $N_\lambda (\lambda \in \Lambda)$ be a non-empty collection of R_Γ -subsemimodule of M such that $\cap_{\lambda \in \Lambda} N_\lambda = \mathbf{0}$. Further, $\lambda \in \Lambda$ there exists an ideal A_λ of R such that $N_\lambda = A_\lambda \Gamma M$. Clearly, $(\cap_{\lambda \in \Lambda} A_\lambda) \Gamma M \subseteq \cap_{\lambda \in \Lambda} N_\lambda$. Thus, $\cap_{\lambda \in \Lambda} A_\lambda = \mathbf{0}$ and by hypothesis there exists a finite subset Λ' of Λ such that $\cap_{\lambda \in \Lambda'} A_\lambda = \mathbf{0}$. By Theorem 3.11, $\cap_{\lambda \in \Lambda'} N_\lambda = \cap_{\lambda \in \Lambda'} (A_\lambda \Gamma M) = (\cap_{\lambda \in \Lambda'} A_\lambda) \Gamma M = \mathbf{0}$. Hence, M is finitely co-generated. \square

4 Conclusion

This study is centered on the development of the theory of multiplication R_Γ -semimodule as a generalization of the concept of multiplication semimodules. Further, we defined the concept of cyclic, finitely generated, essential, and finitely co-generated R_Γ -semimodules. So the concept of the multiplication R_Γ -semimodule is helpful in the field of multiplicatively cancellative multiplication R_Γ -semimodule, additively cancellative multiplication R_Γ -semimodule, content R_Γ -semimodules and content R_Γ -semialgebras by different researchers to explore more in the field of R_Γ -semimodules.

References

- [1] S. E. Atani, M. S. Kohan, *A note on finitely Generated Multiplication Semimodule over Commutative Semirings*, International Journal of Algebra **4** (8) (2010), 389-396. <https://www.m-hikari.com/ija/ija-2010/ija-5-8-2010/ebrahimiIJA5-8-2010.pdf>
- [2] A. Barnard, *Multiplication Modules*, Journal of Algebra **71**(1981), 174-178. <https://www.sciencedirect.com/science/article/pii/0021869381901125/pdf?md5=361f5c80a2a7b379cd20f360ec6b58a21&pid=1-s2.0-0021869381901125-main.pdf>
- [3] W. E. Barnes, *On the Γ -ring of Nobusawa*, Pacific Journal of Mathematics **18**(3)(1966), 411-422. <https://projecteuclid.org/accountAjax/Download?downloadType=journal>
- [4] Zeinab Abd El-Bast and Patrick P. Smith, *Multiplication Modules*, Communications in Algebra, **16**(4) (1998), 755-779. <http://dx.doi.org/10.1080/00927878808823601>
- [5] A. A. Estaji, A. As. Estaji, A.S.Khorasani and S. Baghdari, *On multiplication Γ -modules*, Ratio Mathematica **26**(2014), 21-38. <https://eir.is.it/ojs/index.php/ratiomathematica/article/view/38>
- [6] S. M. A. Galindo and G. C. Petalcorin, *Some properties of Γ -semimodules over Γ -semirings*, Journal of Algebra and Applied Mathematics **13** (2) (2015), 97-113.
- [7] R. D. Jagatap, Y. S. Pawar, *Structure space of Prime Ideals of Γ -Semirings*, Palestine journal of Mathematics **5**(1)(2016), 166-170.
- [8] S. Kyuno, *On Prime Γ -ring*, Pacific Journal of Mathematics **75**(1978), 185-190.
- [9] L. Luh, *On the theory of simple Γ -rings*, Michigan Mathematics Journal **16**(1969), 65-75.
- [10] F. Mehdi, *On Multiplication Modules*, Maths. Student **42**(1974), 149-153.

- [11] R. R. Nazari and S. Ghalandarzadeh, *Multiplication Semimodules*, Acta Universitatis Sapientiae Mathematica **11** (1) (2019), 172-185. doi:10.2478/ausm-2019-0014.
- [12] R. R. Nazari and S. Ghalandarzadeh, *Content Semimodules*, Extracte Mathematicae **32** (2) (2017), 293-254. <https://revista-em.unex.es/index.php/EM/article/view/2605-5686.32.2.239/388>
- [13] N. Nobusawa, *On a generalization of the ring theory*, Osaka Journal of Mathematics **1**(1964), 81-89. <https://projecteuclid.org/accountAjax/Download?downloadType=journal>
- [14] M. M. K Rao, Γ -Semiring-I, Southeast Asian Bulletin of Mathematics **19**(1995), 49-54. <http://www.seams-bull-math.ynu.edu.cn/downloadfile.jsp?filemenu=>
- [15] M. M. K Rao, B. Venkateswarlu, *On k -weakly primary ideals of Γ -semiring*, Palestine journal of Mathematics **6**(Special Issue: II)(2017), 249-288.
- [16] Tilak Raj Sharma and Shweta Gupta, *Ideals in a Γ -semiring*, Journal of Combinatorics Information and System Sciences (JCIS) **42**(2017), 165-177.
- [17] S. Swomin, *k -Content Semimodule*, Palestine Journal of Mathematics **13** (3) (2024), 138-146.
- [18] A. A. Tuganbaev, *Multiplication Modules and Ideals*, Journal of Mathematical Sciences **136** (4) (2006), 4116-4130, 4116-4130. <https://doi.org/10.1007/s10958-006-0223-6>

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