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CERTAIN FUNCTIONALS PROPERTIES OF GENERALIZED CLASS OF ANALYTIC FUNCTIONS SUBORDINATE TO SECANT HYPERBOLIC FUNCTION

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Abstract. The purpose of the present paper is to introduce a new subclass of analytic functions denoted by $\mathcal{G}^{\alpha}_{sech}$ defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ subordinate to secant hyperbolic function. The left hand side of the subclass $\mathcal{G}^{\alpha}_{sech}$ is defined by taking into account the linear combination of starlike function and convex function with respect to symmetric points. We investigate upper bounds of some of the initial coefficients, Fekete-Szegö functional, Hankel determinant of order two and three for the said class. Further, we determine the bounds of logarithmic coefficient, inverse coefficient and logarithmic coefficients of inverse functions for such family. Zalcman conjecture, Krushkal inequality and the bounds of modulo difference of two consecutive logarithmic coefficients of inverse functions are obtained as a particular cases.

1 Introduction and Motivation

Let $\mathfrak{A}$ denote the class of all analytic and normalized functions in the open unit disk $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ and having the Taylor-Maclaurin's series expansion of the form:

$$f(\zeta) = \zeta + \sum_{n=2}^{\infty} a_n \zeta^n \quad (\zeta \in \mathbb{D}). \tag{1.1}$$

Let $\mathcal{S}$ be the subclass of $\mathfrak{A}$ consisting of univalent functions in $\mathbb{D}$. Let $\mathcal{P}$ be the class of analytic functions defined in the unit disk $\mathbb{D}$ of the form:

$$p(\zeta) = 1 + \sum_{n=1}^{\infty} p_n \zeta^n, \quad (\zeta \in \mathbb{D}) \tag{1.2}$$

and satisfy the condition of $Re\{p(\zeta)\} > 0$. So such function in $\mathcal{P}$ is called function with positive real part in $\mathbb{D}$.
Suppose that f and g are two analytic functions in $\mathbb{D}$. The function f is said to be subordinated to g, written symbolically as $f(\zeta) \prec g(\zeta)$ if there exists a Schwarz function $w(\zeta)$ with $w(0) = 0$ and $|w(\zeta)| < 1$ for all $\zeta \in \mathbb{D}$ such that $f(\zeta) = g(w(\zeta)) \quad (\zeta \in \mathbb{D})$. Using the well-known Schwarz lemma, we observe that $f(\zeta) \prec g(\zeta) \implies f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. If g is univalent in $\mathbb{D}$, then

$$f(\zeta) \prec g(\zeta) \Leftrightarrow f(0) = g(0) \; and \; f(\mathbb{D}) \subset g(\mathbb{D}).$$

While proving the Bieberbach conjecture, many scholars have investigated and studied the properties of many interesting subclasses of the class $\mathcal{S}$. Among such class, the class of starlike functions defined by

$$\mathcal{S}^* := \left\{ f \in \mathfrak{A} : Re\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right) > 0 \quad (\zeta \in \mathbb{D})\right\}.$$

In 1992, Ma and Minda [15] extended various subclasses of starlike functions for which the quantity $\frac{\zeta f'(\zeta)}{f(\zeta)}$ is subordinated to a more general function. They considered an analytic function ϕ with positive real part in the unit disk $\mathbb{D}$, with $\phi(0) = 1$, $\phi'(0) > 0$ and ϕ maps $\mathbb{D}$ onto a starlike domain with respect to 1 and symmetric with respect to the real axis. The class of Ma-Minda starlike functions consists of functions $f \in \mathfrak{A}$ satisfying the subordination

$$\frac{\zeta f'(\zeta)}{f(\zeta)} \prec \phi(\zeta). \tag{1.3}$$

A function f is said to be convex in $\mathbb{C}$ if it maps the open disk $\mathbb{D}$ conformally onto a region that is convex. We consider another class which is denoted by $\mathcal{C}_{\mathfrak{S}}$ i.e. a function $f \in \mathfrak{A}$ is said to be convex with respect to symmetric points if and only if,

$$Re\left(\frac{(\zeta f'(\zeta))'}{(f(\zeta) - f(-\zeta))'}\right) > 0 \quad (\zeta \in \mathbb{D}). \tag{1.4}$$

In recent years, finding upper bounds for the modules of Hankel determinants for various subclasses of analytic functions has become a significant area of research in the geometric function theory.
For a function $f \in \mathfrak{A}$ given by (1.1), Pommerenke [23, 24] introduced the k^{th} Hankel determinant as follows:

$$H_{k,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2(k-1)} \end{vmatrix} \quad (a_1 = 1; n, k \in \mathbb{N} = 1, 2, 3 \cdots). \tag{1.5}$$

It is useful to know whether certain coefficient functionals related to the function f are bounded in $\mathbb{D}$ or not. For fixed values of k and n, the growth rate of $H_{k,n}(f)$ as $n \to \infty$ was studied by Noonan and Thomas [18] and Noor [19] for different subfamilies of the univalent functions for the class $\mathcal{S}$. Pommerenke (see [23]) discussed some of the applications of Hankel determinant in the study of singularities.
For $k = 2,\ n = 1$ and $k = n = 2$, we have

$$H_{2,1}(f) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \tag{1.6}$$

and

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{1.7}$$

Note that, $H_{2,1}(f)$ and $H_{2,2}(f)$ are popularly known as Fekete-Szegö functional and second Hankel determinant respectively. The generalization of Fekete-Szegö functional is defined as $|a_3 - \mu a_2^2|$ for $\mu \in \mathbb{C}$.

For, $q = 3$ and $n = 1$, we have $H_{3,1}(f)$ is known as Hankel determinant of third order, defined as:

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \tag{1.8}$$

Various researchers have considered the determinant for different subclasses of $\mathfrak{A}$ in diverse direction and their results are available in literature (see [1, 2, 5, 10, 17, 20, 21, 22]).

Recently, Hu and Deng (see [8]) introduced and studied the class $H(\lambda, \psi)$ defined by,

$$(1-\lambda)(f'(\zeta))^{1-\lambda} + \lambda\left(\frac{2\zeta f'(\zeta)}{f(\zeta)-f(-\zeta)}\right)^{\lambda} \prec \phi(\zeta),$$

and investigate the upper bounds of the third and fourth Hankel determinant and also bound on third Hankel determinant of its inverse function in the special case when $\phi(\zeta) = \frac{2}{1+e^{-\zeta}}$.

Further, Bano et al. [3] (also see [25]) introduced and studied the class $\mathcal{S}_E^*$ defined by

$$\mathcal{S}_E^* = \{f \in \mathfrak{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec sech(\zeta)\},$$

and investigated sharp bounds of initial coefficients results and Hankel determinants of order two and three for the function f and its inverse function f^{-1} for such family.

Motivated by aforementioned works, in this paper with the help of subordination we introduce the class $\mathcal{G}_{sech}^{\alpha}$ as follows:

Definition 1.1. A function $f \in \mathfrak{A}$ given by (1.1) is said to be in the class $\mathcal{G}_{sech}^{\alpha}$ if the following condition hold:

$$F[f^*] = (1-\alpha)\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\alpha} + \alpha\left(\frac{2(\zeta f'(\zeta))'}{(f(\zeta)-f(-\zeta))'}\right)^{\alpha} \prec sech(\zeta) \quad (0 \le \alpha \le 1,\ \zeta \in \mathbb{D}). \tag{1.9}$$

The justification for taking the above left-hand-side expression is based on fact that we could obtain subordination condition for the expression $\frac{\zeta f'(\zeta)}{f(\zeta)}$ and $\frac{2(\zeta f'(\zeta))'}{(f(\zeta)-f(-\zeta))'}$. For particular value of α, some of these functions vanishes or the formula become simple.

Remark 1.2. (i) Letting $\alpha = 0$ in Definition 1.1, we obtain the class $\mathcal{G}_{sech}^{0} \approx \mathcal{S}_E^*$ introduced and studied by Bano et al. [3] (also see [25]).i.e

$$\mathcal{S}_E^* = \{f \in \mathfrak{A} : \frac{\zeta f'(\zeta)}{f(\zeta)} \prec sech(\zeta)\}.$$

(ii) Taking $\alpha = 1$ in the Definition 1.1, we get the class $\mathcal{G}_{sech}^{1} \approx \mathcal{G}_{sech}^{*}$:

$$\mathcal{G}_{sech}^{*} = \{f \in \mathfrak{A} : \frac{2(\zeta f'(\zeta)'}{(f(\zeta)-f(-\zeta))'} \prec sech(\zeta), \quad (\zeta \in \mathbb{D})\}.$$

Remark 1.3. We would like to emphasize that the class $\mathcal{G}_{sech}^{\alpha}$ defined as above is non-empty. Thus, if we consider the function $f^*(\zeta) = a\zeta^2 + \zeta,\ (|a| \le 1)$ then it is easy to check that $f^* \in \mathcal{S}$.

For the particular case $a = 0.20$ and $\alpha = 0.50$, using the *2D plot* of the MAPLE™ computer software, we obtain the image of the boundary $\partial\mathbb{D}$ by the functions $F[f^*]$, shown in Figure 1.

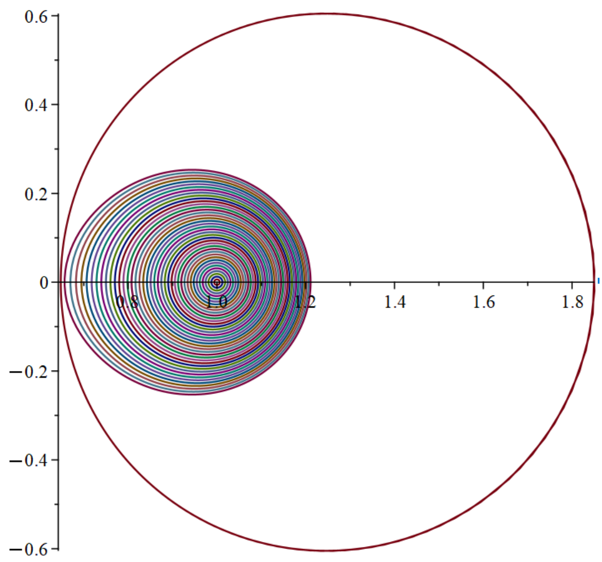

Figure 1. The image of $F[f^*](\mathbb{D})$.

Since $sech\zeta$ is univalent in $\mathbb{D}$, the previous explanation yields that the subordination $F[f^*](\zeta) \prec sech(\zeta)$ hold whenever $F[f^*](0) = sech(0)$ and $F[f^*](\mathbb{D}) \subset sech(\mathbb{D})$. In conclusion, $f^* \in \mathcal{G}^{\alpha}_{sech}$; hence, the class $\mathcal{G}^{\alpha}_{sech}$ is non empty and contains other functions besides the identity.

In the present paper, authors investigate some of the initial coefficient bounds, Fekete-Szegö functional and Hankel determinant of second and third order the class $\mathcal{G}^{\alpha}_{sech}$. Moreover, the bound of logarithmic coefficients, inverse coefficients, logarithmic coefficients of inverse function, Zalcman conjecture, Krushkal inequality and modulo difference for logarithmic coefficients of inverse functions are also obtained.

2 Preliminaries

Here we use the following lemmas for our further investigations.

Lemma 2.1. *(see [13]) Let $p \in \mathcal{P}$ and be of the form (1.2). Then,*

$$2p_2 = p_1^2 + \rho(4 - p_1^2), 4p_3 = p_1^3 + 2p_1(4 - p_1^2)\rho - p_1(4 - p_1^2)\rho^2 + 2(4 - p_1^2)(1 - |\rho|^2)\eta, \quad (2.1)$$

for some ρ and η such that $|\rho| \leq 1$ and $|\eta| \leq 1$.

Lemma 2.2. *(see [7]) Let $p \in \mathcal{P}$ and has the expansion of the form (1.2), then*

$$|p_n| \leq 2 \quad \forall n \in \mathbb{N} = \{1, 2, 3, \cdots\}. \quad (2.2)$$

Lemma 2.3. *(see [15]) If $p \in \mathcal{P}$ then for any $\mu \in \mathbb{C}$, we have*

$$|p_2 - \mu p_1^2| \leq 2max\{1, |2\mu - 1|\}. \quad (2.3)$$

Lemma 2.4. *(see [15]) Let $p \in \mathcal{P}$ be given by (1.2). Then*

$$|p_2 - \mu p_1^2| \leq 2 - \mu|p_1^2| \quad (2.4)$$

for $0 < \mu \leq \frac{1}{2}$.

Lemma 2.5. *(see [1]) Let $p \in \mathcal{P}$ and has the expansion of the form (1.2). Then*

$$|Jp_1^3 - Kp_1p_2 + Lp_3| \leq 2[|J| + |K - 2J| + |J - K + L|], \quad (2.5)$$

where $J,\ K,\ L \in \mathbb{C}$.

Lemma 2.6. *([6]) Let $\overline{\mathbb{D}} := \{\rho \in \mathbb{C} : |\rho| \leq 1\}$, and for J, K, $L \in \mathbb{R}$, let*

$$Y(J,K,L) := max\left\{|J + K\rho + L\rho^2| + 1 - |\rho|^2 : \rho \in \overline{D}\right\}. \tag{2.6}$$

If $JL \geq 0$, then

$$Y(J,K,L) = \begin{cases} |J| + |K| + |L|, & |K| \geq 2(1-|L|), \\ 1 + |J| + \frac{K^2}{4(1-|L|)}, & |K| < 2(1-|L|). \end{cases}$$

If $JL \leq 0$, then

$$\begin{cases} 1 - |J| + \frac{K^2}{4(1-|L|)}, & -4JL(L^{-2}-1) \leq K^2 \wedge |K| < 2(1-|L|), \\ 1 + |J| + \frac{K^2}{4(1+|L|)}, & K^2 < min\{4(1+|L|)^2, -4JL(L^{-2}-1)\}, \\ R(J,K,L), & otherwise, \end{cases}$$

where

$$\begin{cases} |J| + |K| - |L|, & |L|(|K| + 4|J|) \leq |JK|, \\ 1 + |J| + \frac{K^2}{4(1+|L|)}, & |JK| < |L|(|K| - 4|J|), \\ |L| + |J|\sqrt{1 - \frac{K^2}{4JL}}, & otherwise. \end{cases}$$

Lemma 2.7. *([26], Preposition 1) Let $p \in \mathcal{P}$ be given by (1.2). Let $\mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{B}_3$ be numbers such that $\mathcal{B}_1 \geq 0$, $\mathcal{B}_2 \in \mathbb{C}$ and $\mathcal{B}_3 \in \mathbb{R}$. Define $\psi_+(p_1, p_2)$ and $\psi_-(p_1, p_2)$ by*

$$\psi_+(p_1, p_2) = \left|\mathcal{B}_2 p_1^2 + \mathcal{B}_3 p_2\right| - \left|\mathcal{B}_1 q_1\right|,$$

and $\psi_-(p_1, p_2) = -\psi_+(p_1, p_2)$. Then

$$\psi_+(p_1, p_2) \leq \begin{cases} |4\mathcal{B}_2 + 2\mathcal{B}_3| - 2\mathcal{B}_1, & when \quad |2\mathcal{B}_2 + \mathcal{B}_3| \geq |\mathcal{B}_3| + \mathcal{B}_1, \\ 2|\mathcal{B}_3|, & \qquad otherwise, \end{cases} \tag{2.7}$$

and

$$\psi_-(p_1, p_2) \leq \begin{cases} 2\mathcal{B}_1 - \mathcal{B}_4, & when \quad \mathcal{B}_1 \geq \mathcal{B}_4 + 2|\mathcal{B}_3|, \\ 2\mathcal{B}_1\sqrt{\frac{2|\mathcal{B}_3|}{\mathcal{B}_4 + 2|\mathcal{B}_3|}}, & when \quad \mathcal{B}_1^2 \leq 2|\mathcal{B}_3|(\mathcal{B}_4 + 2|\mathcal{B}_3|), \\ 2|\mathcal{B}_3| + \frac{\mathcal{B}_1^2}{\mathcal{B}_4 + 2|\mathcal{B}_3|}, & \qquad otherwise, \end{cases} \tag{2.8}$$

where $\mathcal{B}_4 = |4\mathcal{B}_2 + 2\mathcal{B}_3|$. All the inequalities in (2.7) and (2.8) are sharp.

3 Main Results

In this section, we investigate first four initial coefficient bounds and the upper bounds of the modules of second and third Hankel determinants for the class $\mathcal{G}^{\alpha}_{sech}$.
The following theorem gives the bounds of some of the initial coefficient for the class $\mathcal{G}^{\alpha}_{sech}$.

Theorem 3.1. *If $f \in \mathcal{G}^{\alpha}_{sech}$ has of the form (1.1) and $0 \leq \alpha \leq 1$, then*

$$a_2 = 0, \tag{3.1}$$

$$|a_3| \leq \frac{1}{4(4\alpha^2 - 2\alpha + 1)}, \tag{3.2}$$

$$|a_4| \leq \frac{2}{3\sqrt{3}(19\alpha^2 - 6\alpha + 3)}, \tag{3.3}$$

and

$$|a_5| \leq \frac{1}{8(6\alpha^2 - 2\alpha + 1)}. \tag{3.4}$$

Proof. Let the function $f \in \mathfrak{A}$ given by (1.1) be the member of the class $\mathcal{G}^{\alpha}_{sech}$. Then by Definition 1.1, there exists an analytic function $w(\zeta)$ satisfies the condition of Schwarz lemma such that,

$$(1-\alpha)\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\alpha} + \alpha\left(\frac{2(\zeta f'(\zeta))'}{(f(\zeta)-f(-\zeta))'}\right)^{\alpha} = sech(w(\zeta)). \tag{3.5}$$

Define a function

$$p(\zeta) = \frac{1+w(\zeta)}{1-w(\zeta)} = 1 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + \cdots. \tag{3.6}$$

Clearly, the function $p(\zeta)$ is analytic in $\mathbb{D}$ with $p(0) = 1$ and $Re(p(\zeta)) > 0$ ($\zeta \in \mathbb{D}$). Hence $p \in \mathcal{P}$, the class of Carathèodory function.
From (3.6), we have

$$\begin{aligned} w(\zeta) &= \frac{p(\zeta)-1}{p(\zeta)+1} = \frac{p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + \cdots}{2 + p_1\zeta + p_2\zeta^2 + p_3\zeta^3 + p_4\zeta^4 + p_5\zeta^5 + p_6\zeta^6 + \cdots}, \\ &= \frac{p_1}{2}\zeta + \frac{1}{2}\left(p_2 - \frac{p_1^2}{2}\right)\zeta^2 + \frac{1}{2}\left(p_3 - p_1p_2 + \frac{p_1^3}{4}\right)\zeta^3 + \frac{1}{2}\left(p_4 - p_1p_3 + \frac{3}{4}p_1^2p_2 - \frac{p_1^4}{8} - \frac{p_2^2}{2}\right)\zeta^4 + \cdots. \end{aligned} \tag{3.7}$$

Simple calculation follows from (1.1) that,

$$\begin{aligned} &\left(\frac{\zeta f'(\zeta)}{f(\zeta)}\right)^{1-\alpha} + \alpha\left(\frac{2(\zeta f'(\zeta))'}{(f(\zeta)-f(-\zeta))'}\right)^{\alpha} \\ &= 1 + (5\alpha^2 - 2\alpha + 1)a_2\zeta + \left[2(4\alpha^2 - 2\alpha + 1)a_3 + \left(\frac{15}{2}\alpha^3 - 8\alpha^2 + \frac{3}{2}\alpha - 1\right)a_2^2\right]\zeta^2 \\ &+ \left[(19\alpha^2 - 6\alpha + 3)a_4 + \frac{(1-\alpha)(-65\alpha^3 + 122\alpha^2 + \alpha + 6)}{6}a_2^3 + (22\alpha^3 - 35\alpha^2 + 4\alpha - 3)a_2a_3\right]\zeta^3 \\ &+ \left[4(6\alpha^2 - 2\alpha + 1)a_5 + 2(8\alpha^3 - 17\alpha^2 + \alpha - 1)a_3^2 + (49\alpha^4 - 188\alpha^3 + 137\alpha^2 - 2\alpha + 4)a_2^2a_3\right. \\ &\left.+ (61\alpha^3 - 62\alpha^2 + 5\alpha - 4)a_2a_4 - \frac{(1-\alpha)(255\alpha^4 - 1294\alpha^3 + 1501\alpha^2 + 26\alpha + 24}{24}a_2^4\right]\zeta^4 + \cdots. \end{aligned} \tag{3.8}$$

Using (3.7) in r.h.s. of relation (3.5) we get,

$$sech(w(\zeta)) = 1 - \frac{p_1^2}{8}\zeta^2 - \left(\frac{p_1p_2}{4} - \frac{p_1^3}{8}\right)\zeta^3 - \left(\frac{31}{384}p_1^4 - \frac{3}{8}p_1^2p_2 + \frac{p_2^2}{8} + \frac{p_1p_3}{4}\right)\zeta^4 + \cdots. \tag{3.9}$$

Compairing the coefficients of ζ, ζ^2, ζ^3 *and* ζ^4 on both sides of (3.8) and (3.9) we obtain,

$$a_2 = 0, \tag{3.10}$$

$$a_3 = \frac{-p_1^2}{16(4\alpha^2 - 2\alpha + 1)}, \tag{3.11}$$

$$a_4 = \frac{p_1}{8(19\alpha^2 - 6\alpha + 3)}\left[p_1^2 - 2p_2\right], \tag{3.12}$$

and

$$a_5 = \frac{1}{4(6\alpha^2 - 2\alpha + 1)}\left[\frac{3}{8}p_1^2p_2 - \frac{p_2^2}{8} - \frac{p_1p_3}{4} - \left(\frac{496\alpha^4 - 472\alpha^3 + 321\alpha^2 - 121\alpha + 28}{384(4\alpha^2 - 2\alpha + 1)^2}\right)p_1^4\right]. \tag{3.13}$$

An application of (2.2) of Lemma 2.2 in (3.11) gives the require bounds for $|a_3|$.

Taking modulus on both sides of equation (3.12) and using Lemma 2.4, we obtain

$$\begin{aligned} |a_4| &= \frac{p_1}{4(19\alpha^2-6\alpha+3)}\left|p_2-\frac{1}{2}p_1^2\right| \\ &\leq \frac{1}{4(19\alpha^2-6\alpha+3)}\left[2p_1-\frac{1}{2}p_1^3\right]. \end{aligned}$$

Since the class $\mathcal{G}^{\alpha}_{sech}$ is rotationally invariant, so without loss of any generality we take $p_1 := p \in [0,2]$. Hence

$$|a_4| \leq \frac{1}{4(19\alpha^2-6\alpha+3)}\psi(p), \tag{3.14}$$

where $\psi(p) := 2p - \frac{1}{2}p^3$.
Taking $\psi'(p) = 0$, we obtain

$$p = \frac{2\sqrt{3}}{3}.$$

So for $p = \frac{2\sqrt{3}}{3}$, we get

$$\psi''\left(\frac{2\sqrt{3}}{3}\right) = -2\sqrt{3} < 0.$$

This shows the function $\psi(p)$ has its maxima at $p = \frac{2\sqrt{3}}{3}$ and maximum value is

$$\max_{p\in[0,2]} \psi(p) \leq \psi\left(\frac{2\sqrt{3}}{3}\right) = \frac{8}{9}\sqrt{3}.$$

Thus, From (3.14)

$$|a_4| \leq \frac{2}{3\sqrt{3}(19\alpha^2-6\alpha+3)}.$$

From the relation (3.13), we have

$$a_5 = \frac{1}{384(6\alpha^2-2\alpha+1)}\left[\left(-\frac{496\alpha^4-472\alpha^3+321\alpha^2-121\alpha+28}{4(4\alpha^2-2\alpha+1)^2}\right)p_1^4+36p_1^2p_2-24p_1p_3-12p_2^2\right]. \tag{3.15}$$

Applying Lemma 2.1 in (3.15) and after some computations we may write

$$\begin{aligned} a_5 =& \frac{1}{384(6\alpha^2-2\alpha+1)}\left[\left(9-\frac{496\alpha^4-472\alpha^3+321\alpha^2-121\alpha+28}{4(4\alpha^2-2\alpha+1)^2}\right)p_1^4+6p_1^2(4-p_1^2)\rho^2\right. \\ &\left. -3(4-p_1^2)^2\rho^2-12p_1(4-p_1^2)(1-|\rho|^2)\eta\right], \\ =& \frac{1}{384(6\alpha^2-2\alpha+1)}\left[v_1(p_1,\rho)+v_2(p_1,\rho)\eta\right], \end{aligned} \tag{3.16}$$

where $\rho, \eta \in \bar{\mathbb{D}}$ and $0 \leq \alpha \leq 1$,

$$v_1(p_1,\rho) = \left(9-\frac{496\alpha^4-472\alpha^3+321\alpha^2-121\alpha+28}{4(4\alpha^2-2\alpha+1)^2}\right)p_1^4+6p_1^2(4-p_1^2)\rho^2-3(4-p_1^2)^2\rho^2,$$

$$v_2(p_1,\rho) = -12p_1(4-p_1^2)(1-|\rho|^2).$$

Without loss of generality, we may assume that $p_1 = p \in [0, 2]$. Applying triangle inequality on both sides of (3.16) and letting $t := |\rho|$, and $u := |\eta|$, we get

$$\begin{aligned} |a_5| \leq & \frac{1}{384(6\alpha^2 - 2\alpha + 1)} \left(|v_1(p,\rho)| + |v_2(p,\rho)|u\right), \\ & \leq \mathcal{J}(p,t,u), \end{aligned}$$

where

$$\mathcal{J}(p,t,u) := \frac{1}{384(6\alpha^2 - 2\alpha + 1)} \left(\mathcal{J}_1(p,t) + \mathcal{J}(p,t)u\right), \tag{3.17}$$

with

$$\mathcal{J}_1(p,t) := \left(9 - \frac{496\alpha^4 - 472\alpha^3 + 321\alpha^2 - 121\alpha + 28}{4(4\alpha^2 - 2\alpha + 1)^2}\right) p^4 + 6p^2(4 - p^2)t^2 + 3(4 - p^2)^2 t^2,$$

$$\mathcal{J}_2(p,t) := 12p(4 - p^2)(1 - t^2).$$

Now, we have to find the maximum value in $\mathcal{S} : [0, 2] \times [0, 1] \times [0, 1]$. We determine the maximum value of the function $\mathcal{J}(p,t,u)$ in the interior of $\mathcal{S}$ on the edges and vertices of $\mathcal{S}$.

Consider the interior points of $\mathcal{S}$. Differentiating (3.17) with respect to u and after some simplifications, we obtain

$$\frac{\partial \mathcal{J}(p,t,u)}{\partial u} = \frac{p(4 - p^2)(1 - t^2)}{32(6\alpha^2 - 2\alpha + 1)}. \tag{3.18}$$

Since equation (3.18) is independent of u, we have no maximum value in $(0, 2) \times (0, 1) \times (0, 1)$. Next, we discuss the optimum value in the interior of six faces of $\mathcal{S}$.
On the face $p = 0$, $\mathcal{J}(p,t,u)$ reduces to

$$\mathcal{J}(0,t,u) = l_1(t,u) = \frac{t^2}{8(6\alpha^2 - 2\alpha + 1)}, \quad t, u \in (0, 1). \tag{3.19}$$

The function l_1 has no maximum in $(0, 1) \times (0, 1)$ since

$$\frac{\partial l_1}{\partial t} = \frac{t}{4(6\alpha^2 - 2\alpha + 1)} \neq 0, \quad t \in (0, 1).$$

Assume that, $k = \frac{496\alpha^4 - 472\alpha^3 + 321\alpha^2 - 121\alpha + 28}{4(4\alpha^2 - 2\alpha + 1)^2}$.
On $p = 2$, $\mathcal{J}(p,t,u)$ takes the form

$$\mathcal{J}(2,t,u) = \frac{9 - k}{24(6\alpha^2 - 2\alpha + 1)}, \quad t, u \in (0, 1). \tag{3.20}$$

On $t = 0$, $\mathcal{J}(p,t,u)$ can be written as

$$\mathcal{J}(p,0,u) = l_2(p,u) := \frac{(9 - k)p^4 + 12p(4 - p^2)u}{384(6\alpha^2 - 2\alpha + 1)}, \quad p \in (0, 2), u \in (0, 1). \tag{3.21}$$

The function l_2 has maximum $(0, 2) \times (0, 1)$, since

$$\frac{\partial l_2}{\partial u} = \frac{p(4 - p^2)}{32(6\alpha^2 - 2\alpha + 1)} \neq 0, \quad p \in (0, 2).$$

On $t = 1$, $\mathcal{J}(p,t,u)$ reduces to

$$\mathcal{J}(p,1,u) = l_3(p,u) = \frac{(6 - k)p^4 + 48}{384(6\alpha^2 - 2\alpha + 1)}, \quad p \in (0, 2). \tag{3.22}$$

There is no critical point for the function l_3 in $(0,2)\times(0,1)$, since

$$\frac{\partial l_3}{\partial p} = \frac{(6-k)p^3}{96(6\alpha^2-2\alpha+1)} \neq 0, \quad p\in(0,2).$$

On $u=0$, $\mathcal{J}(p,1,u)$ reduces to

$$\mathcal{J}(p,t,0) = l_4(p,t) = \frac{1}{384(6\alpha^2-2\alpha+1)}\left[(9-k)p^4+6p^2(4-p^2)t^2+3(4-p^2)^2t^2\right].$$

Therefore,

$$\frac{\partial l_4}{\partial t} = \frac{12p^2(4-p^2)t+6t(4-p^2)^2}{384(6\alpha^2-2\alpha+1)},$$

and

$$\frac{\partial l_4}{\partial p} = \frac{4(9-k)p^3-12p^3t^2}{384(6\alpha^2-2\alpha+1)}.$$

By implementing numerical methods, we see that the system $\frac{\partial l_4}{\partial p}=0$ and $\frac{\partial l_4}{\partial t}=0$ has no solution in $(0,2)\times(0,1)$.
On $u=1$, the function $\mathcal{J}(p,t,u)$ takes the form

$$\mathcal{J}(p,t,1) = l_5(p,t) = \frac{(9-k)p^4+3(4-p^2)^2t^2+(4-p^2)(6p^2t^2+12p-12pt^2)}{384(6\alpha^2-2\alpha+1)}.$$

Similarly, on the face $u=0$, the system of equations $\frac{\partial l_5}{\partial p}=0$ and $\frac{\partial l_5}{\partial t}=0$ has no solution in $(0,2)\times(0,1)$.
Now, we investigate the maximum of $\mathcal{J}(p,t,u)$ on the edges of $\mathcal{S}$. From (3.21), we obtain $\mathcal{J}(p,0,0)=s_1(p)=\frac{(9-k)p^4}{384(6\alpha^2-2\alpha+1)}$.
Since, $s_1'(p)=\frac{4(9-k)p^3}{384(6\alpha^2-2\alpha+1)}>0$ for $[0,2]$ and $0\leq\alpha\leq1$. Therefore, s_1 is increasing in $[0,2]$ and hence a maximum is achieved at $p=2$. Therefore,

$$\mathcal{J}(p,0,0) \leq max(2,0,0) = \frac{9-k}{24(6\alpha^2-2\alpha+1)}.$$

Also, from (3.17) at $u=1$, we write

$$\mathcal{J}(p,0,1) = s_2(p) = \frac{(9-k)p^4+12p(4-p^2)}{384(6\alpha^2-2\alpha+1)},$$

since $s_2'=0$ for $p=1.388684545$ in $[0,2]$ and $\alpha=0$; $\mathcal{J}(p,0,1)\leq 0.1092672897$.
Putting, $p=0$ in (3.21), we have

$$\mathcal{J}(0,0,u)=0.$$

Since (3.22) is independent of t, we obtain $\mathcal{J}(p,1,1)=\mathcal{J}(p,1,0)=s_3(p):=\frac{(6-k)p^4+48}{384(6\alpha^2-2\alpha+1)}$.
Since $s_3'(p)\neq0$ for for $p\in[0,2]$ and $\alpha\in[0,1]$.
Thus $s_3(p)$ has no maxima and minima, because it depends upon $6-k$. For $\alpha\in[0,1]$, $6-k$ will give positive as well as negative value.
Putting $p=0$ in (3.22), we obtain

$$\mathcal{J}(0,1,u) = \frac{1}{8(6\alpha^2-2\alpha+1)}.$$

Equation (3.20) is independent variables of u and t. Thus,

$$\mathcal{J}(2,t,0)=\mathcal{J}(2,t,1)=\mathcal{J}(2,0,u)=\mathcal{J}(2,1,u)=\frac{9-k}{24(6\alpha^2-2\alpha+1)}, \quad t,u\in[0,1].$$

As equation (3.19) is independent of the variable u, we have $\mathcal{J}(0,t,0) = \mathcal{J}(0,t,1) = s_4(t) = \frac{t^2}{8(6\alpha^2-2\alpha+1)}$.
Since $s_4'(t) = \frac{t}{4(6\alpha^2-2\alpha+1)}$, which is increasing function for $t \in [0,1]$ and fix α as constant. Thus, $s_4(t)$ is increasing in $[0,1]$ and hence a maximum is achieved at $t = 1$.
Hence

$$|a_5| \leq \mathcal{J}(p,t,u) = \frac{1}{8(6\alpha^2 - 2\alpha + 1)}.$$

This completes the proof of Theorem 3.1. □

By setting $\alpha = 0$ in Theorem 3.1, we obtain the following results, due to Raza et al. [25].

Corollary 3.2. *(see[25]) Let $f \in \mathcal{S}_E^*$ and be of the form (1.1). Then,*

$$a_2 = 0, \quad |a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{2}{27}\sqrt{3} \quad \text{and} \quad |a_5| \leq \frac{1}{8}. \tag{3.23}$$

These bounds are sharp.

By setting $\alpha = 1$ in Theorem 3.1 and proceeding in a similar manner, we obtain the following results:

Corollary 3.3. *Let $f \in \mathcal{G}^*_{sech}$ and be of the form (1.1). Then,*

$$a_2 = 0, \quad |a_3| \leq \frac{1}{12}, \quad |a_4| \leq \frac{1}{24\sqrt{3}} \quad \text{and} \quad |a_5| \leq \frac{1}{40}. \tag{3.24}$$

Next, we determine the upper bounds of the modules for the Fekete-Szegö functional and second and third order Hankel determinant for the functions that belong to the class $\mathcal{G}^\alpha_{sech}$.

Theorem 3.4. *If $f \in \mathcal{G}^\alpha_{sech}$ has of the form (1.1) and for $0 \leq \alpha \leq 1$,*

(i)

$$|H_{2,1}(f)| \leq \frac{1}{4(4\alpha^2 - 2\alpha + 1)}, \tag{3.25}$$

(ii)

$$|H_{2,2}(f)| \leq \frac{1}{16(4\alpha^2 - 2\alpha + 1)^2}, \tag{3.26}$$

(iii)

$$\begin{aligned} |H_{3,1}(f)| \leq\ & \frac{(16b - 8\sqrt{4b^2 - ac})^3}{24576a^3(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)^3(6\alpha^2 - 2\alpha + 1)} \\ & \left(-31600\alpha^8 + 13144\alpha^7 + 8595\alpha^6 - 21551\alpha^5 + 17564\alpha^4 - 8958\alpha^3 + 3027\alpha^2 - 651\alpha + 78\right) \\ & - \frac{(16b - 8\sqrt{4b^2 - ac})^2\left(92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28\right)}{512a^2(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)} \\ & + \frac{(16b - 8\sqrt{4b^2 - ac})(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{128a(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)}, \end{aligned} \tag{3.27}$$

where

$$a = \frac{-31600\alpha^8 + 13144\alpha^7 + 8595\alpha^6 - 21551\alpha^5 + 17564\alpha^4 - 8958\alpha^3 + 3027\alpha^2 - 651\alpha + 78}{(4\alpha^2 - 2\alpha + 1)^2},$$

$$b = 92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28,$$

and

$$c = 407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23.$$

Proof. (i) If $f \in \mathcal{G}^{\alpha}_{sech}$, then from (3.10) and (3.11) we get

$$H_{2,1}(f) = a_3 - a_2^2 = a_3 = \frac{-p_1^2}{16(4\alpha^2 - 2\alpha + 1)}. \tag{3.28}$$

The inequality (3.25) follows by application of (2.2) of Lemma 2.2 in the relation (3.28).

(ii) From the relations (3.10)-(3.12), we have

$$H_{2,2}(f) = a_2 a_4 - a_3^2 = \frac{-p_1^4}{256(4\alpha^2 - 2\alpha + 1)^2}. \tag{3.29}$$

The bounds of (3.26) follows from (3.29) by application of Lemma 2.2.

(iii) Using (3.10) in the expression (1.8) gives,

$$H_{3,1}(f) = -a_3^3 - a_4^2 + a_3 a_5. \tag{3.30}$$

Substituting the values of (3.11)-(3.13) in the relation (3.30), we obtain

$$H_{3,1}(f) = \frac{1}{256(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)}$$
$$\left[\left(\frac{31600\alpha^8 - 13144\alpha^7 - 8595\alpha^6 + 21551\alpha^5 - 17564\alpha^4 + 8958\alpha^3 - 30272\alpha^2 + 651\alpha - 78}{96(4\alpha^2 - 2\alpha + 1)^2(19\alpha^2 - 6\alpha + 3)^2}\right) p_1^6\right.$$
$$\left. - \left(\frac{315\alpha^4 - 44\alpha^3 + 2\alpha^2 + 20\alpha - 5}{2(19\alpha^2 - 6\alpha + 3)^2}\right) p_1^4 p_2 - \left(\frac{407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23}{2(19\alpha^2 - 6\alpha + 3)^2}\right) p_1^2 p_2^2 + p_1^3 p_3\right].$$

As we see that the functional $H_{3,1}(f)$ and the class $\mathcal{G}^{\alpha}_{sech}$ are rotationally invariant, we may take $p_1 := p$, such that $p \in [0, 2]$. Then, by using Lemma 2.1 and after some computations we may write

$$H_{3,1}(f) = \frac{1}{36864(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)}\psi, \tag{3.31}$$

where

$$\psi = -3\left(\frac{80\alpha^4 - 104\alpha^3 + 75\alpha^2 - 11\alpha + 2}{2(4\alpha^2 - 2\alpha + 1)^2}\right) p^6 + 72p^3(4 - p^2)(1 - |\rho|^2)\eta$$
$$- 9p^2(4 - p^2)\rho^2\left(\frac{(630\alpha^4 - 88\alpha^3 + 4\alpha^2 + 40\alpha - 10)p^2 + 8(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{(19\alpha^2 - 6\alpha + 3)^2}\right), \tag{3.32}$$

where ρ and η satisfy the relation $|\rho| \le 1$, $|\eta| \le 1$.

First, we consider the case when $p = 2$. Then,

$$|\psi| \le \frac{96(80\alpha^4 - 104\alpha^3 + 75\alpha^2 - 11\alpha + 2)}{(4\alpha^2 - 2\alpha + 1)^2},$$

therefore from (3.31), we obtain

$$|H_{3,1}(f)| \le \frac{96(80\alpha^4 - 104\alpha^3 + 75\alpha^2 - 11\alpha + 2)}{36864(4\alpha^2 - 2\alpha + 1)^3(6\alpha^2 - 2\alpha + 1)}.$$

Next, we assume that $p = 0$, then $|\psi| = 0$.
Now suppose that $p \in (0, 2)$ and using the well-known triangle inequality in (3.32), we obtain

$$|\psi| \le 72p^3(4 - p^2)\phi(J, K, L),$$

where

$$\phi(J,K,L) = |J + K\rho + +L\rho^2| + 1 - |\rho|^2, \quad \rho \in \overline{\mathbb{D}}, with$$

$$J = \frac{-(80\alpha^4 - 104\alpha^3 + 75\alpha^2 - 11\alpha + 2)p^3}{48(4-p^2)(4\alpha^2 - 2\alpha + 1)^2}, \quad K = 0$$

and

$$L = \frac{(630\alpha^4 - 88\alpha^3 + 4\alpha^2 + 40\alpha - 10)p^2 + 8(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{8p(19\alpha^2 - 6\alpha + 3)^2}.$$

Then clearly, for $p \in (0,2)$ and $\alpha \in (0,1)$,

$$JL = \frac{(80\alpha^4 - 104\alpha^3 + 75\alpha^2 - 11\alpha + 2)}{384(4-p^2)(4\alpha^2 - 2\alpha + 1)^2(19\alpha^2 - 6\alpha + 3)^2} \left[(630\alpha^4 - 88\alpha^3 + 4\alpha^2 + 40\alpha - 10)p^2 + 8(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)\right] p^2 \geq 0.$$

Also,

$$K - 2(1 - |L|) = \frac{(315\alpha^4 - 44\alpha^3 + 2\alpha^2 + 20\alpha - 5)p^2 - 4(19\alpha^2 - 6\alpha + 3)^2 p + 4(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{2(19\alpha^2 - 6\alpha + 3)^2 p}$$

$$\geq 0, \quad p \in (0,2) \; and \; 0 \leq \alpha \leq 1,$$

so that $K \geq 2(1 - |L|)$, and applying Lemma 2.6, we can have

$$|\psi| \leq 72p^3(4-p^2)(|J| + |K| + |L|) = g(p),$$

where

$$g(p) = \frac{-3(31600\alpha^8 - 13144\alpha^7 - 8595\alpha^6 + 21551\alpha^5 - 17564\alpha^4 + 8958\alpha^3 - 3027\alpha^2 + 651\alpha - 78)}{2(4\alpha^2 - 2\alpha + 1)^2(19\alpha^2 - 6\alpha + 3)^2} p^6 - \frac{72(92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28)}{(19\alpha^2 - 6\alpha + 3)^2} p^4 + \frac{288(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{(19\alpha^2 - 6\alpha + 3)^2} p^2.$$

Clearly, $g'(p) = 0$ has roots at $p_1 = 0$ and $p_2 = \sqrt{\frac{16b - \sqrt{256b^2 - 64ac}}{a}}$,
where

$$a = \frac{-31600\alpha^8 + 13144\alpha^7 + 8595\alpha^6 - 21551\alpha^5 + 17564\alpha^4 - 8958\alpha^3 + 3027\alpha^2 - 651\alpha + 78}{(4\alpha^2 - 2\alpha + 1)^2},$$

$$b = 92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28,$$

and

$$c = 407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23.$$

Further $g''(p) < 0$ at $p = p_2$. so maximum value of the function $\psi(p)$ occurs at $p = p_2$. Hence

$$max\ \psi(p) = \psi(p_2) = \frac{3(16b - 8\sqrt{4b^2 - ac})^3}{2a^3(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)^2} \left(-31600\alpha^8 + 13144\alpha^7 + 8595\alpha^6 - 21551\alpha^5 + 17564\alpha^4 - 8958\alpha^3 + 3027\alpha^2 - 651\alpha + 78\right) - \frac{72(16b - 8\sqrt{4b^2 - ac})^2\left(92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28\right)}{a^2(19\alpha^2 - 6\alpha + 3)^2} + \frac{288(16b - 8\sqrt{4b^2 - ac})(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{a(19\alpha^2 - 6\alpha + 3)^2}.$$

Therefore, from (3.31) the required result is

$$|H_{3,1}(f)| \leq \frac{(16b - 8\sqrt{4b^2 - ac})^3}{24576a^3(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)^3(6\alpha^2 - 2\alpha + 1)}$$
$$\left(-31600\alpha^8 + 13144\alpha^7 + 8595\alpha^6 - 21551\alpha^5 + 17564\alpha^4 - 8958\alpha^3 + 3027\alpha^2 - 651\alpha + 78\right)$$
$$- \frac{(16b - 8\sqrt{4b^2 - ac})^2 \left(92\alpha^4 - 368\alpha^3 + 296\alpha^2 + 112\alpha + 28\right)}{512a^2(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)}$$
$$+ \frac{(16b - 8\sqrt{4b^2 - ac})(407\alpha^4 - 412\alpha^3 + 298\alpha^2 - 92\alpha + 23)}{128a(19\alpha^2 - 6\alpha + 3)^2(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)}.$$

Thus, the proof of Theorem 3.4 is completed. □

By setting $\alpha = 0$ in Theorem 3.4, we derive the following results, which coincide with Theorem 2.5 and Theorem 2.6 in the work of M. Raza [25].

Corollary 3.5. *(see[25]) Let $f \in \mathcal{S}_E^*$ and be of the form (1.1). Then,*

(i)

$$|H_{2,2}(f)| \leq \frac{1}{16},$$

and

(ii)

$$|H_{3,1}(f)| \leq \frac{-3115}{164,268} + \frac{671}{657,072}\sqrt{1342}.$$

This inequalities are sharp.

By setting $\alpha = 1$ in Theorem 3.4, we obtain the following results:

Corollary 3.6. *Let $f \in \mathcal{G}^*_{sech}$ and be of the form (1.1). Then,*

$$|H_{2,1}(f)| \leq \frac{1}{12}, \tag{3.33}$$

$$|H_{2,2}(f)| \leq \frac{1}{144}, \tag{3.34}$$

$$|H_{3,1}(f)| \leq -\frac{421}{719104} + \frac{223\sqrt{1338}}{8089920}. \tag{3.35}$$

These bounds are sharp.

4 Logarithmic coefficient bounds for the class $\mathcal{G}^{\alpha}_{sech}$

For function $f \in \mathfrak{A}$, the logarithmic coefficients $\mathfrak{d}_n = \mathfrak{d}_n(f)(n \in \mathbb{N})$ are given by

$$F_l(\zeta) := log\frac{f(\zeta)}{\zeta} = 2\sum_{n=1}^{\infty} \mathfrak{d}_n \zeta^n \quad (\zeta \in \mathbb{D}). \tag{4.1}$$

The logarithmic coefficients play a significant role in the problems of univalent functions coefficients. The importance of logarithmic coefficients is due to the fact that the bounds on logarithmic coefficients of f can be transfer to the Taylor-Maclaurin coefficients of univalent functions themselves to their powers via the Lebedev-Milin inequalities.
In this section, we investigate the upper bounds estimates for the first four initial coefficients of the functions belongs to the class $\mathcal{G}^{\alpha}_{sech}$. Further, we investigate Hankel determinant of order two and three for the same class.

Theorem 4.1. *Suppose that the function $f \in \mathfrak{A}$ given by (1.1) belongs to the class $\mathcal{G}^{\alpha}_{sech}$. Then*

$$\mathfrak{d}_1 = 0, \tag{4.2}$$

$$|\mathfrak{d}_2| \le \frac{1}{8(4\alpha^2 - 2\alpha + 1)}, \tag{4.3}$$

$$|\mathfrak{d}_3| \le \frac{1}{3\sqrt{3}(19\alpha^2 - 6\alpha + 3)}, \tag{4.4}$$

and

$$|\mathfrak{d}_4| \le \frac{64\alpha^4 - 64\alpha^3 + 54\alpha^2 - 18\alpha + 5}{64(4\alpha^2 - 2\alpha + 1)^2(6\alpha^2 - 2\alpha + 1)}. \tag{4.5}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $\mathcal{G}^{\alpha}_{sech}$. Then from (1.1), it follows that,

$$log\frac{f(\zeta)}{\zeta} = a_2\zeta + \left(a_3 - \frac{a_2^2}{2}\right)\zeta^2 + \left(a_4 - a_2a_3 + \frac{a_2^3}{3}\right)\zeta^3 + \left(a_5 - a_2a_4 + a_2^2a_3 - \frac{a_3^2}{2} - \frac{a_2^4}{4}\right)\zeta^4 + \cdots. \tag{4.6}$$

Equating the first four coefficients between the relation (4.1) and (4.6), we obtain

$$\mathfrak{d}_1 = \frac{a_2}{2}, \tag{4.7}$$

$$\mathfrak{d}_2 = \frac{1}{4}(2a_3 - a_2^2), \tag{4.8}$$

$$\mathfrak{d}_3 = \frac{1}{6}(a_2^3 - 3a_2a_3 + 3a_4), \tag{4.9}$$

and

$$\mathfrak{d}_4 = \frac{1}{8}(4a_5 - 4a_2a_4 + 4a_2^2a_3 - 2a_3^2 - a_2^4). \tag{4.10}$$

Using the value $a_2 = 0$ from (3.10) into (4.7)-(4.10), we observe that

$$\mathfrak{d}_1 = 0, \tag{4.11}$$

$$\mathfrak{d}_2 = \frac{a_3}{2}, \tag{4.12}$$

$$\mathfrak{d}_3 = \frac{a_4}{2}, \tag{4.13}$$

and

$$\mathfrak{d}_4 = \frac{1}{4}(2a_5 - a_3^2). \tag{4.14}$$

The estimate (4.3) and (4.4) of Theorem 4.1 follows from (3.2) and (3.3) of Theorem 3.1 respectively.

An application of triangle inequality to the relation (4.14) and use of (3.2) and (3.4) give

$$\begin{aligned} |\mathfrak{d}_4| &\le \frac{1}{2}|a_5| + \frac{1}{4}|a_3|^2 \\ &\le \frac{1}{16(6\alpha^2 - 2\alpha + 1)} + \frac{1}{64(4\alpha^2 - 2\alpha + 1)^2} \\ &= \frac{4(4\alpha^2 - 2\alpha + 1)^2 + (6\alpha^2 - 2\alpha + 1)}{64(6\alpha^2 - 2\alpha + 1)(4\alpha^2 - 2\alpha + 1)^2} \\ &= \frac{64\alpha^4 - 64\alpha^3 + 54\alpha^2 - 18\alpha + 5}{64(6\alpha^2 - 2\alpha + 1)(4\alpha^2 - 2\alpha + 1)^2}. \end{aligned}$$

The proof of Theorem 4.1 is completed. □

By setting $\alpha = 0$ in Theorem 4.1, we obtain the following results:

Corollary 4.2. *Let $f \in \mathcal{S}_E^*$ and be of the form (1.1). Then,*

$$\mathfrak{d}_1 = 0, \quad |\mathfrak{d}_2| \leq \frac{1}{8}, \quad |\mathfrak{d}_3| \leq \frac{\sqrt{3}}{27} \quad \text{and} \quad |\mathfrak{d}_4| \leq \frac{5}{64}. \tag{4.15}$$

When $\alpha = 1$ is applied in Theorem 4.1, the following results is derived:

Corollary 4.3. *Let $f \in \mathcal{G}_{sech}^*$ and be of the form (1.1). Then,*

$$\mathfrak{d}_1 = 0, \quad |\mathfrak{d}_2| \leq \frac{1}{24}, \quad |\mathfrak{d}_3| \leq \frac{1}{48\sqrt{3}} \quad \text{and} \quad |\mathfrak{d}_4| \leq \frac{41}{2880}. \tag{4.16}$$

Theorem 4.4. *Suppose that the function $f \in \mathfrak{A}$ given by (1.1) belongs to the class $\mathcal{G}_{sech}^{\alpha}$. Then for the function F_l given by (4.1) the upper bounds of Fekete-Szegö functional and second order Hankel determinants are*

$$|H_{2,1}(f)| \leq \frac{1024\sqrt{3}\alpha^4 - 1024\sqrt{3}\alpha^3 + (768\sqrt{3} + 171)\alpha^2 - (256\sqrt{3} + 54)\alpha + (64\sqrt{3} + 27)}{576(19\alpha^2 - 6\alpha + 3)(4\alpha^2 - 2\alpha + 1)^2} \tag{4.17}$$

and

$$|H_{2,2}(f)| \leq \frac{820416\alpha^8 - 1378240\alpha^7 + 1605506\alpha^6 - 1148766\alpha^5 + 636227\alpha^4 - 243400\alpha^3 + 72372\alpha^2 - 13330\alpha + 17}{13824(4\alpha^2 - 2\alpha + 1)^3(19\alpha^2 - 6\alpha + 3)^2(6\alpha^2 - 2\alpha + 1)} \tag{4.18}$$

Proof. If $f \in \mathcal{G}_{sech}^{\alpha}$, then from equation (4.12) and (4.13), we have

$$H_{2,1}(f) = \mathfrak{d}_3 - \mathfrak{d}_2^2 = \frac{a_4}{2} - \frac{a_3^2}{4}. \tag{4.19}$$

Taking modulus on both sides of (4.19) and then using the relation (3.2) and (3.3) in the resulting relation, we have

$$\begin{aligned} |H_{2,1}(f)| &\leq \frac{1}{2}|a_4| + \frac{1}{4}|a_3|^2 = \frac{1}{3\sqrt{3}(19\alpha^2 - 6\alpha + 3)} + \frac{1}{64(4\alpha^2 - 2\alpha + 1)^2} \\ &= \frac{1024\sqrt{3}\alpha^4 - 1024\sqrt{3}\alpha^3 + (768\sqrt{3} + 171)\alpha^2 - (256\sqrt{3} + 54)\alpha + (64\sqrt{3} + 27)}{576(19\alpha^2 - 6\alpha + 3)(4\alpha^2 - 2\alpha + 1)^2}. \end{aligned}$$

Similarly from (4.12), (4.13) and (4.14), we can write

$$H_{2,2}(f) = \mathfrak{d}_2\mathfrak{d}_4 - \mathfrak{d}_3^2 = \frac{a_3 a_5}{4} - \frac{a_3^3}{8} - \frac{a_4^2}{4}. \tag{4.20}$$

Applying triangle inequality on both sides of (4.20) and using the relation from (3.2)-(3.4), we obtain

$$\begin{aligned} |H_{2,2}(f)| &\leq \frac{1}{4}|a_3||a_5| + \frac{1}{8}|a_3|^3 + \frac{1}{4}|a_4|^2 \\ &\leq \frac{1}{128(4\alpha^2 - 2\alpha + 1)(6\alpha^2 - 2\alpha + 1)} + \frac{1}{512(4\alpha^2 - 2\alpha + 1)^3} + \frac{1}{27(19\alpha^2 - 6\alpha + 3)^2}. \end{aligned}$$

Simplifying the above expression we obtain the desire estimate as studied in (4.18).

This completes the proof of Theorem 4.4. □

By taking $\alpha = 0$ in Theorem 4.4 yields the following results:

Corollary 4.5. *Let $f \in \mathcal{S}^*_E$ and be of the form (1.1). Then,*

$$|H_{2,1}(f)| \leq \frac{1}{64} + \frac{\sqrt{3}}{27} \tag{4.21}$$

and

$$|H_{2,2}(f)| \leq \frac{1727}{124416}. \tag{4.22}$$

By setting $\alpha = 1$ in Theorem 4.4, we derive the following result.

Corollary 4.6. *Let $f \in \mathcal{G}^*_{sech}$ and be of the form (1.1). Then,*

$$|H_{2,1}(f)| \leq \frac{1 + 4\sqrt{3}}{576}, \tag{4.23}$$

and

$$|H_{2,2}(f)| \leq \frac{17}{23040}. \tag{4.24}$$

5 Inverse coefficient bounds for the class $\mathcal{G}^{\alpha}_{sech}$

For every univalent functions f defined on the domain of open unit disk $\mathbb{D}$, the famous Koebe one-quarter theorem asserts that its inverse f^{-1} exists at least on a disk of radius $1/4$ with the Taylor-Maclaurin series expansion of the form:

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} \mathfrak{c}_n w^n \quad (|w| < \frac{1}{4}), \tag{5.1}$$

where

$$\mathfrak{c}_2 = -a_2, \tag{5.2}$$

$$\mathfrak{c}_3 = 2a_2^2 - a_3, \tag{5.3}$$

$$\mathfrak{c}_4 = -(a_4 - 5a_2a_3 + 5a_2^3), \tag{5.4}$$

and

$$\mathfrak{c}_5 = -(a_5 - 6a_4a_2 + 21a_3a_2^2 - 3a_3^2 - 14a_2^4). \tag{5.5}$$

Researchers have shown their interest for finding inverse function where the function f belongs to some specific subfamilies of univalent functions. For instance, Krzyz et al. [12] investigated the upper bound of the some of initial coefficients of inverse function for certain families of analytic functions and later these bounds were improved by Kapoor and Mishra [9].

Theorem 5.1. *Suppose that the function $f \in \mathfrak{A}$ given by (1.1) belongs to the class $\mathcal{G}^{\alpha}_{sech}$. Then*

$$\mathfrak{c}_2 = 0, \tag{5.6}$$

$$|\mathfrak{c}_3| \leq \frac{1}{4(4\alpha^2 - 2\alpha + 1)}, \tag{5.7}$$

$$|\mathfrak{c}_4| \leq \frac{2}{3\sqrt{3}(19\alpha^2 - 6\alpha + 3)}, \tag{5.8}$$

and

$$|\mathfrak{c}_5| \leq \frac{32\alpha^4 - 32\alpha^3 + 42\alpha^2 - 14\alpha + 5}{16(4\alpha^2 - 2\alpha + 1)^2(6\alpha^2 - 2\alpha + 1)}. \tag{5.9}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $\mathcal{G}^{\alpha}_{sech}$. Then using the relation from (3.10)-(3.13) in (5.2)-(5.5) we get,

$$\mathfrak{c}_2 = 0,$$

$$\mathfrak{c}_3 = -a_3,$$

$$\mathfrak{c}_4 = -a_4,$$

and

$$\mathfrak{c}_5 = -a_5 + 3a_3^2. \tag{5.10}$$

The bounds of $|\mathfrak{c}_2|$, $|\mathfrak{c}_3|$ and $|\mathfrak{c}_4|$ follows from the bounds of $|a_2|$, $|a_3|$ and $|a_4|$.
Taking modulus on both sides of (5.10) and then applying the triangle inequality, we obtain

$$\begin{aligned}|\mathfrak{c}_5| \leq |a_5| + 3|a_3|^2 &= \frac{1}{8(6\alpha^2-2\alpha+1)} + \frac{3}{16(4\alpha^2-2\alpha+1)^2}\\ &= \frac{32\alpha^4-32\alpha^3+42\alpha^2-14\alpha+5}{16(4\alpha^2-2\alpha+1)^2(6\alpha^2-2\alpha+1)}.\end{aligned}$$

This completes the proof of the Theorem 5.1.

By setting $\alpha = 0$ in Theorem 5.1, we obtain the following result.

Corollary 5.2. *Let $f \in \mathcal{S}^*_E$ and be of the form (1.1). Then,*

$$\mathfrak{c}_2 = 0, \quad |\mathfrak{c}_3| \leq \frac{1}{4}, \quad |\mathfrak{c}_4| \leq \frac{2\sqrt{3}}{27} \quad \textit{and} \quad |\mathfrak{c}_5| \leq \frac{5}{16}. \tag{5.11}$$

By setting $\alpha = 1$ in Theorem 5.1, the following result is derived:

Corollary 5.3. *Let $f \in \mathcal{G}^*_{sech}$ and be of the form (1.1). Then,*

$$\mathfrak{c}_2 = 0, \quad |\mathfrak{c}_3| \leq \frac{1}{12}, \quad |\mathfrak{c}_4| \leq \frac{\sqrt{3}}{72} \quad \textit{and} \quad |\mathfrak{c}_5| \leq \frac{11}{240}. \tag{5.12}$$

□

Theorem 5.4. *If the function $f \in \mathcal{G}^{\alpha}_{sech}$ given by (1.1) and $f^{-1}(w) = w + \sum_{n=2}^{\infty} \mathfrak{c}_n w^n$ is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \geq \frac{1}{4}$ is the radius of the Koebe domain, then*

$$|H_{2,1}(f^{-1})| \leq \frac{1}{4(4\alpha^2-2\alpha+1)}, \tag{5.13}$$

$$|H_{2,2}(f^{-1})| \leq \frac{1}{16(4\alpha^2-2\alpha+1)^2}, \tag{5.14}$$

and

$$|H_{3,1}(f^{-1})| \leq \frac{102552\alpha^8-172280\alpha^7+222619\alpha^6-164757\alpha^5+96913\alpha^4-37958\alpha^3+11841\alpha^2-2213\alpha+307}{432(6\alpha^2-2\alpha+1)(4\alpha^2-2\alpha+1)^3(19\alpha^2-6\alpha+3)^2}. \tag{5.15}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $\mathcal{G}^{\alpha}_{sech}$. The inequalities (5.13) and (5.14) follows from (5.7).
Now the third order Hankel determinant of inverse function is written as,

$$H_{3,1}(f^{-1}) = \mathfrak{c}_3(\mathfrak{c}_2\mathfrak{c}_4 - \mathfrak{c}_3^2) - \mathfrak{c}_4(\mathfrak{c}_4 - \mathfrak{c}_2\mathfrak{c}_3) + \mathfrak{c}_5(\mathfrak{c}_3 - \mathfrak{c}_2^2).$$

Since $\mathfrak{c}_2 = 0$ we have,

$$H_{3,1}(f^{-1}) = -\mathfrak{c}_3^3 - \mathfrak{c}_4^2 + \mathfrak{c}_3\mathfrak{c}_5. \tag{5.16}$$

Applying the relation (5.3) to (5.5) in (5.16), we have

$$H_{3,1}(f^{-1}) = a_3a_5 - a_4^2 - 2a_3^3. \tag{5.17}$$

Taking the modulus on both sides of (5.17), applying the triangle inequality and using the values from (3.2) to (3.4), we obtain

$$\begin{aligned}|H_{3,1}(f^{-1})| &\leq 2|a_3|^3 + |a_4|^2 + |a_3||a_5| \\ &= \frac{1}{32(4\alpha^2-2\alpha+1)^3} + \frac{4}{27(19\alpha^2-6\alpha+3)^2} + \frac{1}{32(6\alpha^2-2\alpha+1)(4\alpha^2-2\alpha+1)}.\end{aligned}$$

Simplifying the above expression gives the required estimate. This completes the proof of the Theorem 5.4. □

By setting $\alpha = 0$ in Theorem 5.4, we derive the following result.

Corollary 5.5. *Let $f \in \mathcal{S}_E^*$ and be of the form (1.1). Then,*

$$|H_{2,1}(f^{-1})| \leq \frac{1}{4}, \tag{5.18}$$

$$|H_{2,2}(f^{-1})| \leq \frac{1}{16}, \tag{5.19}$$

$$|H_{3,1}(f^{-1})| \leq \frac{307}{3888}. \tag{5.20}$$

By setting $\alpha = 1$ in Theorem 5.4 yields the following result.

Corollary 5.6. *Let $f \in \mathcal{G}_{sech}^*$ and be of the form (1.1). Then,*

$$|H_{2,1}(f^{-1})| \leq \frac{1}{12}, \tag{5.21}$$

$$|H_{2,2}(f^{-1})| \leq \frac{1}{144}, \tag{5.22}$$

$$|H_{3,1}(f^{-1})| \leq \frac{11}{2880}. \tag{5.23}$$

In 2023 Mandal et al.[16] obtained the sharp bounds of second order Hankel and Toeplitz determinants of logarithmic coefficient of inverse functions lying in starlike functions and convex functions with respect to symmetric point. However, no work has been carried out for bounds of third order Hankel determinant, Zalcman conjecture and Krushkal inequality and coefficient modulo difference of logarithmic coefficients of inverse functions for the class $\mathcal{G}_{sech}^*$.

Now we determine the bounds of Hankel determinants of order two and three for logarithmic coefficient of inverse functions for the class $\mathcal{G}_{sech}^*$. Further the bounds for Zalcman conjecture, Krushkal inequality and the coefficient modulo difference for logarithmic coefficient inverse functions for such family are determined.

The remaining results of the paper are developed for only fixing the bounds for the class $\mathcal{G}_{sech}^*$ with $\alpha = 1$. Putting $\alpha = 1$ in the relations (3.10)–(3.13), we get

$$a_2 = 0, \tag{5.24}$$

$$a_3 = \frac{-p_1^2}{48}, \tag{5.25}$$

$$a_4 = \frac{-p_1}{64}\left[p_2 - \frac{1}{2}p_1^2\right], \tag{5.26}$$

$$a_5 = \frac{-28}{7680}p_1^4 + \frac{3}{160}p_1^2 p_2 - \frac{1}{160}p_2^2 - \frac{1}{80}p_1 p_3, \tag{5.27}$$

and

$$a_6 = \frac{1}{144}\left[\frac{11}{96}p_1^5 - \frac{59}{48}p_1^3 p_2 + \frac{3}{2}p_1^2 p_3 + \frac{3}{2}p_1 p_2^2 - p_1 p_4 - p_2 p_3\right]. \tag{5.28}$$

6 Logarithmic Coefficients of Inverse Functions for the Class $\mathcal{G}^*_{sech}$

Theorem 6.1. *If the function $f \in \mathcal{G}^*_{sech}$ given by (1.1) and log $\left(\frac{f^{-1}(w)}{w}\right) = 2\sum_{n=2}^{\infty}\gamma_n w^n$ is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \geq \frac{1}{4}$ is the radius of the Koebe domain, then*

$$\gamma_1 = 0, \tag{6.1}$$

$$|\gamma_2| \leq \frac{1}{24}, \tag{6.2}$$

$$|\gamma_3| \leq \frac{1}{48\sqrt{3}}, \tag{6.3}$$

$$|\gamma_4| \leq \frac{61}{2880}, \tag{6.4}$$

and

$$|\gamma_5| \leq \frac{1}{12}. \tag{6.5}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $\mathcal{G}^*_{sech}$ and

$$log\left(\frac{f^{-1}(w)}{w}\right) = 2\sum_{n=2}^{\infty}\gamma_n w^n \quad \left(|w| < \frac{1}{4}\right). \tag{6.6}$$

Substituting the value of (5.1) in (6.6), then differentiating and using inverse coefficient (5.2)-(5.5), we obtain

$$\gamma_1 = -\frac{1}{2}a_2, \tag{6.7}$$

$$\gamma_2 = -\frac{1}{2}\left(a_3 - \frac{3}{2}a_2^2\right), \tag{6.8}$$

$$\gamma_3 = -\frac{1}{2}\left(a_4 - 4a_2 a_3 + \frac{10}{3}a_2^3\right), \tag{6.9}$$

$$\gamma_4 = -\frac{1}{2}\left(a_5 - 5a_4 a_2 + 15a_3 a_2^2 - \frac{5}{2}a_3^2 - \frac{35}{4}a_2^4\right), \tag{6.10}$$

$$\gamma_5 = -\frac{1}{2}\left(a_6 - 6a_2 a_5 - 6a_3 a_4 + 21a_4 a_2^2 + 21a_2 a_3^2 - 56a_2^3 a_3 + \frac{126}{5}a_2^5\right). \tag{6.11}$$

Applying the value of $a_2 = 0$ from (5.24) in the relation from (6.7) to (6.11), we obtain

$$\gamma_1 = 0,$$

$$\gamma_2 = -\frac{a_3}{2}, \tag{6.12}$$

$$\gamma_3 = -\frac{a_4}{2}, \tag{6.13}$$

$$\gamma_4 = -\frac{1}{2}\left(a_5 - \frac{5}{2}a_3^2\right), \tag{6.14}$$

$$\gamma_5 = -\frac{1}{2}(a_6 - 6a_3a_4). \tag{6.15}$$

Taking modulus on both sides of (6.12) and (6.13) and using the bounds of $|a_3|$ and $|a_4|$ from Corollary 3.3 we obtain the estimates for $|\gamma_2|$ and $|\gamma_3|$ as stated in the theorem.
By taking the modulus of both sides in (6.14) and applying the triangle inequality and the results of Corollary 3.3 give

$$\begin{aligned}|\gamma_4| &\leq \frac{1}{2}\left[|a_5| + \frac{5}{2}|a_3|^2\right]\\ &= \frac{1}{2}\left[\frac{1}{40} + \frac{5}{2} \times \frac{1}{144}\right]\\ &= \frac{61}{2880}.\end{aligned}$$

Similarly using the value of (5.25), (5.26) and (5.28) in (6.15), we get

$$\begin{aligned}\gamma_5 &= \frac{-49}{55296}p_1^5 + \frac{145}{27648}p_1^3p_2 - \frac{3}{576}p_1^2p_3 - \frac{3}{576}p_1p_2^2 + \frac{1}{288}p_1p_4 + \frac{1}{288}p_2p_3\\ &= \frac{-p_1^2}{192}\left(\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right) + \frac{p_2}{288}\left(p_3 - \frac{3}{2}p_1p_2\right) + \frac{1}{288}p_1p_4.\end{aligned}$$

Taking modulus on both sides of the above equation and applying Lemma 2.2, Lemma 2.3 and Lemma 2.5, we obtain

$$\begin{aligned}|\gamma_5| &\leq \frac{|p_1|^2}{192}\left|\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right| + \frac{|p_2|}{288}\left|p_3 - \frac{3}{2}p_1p_2\right| + \frac{1}{288}|p_1p_4|\\ &= \frac{1}{24}\left[\frac{49}{288} + \frac{96}{144} + \frac{47}{288}\right] + \frac{1}{72}max\{1,2\} + \frac{1}{72} = \frac{1}{24} + \frac{1}{36} + \frac{1}{72} = \frac{1}{12}.\end{aligned}$$

This completes the proof of Theorem 6.1. □

7 Upper bounds of Hankel determinants for logarithmic inverse functions for the class $\mathcal{G}^*_{sech}$

Mandal et al. [16] introduce the investigation of the Hankel determinant $H_{k,n}(F_{f^{-1}}/2)$, where the elements of logarithmic coefficients of inverse function of $f^{-1} \in \mathcal{S}$. the determinant $H_{k,n}(F_{f^{-1}}/2)$ is expressed as follows:

$$H_{k,n}(F_{f^{-1}}/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+k-1}\\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+k}\\ \vdots & \vdots & \vdots & \vdots\\ \gamma_{n+k-1} & \gamma_{n+k} & \cdots & \gamma_{n+2(k-1)} \end{vmatrix} \quad (n,k \in \mathbb{N} = 1,2,3\cdots). \tag{7.1}$$

Theorem 7.1. *If the function $f \in \mathcal{G}^*_{sech}$ given by (1.1) and* $\log\left(\frac{f^{-1}(w)}{w}\right) = 2\sum_{n=2}^{\infty}\gamma_n w^n$ *is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \geq \frac{1}{4}$ is the radius of the Koebe domain, then*

$$|H_{2,1}(F_{f^{-1}}/2)| \leq \frac{1}{576}, \tag{7.2}$$

$$|H_{2,2}(F_{f^{-1}}/2)| \leq \frac{23}{15360}, \tag{7.3}$$

$$|H_{3,1}(F_{f^{-1}}/2)| \leq \frac{3946291541}{31310112522240}. \tag{7.4}$$

Proof. If $f \in \mathcal{G}^*_{sech}$, then from (7.1), we estimates

$$H_{2,1}(F_{f^{-1}}/2) = \gamma_1\gamma_3 - \gamma_2^2, \tag{7.5}$$

$$H_{2,2}(F_{f^{-1}}/2) = \gamma_2\gamma_4 - \gamma_3^2, \tag{7.6}$$

$$H_{3,1}(F_{f^{-1}}/2) = \gamma_1\gamma_3\gamma_5 - \gamma_1\gamma_4^2 - \gamma_2^2\gamma_5 + 2\gamma_2\gamma_3\gamma_4 - \gamma_3^3. \tag{7.7}$$

Since $\gamma_1 = 0$, using the value of (6.12)-(6.15) in (7.5)-(7.7), we obtain

$$H_{2,1}(F_{f^{-1}}/2) = -\gamma_2^2 = -\frac{a_3^2}{4}, \tag{7.8}$$

$$H_{2,2}(F_{f^{-1}}/2) = \gamma_2\gamma_4 - \gamma_3^2 = \frac{1}{4}\left[a_3a_5 - \frac{5}{2}a_3^3 - a_4^2\right], \tag{7.9}$$

$$H_{3,1}(F_{f^{-1}}/2) = \gamma_1\gamma_3\gamma_5 - \gamma_1\gamma_4^2 - \gamma_2^2\gamma_5 + 2\gamma_2\gamma_3\gamma_4 - \gamma_3^3 = \frac{1}{8}\left[a_3^2a_6 - a_3^3a_4 - 2a_3a_4a_5 + a_4^3\right]. \tag{7.10}$$

Using the value of (5.25) in (7.8), we have

$$H_{2,1}(F_{f^{-1}}/2) = -\frac{p_1^4}{9216}.$$

Applying Lemma 2.2 in the above, we have

$$|H_{2,1}(F_{f^{-1}}/2)| \leq \frac{1}{576}.$$

Using the relation from (5.25) to (5.27) in (7.9), we get

$$\begin{aligned} H_{2,2}(F_{f^{-1}}/2) &= \frac{1}{4}\left[\frac{83}{2211840}p_1^6 - \frac{9}{61440}p_1^4p_2 - \frac{7}{61440}p_1^2p_2^2 + \frac{1}{3840}p_1^3p_3\right] \\ &= \frac{p_1^3}{15360}\left(\frac{83}{576}p_1^3 - \frac{9}{16}p_1p_2 + p_3\right) - \frac{7}{245760}p_1^2p_2^2. \end{aligned}$$

By taking the modulus of both sides in (7.9) and applying the triangle inequality along with the relation in (3.24), it follows that

$$\begin{aligned} |H_{2,2}(F_{f^{-1}}/2)| &\leq \frac{1}{4}\left[|a_3||a_5| + \frac{5}{2}|a_3|^3 + |a_4|^2\right] \\ &= \left[\frac{1}{12} \times \frac{1}{40} + \frac{5}{2} \times \frac{1}{1728} + \frac{1}{1728}\right] \\ &= \frac{1}{4} \times \frac{71}{17280} \\ &= \frac{71}{69120}. \end{aligned}$$

Thus,

$$|H_{2,2}(F_{f^{-1}}/2)| \leq \frac{71}{69120}.$$

Similarly substitute the value of (5.25), (5.26) and (5.28) in (7.10), we have

$$\begin{aligned}
H_{3,1}(F_{f^{-1}}/2) &= \frac{a_3^2 a_6}{8} - \frac{a_3^3 a_4}{8} - \frac{a_3 a_4 a_5}{4} + \frac{a_4^3}{8} \\
&= -\frac{749}{20384317440} p_1^9 + \frac{451}{2038431744} p_1^7 p_2 + \frac{1}{17694720} p_1^6 p_3 - \frac{1}{2654208} p_1^5 p_4 \\
&+ \frac{1}{31457280} p_1^3 p_2^3 + \frac{17}{26542080} p_1^4 p_2 p_3 - \frac{28}{566231040} p_1^5 p_2^2 \\
&= \frac{451 p_1^7}{2038431744}\left[p_2 - \frac{749}{4510} p_1^2\right] - \frac{p_1^5}{2654208}\left[p_4 - \frac{3}{20} p_1 p_3\right] + \frac{p_1^3 p_2^2}{31457280}\left[p_2 - \frac{14}{9} p_1^2\right] + \frac{17}{26542080} p_1^4 p_2 p_3.
\end{aligned}$$

Now taking modulus on both sides of the above equation and applying Lemma 2.2 and Lemma 2.3, we get

$$\begin{aligned}
|H_{3,1}(F_{f^{-1}}/2)| &\leq \frac{451 |p_1|^7}{2038431744}\left|p_2 - \frac{749}{4510} p_1^2\right| - \frac{|p_1|^5}{2654208}\left|p_4 - \frac{3}{20} p_1 p_3\right| + \frac{|p_1|^3 |p_2|^2}{31457280}\left|p_2 - \frac{14}{9} p_1^2\right| \\
&+ \frac{17}{26542080} |p_1|^4 |p_2| |p_3| \\
&= \frac{451}{2038431744} \cdot 128 \left[2max\left\{1, \frac{1506}{2255}\right\}\right] + \frac{1}{2654208} \cdot 32 \left[2max\left\{1, \frac{7}{10}\right\}\right] \\
&+ \frac{1}{31457080} \cdot 32 \left[2max\left\{1, \frac{19}{9}\right\}\right] + \frac{17}{26542080} \cdot 80 \\
&= \frac{451}{7962624} + \frac{1}{41472} + \frac{152}{35389215} + \frac{17}{414720} \\
&= \frac{3946291541}{31310112522240}.
\end{aligned}$$

Hence this completes the proof of Theorem 7.1. □

8 Krushkal Inequality and Zalcman conjecture of logarithmic coefficients of inverse functions for the class $\mathcal{G}^*_{sech}$

In this section, we will give a direct proof of the inequality

$$\left|\gamma_n^p - \gamma_2^{p(n-1)}\right| \leq 2^{p(n-1)-n^p}, \tag{8.1}$$

over the class $\mathcal{G}^*_{sech}$ for the choice of $n = 4,\ 5$ and $p = 1$. Kruskal introduced and proved this inequality for the whole class of univalent functions in [11].

In the 1960's, Lawrence Zalcman conjectured that the coefficients of univalent functions having the form (1.1) satisfy the inequality

$$\left|a_n^2 - a_{2n-1}\right| \leq (n-1)^2,\ n \geq 2, \tag{8.2}$$

and the equality holds only for the Koebe function $\kappa(\zeta) = \frac{\zeta}{(1-\zeta)^2}$ and its rotations. In the literature the Zalcman functional has been studied by many researchers (see, for example, [4], [10], [14], [17]).

Theorem 8.1. *If the function $f \in \mathcal{G}^*_{sech}$ given by (1.1), then*

$$|\gamma_4 - \gamma_2^3| \leq \frac{3017}{69120}, \tag{8.3}$$

and

$$|\gamma_5 - \gamma_2^4| \leq \frac{27649}{331776}. \tag{8.4}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $f \in \mathcal{G}^*_{sech}$. Then for the choice of $n = 4,\ p = 1$ and $n = 5,\ p = 1$, equation (8.1) reduces to

$$|\gamma_4 - \gamma_2^3| \leq \frac{1}{2},$$

and

$$|\gamma_5 - \gamma_2^4| \leq \frac{1}{2}.$$

From (6.12) and (6.14), we can write

$$\gamma_4 - \gamma_2^3 = \frac{1}{8}a_3^3 + \frac{5}{4}a_3^2 - \frac{1}{2}a_5$$

From (5.25) and (5.27), we have

$$\gamma_4 - \gamma_2^3 = \frac{p_1}{160}\left[\frac{109}{288}p_1^3 - \frac{3}{2}p_1p_2 + p_3\right] + \frac{1}{320}p_2^2 - \frac{1}{884736}p_1^6.$$

Taking modulus on both sides of the above equation and applying Lemma 2.2 and Lemma 2.5, we have

$$\begin{aligned}|\gamma_4 - \gamma_2^3| &\leq \frac{2}{160}\cdot 2\left[\left|\frac{109}{288}\right| + \left|\frac{3}{2} - \frac{109}{144}\right| + \left|\frac{109}{288} - \frac{3}{2} + 1\right|\right] + \frac{4}{320} + \frac{64}{884736}\\ &= \frac{179}{5760} + \frac{1}{80} + \frac{1}{13824} = \frac{3017}{69120}.\end{aligned}$$

Thus

$$|\gamma_4 - \gamma_2^3| \leq \frac{3017}{69120}.$$

Similarly from (6.12) and (6.15), we can write

$$\gamma_5 - \gamma_2^4 = -\frac{a_6}{2} + 3a_3a_4 - \frac{1}{16}a_3^4,$$

using (5.25), (5.26) and (5.28) in the above equation, we get

$$\begin{aligned}\gamma_5 - \gamma_2^4 &= -\frac{49}{55296}p_1^5 + \frac{145}{27648}p_1^3p_2 - \frac{1}{192}p_1^2p_3 + \frac{1}{288}p_2p_3 - \frac{1}{192}p_1p_2^2 + \frac{1}{288}p_1p_4 - \frac{1}{84934656}p_1^8\\ &= -\frac{p_1^2}{192}\left(\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right) + \frac{p_2}{288}\left(p_3 - \frac{3}{2}p_1p_2\right) + \frac{p_1p_4}{288} - \frac{p_1^8}{84934656}.\end{aligned}$$

Taking modulus on both sides of the above equation and applying Lemma 2.2, Lemma 2.3 and Lemma 2.5, we have

$$\begin{aligned}|\gamma_5 - \gamma_2^4| &\leq \frac{|p_1|^2}{192}\left|\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right| + \frac{|p_2|}{288}\left|p_3 - \frac{3}{2}p_1p_2\right| + \frac{|p_1p_4|}{288} + \frac{|p_1|^8}{84934656}\\ &= \frac{4}{192}\cdot 2\left[\left|\frac{49}{288}\right| + \left|\frac{145}{144} - \frac{49}{144}\right| + \left|\frac{49}{288} - \frac{145}{144} + 1\right|\right] + \frac{1}{72}max\{1,2\} + \frac{4}{288} + \frac{256}{84934656}\\ &= \frac{27649}{331776}.\end{aligned}$$

Hence

$$|\gamma_5 - \gamma_2^4| \leq \frac{27649}{331776}.$$

This completes the proof of Theorem 8.1. □

Theorem 8.2. *If the function $f \in \mathcal{G}^*_{sech}$ given by (1.1), then*

$$|\gamma_2^2 - \gamma_3| \leq \frac{19}{576} \tag{8.5}$$

and

$$|\gamma_3^2 - \gamma_5| \leq \frac{265}{3072}. \tag{8.6}$$

Proof. Let the function $f \in \mathfrak{A}$ be in the class $f \in \mathcal{G}^*_{sech}$. Then from (6.12) and (6.13), we have

$$\begin{aligned}\gamma_2^2 - \gamma_3 &= \frac{a_3^2}{4} + \frac{a_4}{2} \\ &= \frac{p_1^4}{9216} + \frac{p_1}{128}\left(p_2 - \frac{1}{2}p_1^2\right).\end{aligned}$$

Taking modulus on both sides of the above equation and applying Lemma 2.2 and Lemma 2.3, we obtain

$$|\gamma_2^2 - \gamma_3| \leq \frac{|p_1|^4}{9216} + \frac{|p_1|}{128}\left|p_2 - \frac{1}{2}p_1^2\right| = \frac{16}{9216} + \frac{2}{128}\cdot 2max\{1,0\} = \frac{19}{576}.$$

Similarly from (6.13) and (6.15), we have

$$\begin{aligned}\gamma_3^2 - \gamma_5 &= \frac{a_4^2}{4} - 3a_3a_4 + \frac{a_6}{2} \\ &= \frac{p_1^6}{65536} - \frac{p_1^4p_2}{16384} + \frac{49p_1^5}{55296} - \frac{145p_1^3p_2}{27648} + \frac{p_1^2p_3}{192} + \frac{p_1p_2^2}{192} - \frac{p_2p_3}{288} - \frac{p_1p_4}{288} + \frac{p_1^2p_2^2}{16384} \\ &= \frac{p_1^2}{192}\left[\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right] - \frac{p_1^4}{16384}\left[p_2 - \frac{1}{4}p_1^2\right] - \frac{p_2}{288}\left[p_3 - \frac{3}{2}p_1p_2\right] + \frac{p_1^2p_2^2}{16384} - \frac{p_1p_4}{288}.\end{aligned}$$

Now

$$|\gamma_3^2 - \gamma_5| \leq \frac{|p_1|^2}{192}\left|\frac{49}{288}p_1^3 - \frac{145}{144}p_1p_2 + p_3\right| + \frac{|p_1|^4}{16384}\left|p_2 - \frac{1}{4}p_1^2\right| + \frac{|p_2|}{288}\left|p_3 - \frac{3}{2}p_1p_2\right| + \frac{|p_1^2p_2^2|}{16384} + \frac{|p_1p_4|}{288}.$$

Applying Lemma 2.2, Lemma 2.3 and Lemma 2.4 in the above equation, we estimate

$$\begin{aligned}|\gamma_3^2 - \gamma_5| &\leq \frac{1}{24}\left[\frac{49}{288} + \frac{96}{144} + \frac{47}{288}\right] + \frac{16}{16384}\cdot 2max\{1, \frac{1}{2}\} + \frac{2}{288}\cdot 2max\{1,2\} + \frac{16}{16384} + \frac{4}{288} \\ &= \frac{1}{24} + \frac{1}{512} + \frac{1}{36} + \frac{1}{1024} + \frac{1}{72} = \frac{265}{3072}.\end{aligned}$$

Thus

$$|\gamma_3^2 - \gamma_5| \leq \frac{265}{3072}.$$

Hence this completes the proof of the Theorem 8.2. □

9 Coefficient modulo difference of logarithmic coefficients of inverse functions for the class $\mathcal{G}^*_{sech}$

Theorem 9.1. *If $f \in \mathcal{G}^*_{sech}$ has of the form (1.1), then*

$$-\frac{1}{24} \leq |\gamma_3| - |\gamma_2| \leq \frac{1}{32}. \tag{9.1}$$

Proof. If $f \in \mathcal{G}^*_{sech}$, from (6.8) and (6.9) we have

$$|\gamma_3| - |\gamma_2| = \left|\frac{-a_4}{2}\right| - \left|\frac{-a_3}{2}\right|,$$

using the value of a_3 and a_4 from (5.25) and (5.26), we have

$$|\gamma_3| - |\gamma_2| = \left|\frac{p_1 p_2}{128} - \frac{p_1^3}{256}\right| - \left|\frac{p_1^2}{96}\right| = |p_1|\psi_+(p_1, p_2), \tag{9.2}$$

where

$$\psi_+(p_1, p_2) = \left|\frac{p_1^2}{256} - \frac{p_2}{128}\right| - \left|\frac{p_1}{96}\right|.$$

Applying Lemma 2.2 in (9.2), we obtain

$$|\gamma_3| - |\gamma_2| \leq 2\psi_+(p_1, p_2). \tag{9.3}$$

Here, $\mathcal{B}_1 = \frac{1}{96}$, $\mathcal{B}_2 = \frac{1}{256}$ and $\mathcal{B}_3 = \frac{-1}{128}$.
Since $|2\mathcal{B}_2 + \mathcal{B}_3| \ngeq |\mathcal{B}_3| + \mathcal{B}_1$, then from Lemma 2.5, we have

$$\psi_+(p_1, p_2) \leq \frac{1}{64}.$$

Thus from (9.3), we get

$$|\gamma_3| - |\gamma_2| \leq \frac{1}{32}. \tag{9.4}$$

From (9.2) we have

$$|\gamma_2| - |\gamma_3| = -|p_1|\psi_+(p_1, p_2) = |p_1|\psi_-(p_1, p_2). \tag{9.5}$$

Here, $\mathcal{B}_1 = \frac{1}{96}$, $\mathcal{B}_2 = \frac{1}{256}$ and $\mathcal{B}_3 = \frac{-1}{128}$, then $\mathcal{B}_4 = |4\mathcal{B}_2 + 2\mathcal{B}_3| = 0$, and $\mathcal{B}_1^2 \leq 2|\mathcal{B}_3(\mathcal{B}_4 + 2|\mathcal{B}_3|)$.
Hence, from Lemma 2.5 we have

$$\psi_-(p_1, p_2) \leq 2\mathcal{B}_1\sqrt{\frac{2|\mathcal{B}_3|}{\mathcal{B}_4 + 2|\mathcal{B}_3|}} = \frac{1}{48}.$$

Applying lemma 2.2 in (9.5), we have

$$|\gamma_2| - |\gamma_3| = |p_1|\psi_-(p_1, p_2) \leq 2 \cdot \frac{1}{48} = \frac{1}{24}. \tag{9.6}$$

Therefore from (9.4) and (9.6), we get

$$-\frac{1}{24} \leq |\gamma_3| - |\gamma_2| \leq \frac{1}{32}.$$

This completes the proof of Theorem 9.1. □

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