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Notes on Irreducible Laman Torus Graphs

Qays R. Shakir

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Abstract We investigate specific Laman graphs that are embedded in the torus. We present some important structural properties of irreducible Laman torus graphs. Specifically, we partially provide a road map that may help in extracting all irreducible Laman torus graphs.

1 Introduction

Surface graph theory concerns drawing graphs on surfaces without edge crossing. This theory has a long period of development dates back to the work of Leonhard Euler. Surface graphs appear in various topics of mathematics and computer science. For example, combinatorially, surface graphs are used to study problems that arise from colorability, [13]. Much interest has been spent on studying triangulations on surfaces. The key point in generating triangulations on a surface is to find specific triangulations on the same surface such that no contractible edge exists in such triangulations. Usually, such triangulations are called irreducible triangulations. Many studies have been conducted to find such irreducible triangulations on various surfaces, [12], [2] and [17]. The existence of a finite number of triangulations on surfaces was investigated in [3]. In this work, we try to initiate a systematic way to search for specific graphs, called Laman graphs, that are embedded in the torus where such graphs are irreducible.

A surface S is a connected compact Hausdorff topological space such that each point of S has an open neighbourhood homeomorphic to the open unit disc in $\mathbb{R}^2$. A surface graph, S-graph, G is a pair (Γ, ϕ) where Γ is a graph and ϕ is a continuous injective map between the geometric realisation of Γ and a surface S, [1]. A face of G is a component of the complement of the image of ϕ. A face in a surface graph is cellular if it is homeomorphic to an open disc. A surface graph is cellular if all its faces are cellular. For a cellular face F, the degree of F, denoted by $|F|$, is the edge length of its unique boundary walk. We write f_i for the number of cellular faces of degree i. Let F be a face of an S-subgraph H of G. We recall the following concepts from [4]. An S-subgraph of G that corresponds to the vertices and the edges of Γ and whose images lie in the topological closure of F is denoted by $int_G(F)$. Similarly, $ext_G(F)$ represents the S-subgraph of G corresponding to those vertices and edges of Γ whose images lie in $S - F$. It is clear that $\partial F = int_G(F) \cap ext_G(F) = H \cap int_G(F)$ where ∂F is the S-subgraph $int_G(F) \cap ext_G(F)$ of G. Sparse and consequently tight graphs have been extensively studied due to their contribution in different topics. Among the topics in which such graphs have been used are the decomposition of graphs [18] and the combinatorial aspect of the geometric rigidity theory, [9, 8]. Let l, k be nonnegative integers with $l \leq k$. A graph $\Gamma = (V, E)$ is called (k, l)- sparse if, for every nonempty subgraph Ω of Γ, $|E_\Omega| \leq k|V_\Omega| - l$. Γ is called (k, l)-tight if it is (k, l)-sparse and $|E| = k|V| - l$, [7]. This article presents a road map to find specific graphs in the class of Laman graphs that are embedded in the torus where Laman graphs are basically $(2, 3)$-tight graphs. Such graphs play a significant role in investigating the rigidity status of frameworks in the plane. Specifically, Laman's result states that a generic framework in the plane is minimally rigid if and only if the underlying graph of the framework is $(2, 3)$-tight, [11].

In graph theory, an inductive construction for a class of graphs consists of two main tools; a set of inductive operations and a set of small graphs in which there is no way to conduct the inverse of any operation under consideration. The small graphs here will be called irreducible.

Graph operations are well studied in many contexts in graph theory as they play significant roles in extending and reducing graphs, see [10] and [15]. An inductive construction related to Laman plane graph was given in [6] by presenting the set of irreducible graphs which is one edge and one operation called plane vertex splitting. Various other inductive constructions for surface graphs have been initiated, such as in [4] and [14].

The rest of this article is structured as follows: Section 2 of this article is devoted to presenting basic and necessary concepts of cycles and their associated loops on surfaces. Section 3 presents a road map for finding irreducible torus graphs that are $(2, 3)$-tight.

2 Cycles and their associated loops in surface graphs

Some types of cycles in surface graphs can be used to extract some structural properties of specific kinds of surface graphs. For example, in searching for irreducible triangulations on various surfaces, cycles have been used to recognise that the sets of irreducible triangulations on some surfaces are finite. See for example [17]. In the following, we review some technical concepts related to cycles on surface graphs. A loop α in a surface $\mathcal{S}$ is a continuous function $\alpha : S^1 \longrightarrow \mathcal{S}$. α is called simple if it is injective. A simple loop α is nonseparating if the complement of the image of α has the same number of connected components as $\mathcal{S}$, otherwise α is separating. Two loops α and β are said to be homotopic if there is a continuous function $F : [0, 1] \times S^1 \longrightarrow S$ such that $F(0, t) = \alpha(t)$ and $F(1, t) = \beta(t)$ for all $t \in S^1$. A loop α is null homotopic if α is homotopic to a constant map. α is called essential if it is not null homotopic otherwise it is called inessential. Let c be a cycle in an S-graph $G = (\Gamma, \phi)$. The geometric realisation of c is a subspace in $\mathcal{S}$ which is homeomorphic to the circle S^1. Notice that c has two possible orientations. We can find a loop α homotopic to S^1 with an orientation coincides with the orientation of the cycle c. Such a loop is called the associated loop of c. A separating and nonseparating cycle c in an S-graph G is a simple closed walk whose associated simple loop is separating and nonseparating in $\mathcal{S}$, respectively.

Suppose that α is a nonseparating loop in a surface $\mathcal{S}$ of genus g that is contained within some face F of a surface graph G. By cutting S along α and filling in the two resulting boundary curves with open discs, a $\widetilde{S}$-graph is obtained. We denote such a graph by $\widetilde{G}$. Notice that the surface $\widetilde{S}$ is of genus $g - 1$. This procedure is called cutting and capping along α. Notice that if α is a separating loop in $\mathcal{S}$, then the cutting and capping along α results in two surfaces such that the sum of the genera of the resulting surfaces is equal to the genus of the original surface in the case of a separating loop. For more details, one can consult for example, [5].

3 Exploring irreducible Laman torus graphs

This section presents some results concerning irreducible torus graphs that are $(2, 3)$-tight. In the following we review some of Laman properties. We focus mainly on specific Laman graphs that are embedded in the torus.

Lemma 3.1. *If L_1 and L_2 are two subgraphs of a Laman graph Γ, then*

$$2|V_{L_1 \cup L_2}| - |E_{L_1 \cup L_2)}| = (2|V_{L_1}| - |E_{L_1}|) + (2|V_{L_2}| - |E_{L_2}|) - (2|V_{L_1 \cap L_2}| - |E_{L_1 \cap L_2)}|)$$

Proof.

$$\begin{aligned} 2|V_{L_1 \cup L_2}| - |E_{L_1 \cup L_2)}| =& 2(|V_{L_1}| + |V_{L_2}| - |V_{L_1 \bigcap L_2}|) - (|E_{L_1}| + |E_{L_2}| - |E_{L_1 \bigcap L_2}|) \\ =& 2|V_{L_1}| - |E_{L_1}| + 2|V_{L_2}| - |E_{L_2}| - (2|V_{L_1 \bigcap L_2}| - |E_{L_1 \bigcap L_2}|) \end{aligned}$$

□

Proposition 3.2. *Let L_1 and L_2 be two Laman subgraphs of a Laman graph Γ. Then the subgraphs $L_1 \bigcup L_2$ and $L_1 \bigcap L_2$ are Laman.*

Proof. By Lemma 3.1,

$$2|V_{L_1 \cup L_2}| - |E_{L_1 \cup L_2)}| = 6 - (2|V_{L_1 \cap L_2}| - |E_{L_1 \cap L_2)}|)$$

But Γ is Laman, thus, $L_1 \bigcup L_2$ and $L_1 \bigcap L_2$ are $(2,3)$-sparse. It follows that $2|V_{L_1 \cup L_2}| - |E_{L_1 \cup L_2)}| = 3$ and $2|V_{L_1 \cap L_2}| - |E_{L_1 \cap L_2)}| = 3$. □

Now, we suggest an inductive construction for the class of Laman torus graphs. However, this work will not present all the irreducible graphs in this class of Laman torus graphs.

Two fundamental operations in graph theory are edge contraction and edge deletion. We re-examine them in some detail via surface graph terms. Let e be an edge in a surface graph G, then G/e represents a surface graph that is obtained from G by contracting e and identifying its end-nodes. $G - e$ represents a surface graph that is obtained from G by deleting e. In specific cases, these two operations are performed together to maintain specific properties of the resulting graph. For example, when specific edges are contracted in a sparse graph, it needs to delete some edges to preserve the sparsity of the graph. Triangle contraction operation is a combination of contracting and deleting two edges. This operation has been used to construct Laman graphs. Here, we also employ the operation of triangle contraction to explore irreducible Laman torus graphs. Let T be a triangle face in G with a boundary walk $v_1, e_1, v_2, e_2, v_3, e_3, v_1$. Then $G_{T_{e_1}} = (G/e_1) - e_2$ represents the surface graph that is obtained from G by contracting the triangle T. Now, let Q be a quadrilateral face in G with boundary walk $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_1$, then $G_{Q,v_1,v_3} = (G/v_1 \sim v_3) - \{e_1, e_3\}$ represents the surface graph that is obtained from G by contracting the quadrilateral Q.

Conducting triangle or quadrilateral contractions is not always possible. This is because sometimes the desired edges are part of subgraphs and such subgraphs have reached their top sparsity counts. More specifically, in Laman graphs, if it happens that one edge of a triangle located in a Laman subgraph differs from that triangle, then it is not possible to contract this edge. Such a subgraph is called a blocker. In the quadrilateral case, in the torus, there are two kinds of blockers, which can be described as follows. Suppose Q be a quadrilateral face with the boundary walk given above, a subgraph H that is Laman and containing v_1 and v_3 and exactly one of v_2 or v_4 is called a blocker of type 1 for conducting $G_{Q,v_1,v_3} = (G/v_1 \sim v_3) - \{e_1, e_3\}$, Figure 1(a). If H is a subgraph of G such that $2|V_H| - |E_H| = 4$ and H containing v_1 and v_3 but not v_2 and v_4 then H is called a blocker of type 2 for conducting $G_{Q,v_1,v_3} = (G/v_1 \sim v_3) - \{e_1, e_3\}$, Figure 1(b).

Lemma 3.3. *If T is a triangle of a Laman torus graph G, then at least two of the edges of T are contractible.*

Proof. Let $v_1, e_1, v_2, e_2, v_3, e_3, v_1$ be the boundary walk of T. On the contrary, suppose that L_1 and L_2 is a blocker for $G_{T_{e_1}}$ and $G_{T_{e_2}}$, respectively. It is clear that $v_1, v_2, v_3 \in L_1 \bigcup L_2$ and $e_3 \notin L_1 \bigcup L_2$. By Proposition 3.2, $L_1 \bigcup L_2$ is Laman. Hence,

$$\begin{aligned} 2|V_{L_1 \cup L_2 \cup \{e_3\}}| - |E_{L_1 \cup L_2 \cup \{e_3\}}| =& 2|V_{L_1 \cup L_2}| - |E_{L_1 \cup L_2}| + 2|V_{\{e_1\}}| - |E_{\{e_1\}}| \\ &-(2|V_{L_1 \cap L_2 \cap \{e_1\}}| - |E_{L_1 \cap L_2 \cap \{e_1\}}|) \\ =& 2 \end{aligned}$$

It is clear that the subgraph $L_1 \cup L_2 \cup \{e_3\}$ is not $(2,3)$-sparse which contradicts the sparsity of G. □

Our main aim is to present a road map for exploring specific Laman torus graphs from which one can construct the class of Laman torus graphs. Such graphs are called irreducible. The irreducibility here is subject to conducting the triangle and quadrilateral contractions. Therefore, we define an irreducible Laman torus graph to be a Laman torus graph such that there is no triangle or quadrilateral contraction resulting also Laman torus graph. Now, we recall the cutting and capping procedure on curves on the torus. We notice that if α is a separating loop in the torus, then the cutting and capping along α results in a spherical graph and a punctured torus. Recall that in the torus, a simple loop is nonseparating if and only if it does not bound an embedded disc.

Our next aim is to get some structural information on the irreducible Laman torus graphs to understand much better how to extract all such graphs. Although, finding the complete list of irreducible Laman torus graphs is out of the scope of this work. However, we present some of such graphs later on in this section.

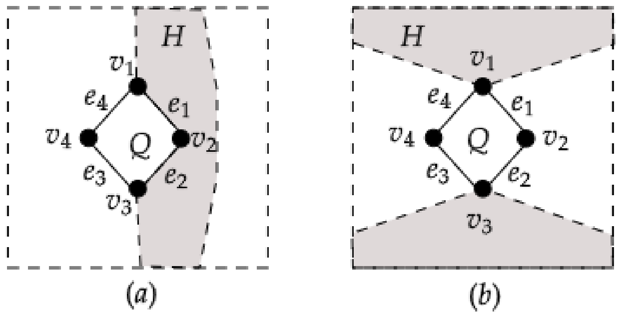

Figure 1. (a) H is a blocker of type 1 for the contraction G_{Q,v_1,v_3}. (b) H is a blocker of type 2 for the contraction G_{Q,v_1,v_3}

Cycles of length three appear in Laman graphs and knowing how they can be embedded in the torus can reduce the search space for the irreducible Laman graphs. Theorem 3.7 shows how cycles of length three can be embedded in the torus if such cycles are subgraphs of the irreducible Laman torus graphs.

Theorem 3.4. *[4] Let G be a (k,l)-tight surface graph and $l \leq k$ and H be a surface subgraph of G. Let F be a face in H. Then*

$$k|V_{H\cup int_G(F)}| - |E_{H\cup int_G(F)}| \leq k|V_H| - |E_H|$$

Lemma 3.5. *Let H be a torus subgraph of a Laman torus graph G. If H is a separating cycle of length 3 and F be the face of H that is homeomorphic to the punctured torus. Then $K = \widetilde{ext_G(F)}$ is Laman.*

Proof. First, notice that $2|V_{ext_G}(F)| - |E_{ext_G(F)}| = 2|V_{\widetilde{ext_G(F)}}| - |E_{\widetilde{ext_G(F)}}|$.

$$\begin{aligned}3 = 2|V_G| - |E_G| &= 2|V_{int_G(F)\cup ext_G(F)}| - |E_{int_G(F)\cup ext_G(F)}| \\ &= 2|V_{int_G(F)}| - |E_{int_G(F)}| + 2|V_{ext_G(F)}| - |E_{ext_G(F)}| - 2|V_{\partial F}| + |E_{\partial F}| \\ &= 2|V_{int_G(F)}| - |E_{int_G(F)}| + 2|V_{ext_G(F)}| - |E_{ext_G(F)}| - 3\end{aligned}$$

So

$$2|V_{int_G(F)}| - |E_{int_G(F)}| + 2|V_{ext_G(F)}| - |E_{ext_G(F)}| = 6 \tag{3.1}$$

By Theorem 3.4, we get that $2|V_{int_G(F)}| - |E_{int_G(F)}| \leq 3$. Then the sparsity of G implies $2|V_{int_G(F)}| - |E_{int_G(F)}| = 3$. From 3.1, we get that $2|V_{ext_G(F)}| - |E_{ext_G(F)}| = 3$. It follows easily that K is Laman. □

The following lemma emphasises existing at least two triangles in any Laman spherical graph with at least four vertices.

Lemma 3.6. *[6] Every Laman spherical graph with at least four vertices contains at least two triangular faces.*

Proof. By Euler's formula, we have $|V| - |E| + |F| = 2$. But G is Laman, so $3 - |E| + 2|F| = 4$. But $2|E| = \sum_{i\geq 0} i f_i$, thus we get $-\sum_{i\geq 0} i f_i + 4\sum_{i\geq 0} f_i = 2$. Rearranging the last equation leads to $\sum_{i\geq 0}(4-i)f_i = 2$. Expanding the last equation results in $4f_0+3f_1+2f_2+3f_3-f_5-2f_6-\cdots = 2$. It is clear that $f_0 = f_1 = f_2 = 0$. Thus, $f_3 = 2 + f_5 + 2f_6 + 3f_7 + \ldots$. Consequently, $f_3 \geq 2$. □

Theorem 3.7. *Let G be an irreducible Laman torus graph. Then any 3-cycle in Γ forms a nonseparating cycle in the torus.*

Proof. On the contrary, suppose that Ω is a 3-cycle in Γ which forms $\widetilde{\text{a separating}}$ cycle H in the torus, where $H = \phi(\Omega)$. Now, consider the spherical graph $K = ext_G(F)$. By Lemma 3.5, $2|V_K| - |E_K| = 3$. It follows that K is Laman. By Lemma 3.6, K has at least two triangles. So one of these triangles is also a face of G which contradicts the irreducibility of G. □

It is clear that there is only one Laman graph with two vertices, Figure 2(a). While there is only one irreducible Laman torus graph with three vertices, Figure 2(b). Notice that both of these torus graphs are non-cellular.

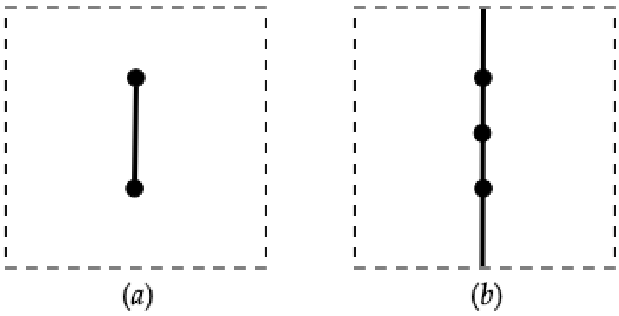

Figure 2. Non-cellular irreducible Laman torus graphs: (a) Irreducible Laman graph with two vertices. (b) Irreducible Laman graph with three vertices.

In the following, we try to explore irreducible Laman torus graphs that are cellular.

Proposition 3.8. *Let G be a cellular irreducible Laman torus graph with no quadrilateral face. Then G satisfies:*

$$\begin{aligned}(f_5, f_6, f_7, f_8, f_9, f_{10}) \in \{&(6,0,0,0,0,0), (4,1,0,0,0,0), (3,0,1,0,0,0), (2,2,0,0,0,0), (2,0,0,1,0,0),\\ &(1,1,1,0,0,0), (1,0,0,0,1,0), (0,1,0,1,0,0), (0,0,0,0,0,1), (0,0,2,0,0,0),\\ &(0,3,0,0,0,0)\}\end{aligned}$$

Proof. By Euler's formula, we have $|V|-|E|+|F| = 0$. But G is Laman, so $3-|E|+2|F| = 0$. But $2|E| = \sum_{i\geq 0} if_i$, thus we get $-\sum_{i\geq 0} if_i + 4\sum_{i\geq 0} f_i = -6$. Rearranging the last equation leads to $\sum_{i\geq 0}(i-4)f_i = 6$. Expanding the last equation results in $-4f_0-3f_1-2f_2-3f_3+f_5+2f_6+\cdots =$ 6. It is clear that $f_0 = f_1 = f_2 = f_3 = 0$ and also, $f_i = 0$ for $i = 11, 12, \ldots$. Thus, $f_5 + 2f_6 + 3f_7 + 4f_8 + 5f_9 + 6f_{10} = 6$. Therefore, the required conclusion follows easily. □

Knowing the bound of the number of vertices of the irreducible Laman torus graphs can also ease the search for such graphs. The following proposition presents bounds for the number of vertices of the irreducible Laman torus graphs with no quadrilateral faces.

Proposition 3.9. *Let G be an irreducible Laman torus graph with no quadrilateral face. Then G has at least 2 vertices and at most 9 vertices.*

Proof. If G is noncellular, then it either has two vertices or three vertices. Now, if G is cellular, by a similar argument to that one in the proof of Proposition 3.8, we have that $f_5 + 2f_6 + 3f_7 + 4f_8 + 5f_9 + 6f_{10} = 6$ and $f_i = 0$ for all $i \in \{1, 2, 3, 4, 11, 12, \ldots\}$. So

$$\begin{aligned}2|E| &= 5f_5 + 6f_6 + 7f_7 + 8f_8 + 9f_9 + 10f_{10}\\ &\leq 5(f_5 + 2f_6 + 3f_7 + 4f_8 + 5f_9 + 6f_{10})\\ &\leq 5(6) = 30\end{aligned}$$

Hence, $|E| \leq 15$ and $|V| = \frac{|E|+3}{2} \leq 9$. □

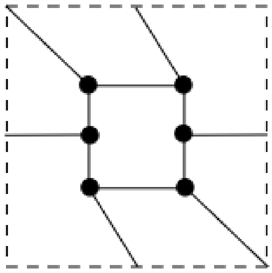

Figure 3. Embedding $K_{3,3}$ as an irreducible Laman torus graph.

One interested example of irreducible Laman torus graph is depicted in Figure 3. The underline graph of this torus graph is $K_{3,3}$.

Algorithmically, we can use Theorem 3.7 in the search of irreducible graphs. Specifically, for any Laman graph with a number of vertices within the bound that we gave in Proposition 3.9, its cycles of length 3 have to be embedded as nonseparating cycles in the torus. On the other hand, It is also possible to use an interesting method, polygon representation, of extracting some cellular irreducible Laman torus graphs. This method represents cellular surface graphs by a collection of plane polygons. An example of irreducible Laman torus graph is depicted in Figure 3. The underline graph of this torus graph is $K_{3,3}$.

The following irreducible Laman torus graphs as in Figure 4 were presented in [16] and extracted using the polygon representation method.

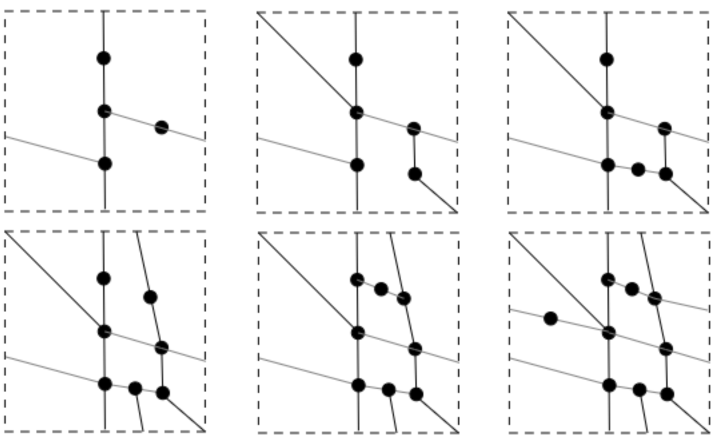

Figure 4. Some examples of irreducible Laman torus graphs with no quadrilateral face.

Conclusion

In this work, we explored Laman graphs that are embedded in the torus. We investigated the structure of such irreducible graphs by presenting some results regarding the counting of their faces and the counting of the vertices of such irreducible torus graphs. Also, we stated the operations that can be used to construct the class of Laman torus graphs. Consequently, the obtained results can help in searching for all these irreducible graphs in the class.

CONFLICTS OF INTEREST

The author declares no conflict of interest.

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Author information

Qays R. Shakir, Information Technology Department, Middle Technical University, Iraq.
E-mail: qays.shakir@mtu.edu.iq

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