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SOME RESULTS ON IRREDUCIBLE IDEALS OF MONOIDS

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Abstract The purpose of this note is to study some algebraic properties of s-irreducible s-ideals of monoids. We establish relations between s-irreducible, s-prime, and s-semiprime s-ideals. We explore some properties of s-irreducible s-ideals in local monoids.

1 Introduction and preliminaries

A comprehensive ideal theory for monoids was originally introduced in [1], and has since been extended through numerous studies (see [8, 10, 11, 12, 13, 14, 15, 17, 5]). In the context of algebraic geometry, the recent advancements in logarithmic algebraic geometry and toric varieties have shown interest in the study of ideals associated with monoids, as indicated in [21, 20, 22, 23, 16, 6].

In this paper, we study a specific type of s-ideals of monoids, namely, s-irreducible s-ideals. We extend certain algebraic properties observed in strongly irreducible ideals of rings or semirings to s-irreducible s-ideals of monoids.

Throughout, a monoid is a commutative cancellative semigroup with identity element, and we use multiplicative notation. A monoid is called *pointed* if there exists an element $0 \in M$ such that $x0 = 0$, for all $x \in M$. In this paper, all our monoids are pointed.

Let us recall some central notions from [17]. Suppose M is a monoid. An *ideal system* on a monoid M is a map $r\colon \mathcal{P}(M) \to \mathcal{P}(M)$ defined by $X \mapsto X_r$ such that the following conditions are satisfied for all subsets X, $Y \subseteq M$ and all elements $m \in M$:

- $X \cup \{0\} \subseteq X_r$,
- $X \subseteq Y_r$ implies that $X_r \subseteq Y_r$,
- $mM \subseteq \{m\}_r$, and
- $mX_r = (mX)_r$.

Let r be an ideal system. A subset $I \subseteq M$ is called an *r-ideal* if $I = I_r$. Let $Q(M)$ be the quotient group of M. For a subset X of $Q(M)$, we define $X^{-1} := (M : X) = \{z \in Q(M) \mid zX \subseteq M\}$. The most important ideal systems are the s-system, the v-system, and the t-system. For $X \subseteq M$, they are respectively defined by

$$X_s := XM, \quad X_v := (X^{-1})^{-1}, \quad t := v_s.$$

If for an ideal system r on M, we denote by

$$\mathcal{I}_r(M) := \{X_r \mid X \subseteq M\},$$

the set of all r-ideals of M, then $\mathcal{I}_v(M) \subseteq \mathcal{I}_t(M) \subseteq \mathcal{I}_s(M)$.

Let us record some elementary definitions about s-ideals of monoids. If I and J are s-ideals of a monoid M, then their *product* is defined by

$$IJ := \{ij \mid i \in I, j \in J\},$$

which is also an s-ideal of M. Let S be a nonempty subset of a monoid M. An *s-ideal quotient* or a *colon s-ideal* is defined by

$$(I : S) := \{m \in M \mid mS \subseteq I\}.$$

It is easy to check that $(I : S)$ is indeed an s-ideal of M. A proper s-ideal P is called *s-prime* if $xy \in P$ implies that $x \in P$ or $y \in P$ for all $x,\, y \in M$. By $\mathrm{Spec}(M)$, we shall denote the set of s-prime s-ideals of M. A proper s-ideal $\mathfrak{m}$ of M is called *s-maximal* if it is not properly contained in another proper s-ideal of M. If I is an s-ideal of M, the *s-radical* of I is defined by

$$\sqrt{I} := \left\{ m \in M \mid m^k \in I \text{ for some } k \in \mathbb{Z}^+ \right\}.$$

An s-ideal I is said to be a *s-radical s-ideal* (or to be a *s-semiprime*) if $\sqrt{I} = I$. A proper s-ideal I of M is called *s-primary* if $xy \in I$ implies $x \in I$ or $y^k \in I$ for some $k \in \mathbb{Z}^+$. An s-ideal I is called *P-primary* if $P = \sqrt{I}$, for some s-prime ideal P of M. Similarly, we define *$\mathfrak{m}$-primary* ideal for an s-maximal ideal $\mathfrak{m}$ of M.

An s-ideal L of M is called *s-irreducible* (*s-strongly irreducible*) if for s-ideals $I,\, J$ of M and $L = I \cap J$ ($L \subseteq I \cap J$) implies that $L = I$ ($L \subseteq I$) or $L = J$ ($L \subseteq J$) .We shall denote the sets of all s-irreducible and s-strongly irreducible s-ideals of M respectively by $\mathcal{I}_{\mathrm{s}}^{+}(M)$ and $\mathcal{I}_{\mathrm{s}}^{++}(M)$. It is easy to see that every s-strongly irreducible s-ideal is s-irreducible. The converse of this also holds in monoids, which follows from the fact that every monoid is *arithmetic*, that is, the set $\mathcal{I}_{\mathrm{s}}(M)$ of all s-ideals of M forms a distributive lattice under set inclusion as the partial order.

To prove a property for an s-irreducible s-ideal, thanks to the equivalence between $\mathcal{I}_{\mathrm{s}}^{+}(M)$ and $\mathcal{I}_{\mathrm{s}}^{++}(M)$, it is sufficient to check it for the s-strongly irreducible condition.

In the following lemma, we gather some properties of s-ideals of monoids that will be used in sequel.

Lemma 1.1. *Let M be monoid.*

(i) The unique maximal s-ideal of I is the set of non-invertible elements of M.

(ii) If I and J are two s-ideals of M, then $IJ \subseteq I \cap J$.

(iii) If $i,\, j \in M$, then $\langle i \rangle \langle j \rangle = \langle ij \rangle$.

(iv) An s-ideal P of M is s-prime if and only if, for all $I,\, J \in \mathcal{I}_{\mathrm{s}}(M)$ and $IJ \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

(v) An s-ideal I of M is s-semiprime if and only if $J^2 \subseteq I$ implies $J \subseteq I$, for all $J \in \mathcal{I}_{\mathrm{s}}(M)$.

Proof. (i)–(iii) Straightforward.

(iv) Let P be a s-prime s-ideal of M and $IJ \subseteq P$ for some $I, J \in \mathcal{I}_{\mathrm{s}}(M)$. Let $I \nsubseteq P$. Then $i \notin P$ for some $i \in I$. However, for all $j \in J$, we have $ij \in IJ \subseteq P$. This implies $j \in P$ for all $j \in J$. Hence $J \subseteq P$. To show the converse, let $ij \in P$ for some $i, j \in M$. Then $\langle i \rangle \langle j \rangle = \langle ij \rangle$, by (iii); and therefore, $i \in \langle i \rangle \subseteq P$ or $j \in \langle j \rangle \subseteq P$.

(v) If I is s-semiprime, then the condition holds trivially. The converse follows by induction. □

2 Irreducible ideals

Here is an elementwise equivalent definition of s-irreducible s-ideals of monoids, and as mentioned before, we shall check only the s-strongly irreducible condition.

Lemma 2.1. *An s-ideal I of M is s-irreducible if and only if $\langle m \rangle \cap \langle m' \rangle \subseteq I$ implies $m \in I$ or $m' \in I$, for all $m, m' \in M$.*

Proof. Let I be an s-irreducible s-ideal of M and $\langle m \rangle \cap \langle m' \rangle \subseteq I$, for some $m, m' \in M$. Then $m \in \langle m \rangle \subseteq I$ or $m' \in \langle m' \rangle \subseteq I$. Conversely, let I be an s-ideal of M with $J \cap K \subseteq I$ for some s-ideals J and K of M. Let $J \nsubseteq I$. This implies $j \notin I$ for some $j \in J$, and hence $\langle j \rangle \nsubseteq I$. Therefore, for all $k \in K$, we have

$$\langle j \rangle \cap \langle k \rangle \subseteq J \cap K \subseteq I,$$

and by hypothesis, we obtain $k \in I$ for all $k \in K$. In other words, $K \subseteq I$. □

The following result shows when all proper s-ideals of a monoid are s-irreducible. Since the proof is straightforward, we skip it. This result generalizes [2, Lemma 3.5].

Lemma 2.2. *Let M be a monoid. Then the following are equivalent.*

(i) Every proper s-ideal of M is an s-irreducible s-ideal.

(ii) Every two s-ideals of M are comparable.

In the following proposition we shall see some relations of s-irreducible s-ideals with s-prime and s-semiprime s-ideals of a monoid. This result extend [3, Proposition 2], [2, Theorem 2.1], and [9, Proposition 7.36].

Proposition 2.3. *Let M be a monoid. Then the following hold.*

(i) Every s-prime s-ideal of M is s-irreducible.

(ii) An s-ideal P of M is s-prime if and only if it is s-semiprime and s-irreducible.

Proof. To obtain (i), it suffices to notice that by Lemma 1.1(ii), the condition: $IJ \subseteq I \cap J$ holds for any two s-ideals I and J of M. For (ii), let P be a s-prime s-ideal. Then P is s-irreducible, by (i). To show P is s-semiprime, it suffices to show $\sqrt{P} \subseteq P$, which follows immediately; indeed, $m \in \sqrt{P}$ implies $m^k \in P$, for some $k \in \mathbb{Z}^+$, and hence $m \in P$ as P is a s-prime s-ideal. For the converse, let P be a s-irreducible s-ideal and $IJ \subseteq P$, for some s-ideals I and J of M. Since

$$(I \cap J)^2 \subseteq IJ \subseteq P,$$

and since P is s-semiprime, $I \cap J \subseteq P$. Since P is s-irreducible , $I \subseteq P$ or $J \subseteq P$. □

By Lemma 1.1(i), every proper s-ideal of a monoid M is contained in its unique s-maximal s-ideal. We can further generalize this containment to s-irreducible s-ideals as we see in the following proposition, which also extends the corresponding property of rings (see [2, Theorem 2.1(ii)]).

Proposition 2.4. *Let M be a monoid. For each proper s-ideal J of M, there is an s-minimal s-irreducible s-ideal over J.*

Proof. Suppose $\mathcal{E} := \{I \in \mathcal{I}_s^+(M) \mid I \supseteq J\}$. Since the unique s-maximal s-ideal of M is an s-irreducible s-ideal, obviously $\mathcal{E} \neq \emptyset$. By Zorn's lemma, $\mathcal{E}$ has an s-minimal element, which is our desired s-ideal. □

Before we study further properties of s- irreducible s-ideals of monoids, let us pause for some examples of them. Every s-prime s-ideal of a monoid M is an s-irreducible s-ideal (see Proposition 2.3(i)), and so is the unique maximal s-ideal of M. In the monoids $(\mathbb{N}, \cdot, 1)$ and $(\mathbb{Z}, \cdot, 1)$, the s-ideals generated by a set of s-prime numbers are s-irreducible. Using the notion of colon s-ideals, the following two results show how to generate further examples of s-irreducible s-ideals.

Proposition 2.5. *If I is an s-irreducible s-ideal and J is an s-ideal of a monoid M, then $(I : J)$ is an s-irreducible s-ideal of M.*

Proof. Let $K \cap L \subseteq (I : J)$ for some $K, L \in \mathcal{I}_s(M)$. This implies $KJ \cap LJ = (K \cap L)J \subseteq I$. From this we have $KJ \subseteq I$ or $LJ \subseteq I$, since I is s-irreducible . □

Corollary 2.6. *Suppose that J, $\{J_\lambda\}_{\lambda \in \Lambda}$, K are s-ideals and I, $\{I_\omega\}_{\omega \in \Omega}$ are s-irreducible s-ideals of a monoid M. Then $(I \colon J)$, $((I \colon J) \colon K)$, $(I \colon JK)$, $((I \colon K) \colon J)$, $(\bigcap_\omega I_\omega \colon J)$, $\bigcap_\omega (I_\omega \colon J)$, $(I \colon \sum_\lambda J_\lambda)$, and $\bigcap_\lambda (I \colon J_\lambda)$ are all s-irreducible s-ideals of M.*

Recall that a *monoid homomorphism* $\phi \colon M \to M'$ is a map ϕ from M to M' with the property:

- $\phi(1) = 1$,
- $\phi(mm') = \phi(m)\phi(m')$,

for all $m, m' \in M$. If J is a an s-ideal of M', then we denote inverse image $\phi^{-1}(J)$ by J^c. If $I \in \mathcal{I}_s(M)$, then the s-ideal generated by $\phi(I)$ is denoted by I^e. The *kernel of* ϕ is defined by

$$\ker(\phi) := \{(m, m') \in M \times M \mid \phi(m) = \phi(m')\}.$$

It is well-known that inverse image of a prime ideal under a ring homomorphism is a prime ideal. However, that fails to hold for strongly irreducible ideals. A sufficient condition for the preservation of strongly irreducible ideals (of rings) under inverse image is given in [24, Proposition 1.4]. The following proposition generalizes that result to monoids for s-irreducible s-ideals.

Proposition 2.7. *Let* $\phi\colon M \to M'$ *be a surjective monoid homomorphism such that* $\ker(\phi) \subseteq \langle x\rangle$ *for each* $x \notin \ker(\phi)$. *If J is an s-irreducible s-ideal of M', then J^c is an s-irreducible s-ideal of M.*

Proof. Let $x, x' \in M \setminus J^c$. Then $\phi(x), \phi(x') \in M' \setminus J$. Since J is s-irreducible , by Proposition 2.1, there exist $y, y' \in M'$ such that

$$\phi(x)y = \phi(x')y' \in M' \setminus J.$$

Since ϕ is surjective, $y = \phi(z)$ and $y' = \phi(z')$ for some $z, z' \in M$. From this, we obtain $xz, x'z' \notin J^c$ and $(xz, x'z') \in \ker(\phi) \subseteq C$, where C is the smallest congruence containing $\{(mx, m'x) \mid m, m' \in M\}$. From this, it follows that $x'z' \notin J^c$. □

Like localization of rings, one can also construct local monoids. Here we briefly recall some essential facts about it, and for further details, we refer to [7] and [4]. Let M be a monoid. A subset S of M is called *multiplicatively closed* if

- $1 \in S$, and
- $ss' \in S$, whenever $s, s' \in S$.

For any $m \in M$ and $s \in S$, define a set

$$M_S := \{m/s \mid m/s = m'/s' \text{ whenever there exists } u \in S \text{ such that } (ms')u = (m's)u\}.$$

The multiplication on M_S is defined by $(m/s) \cdot (m'/s') := (mm')(ss')$ and the multiplicative identity of M_S is $1/1$. The system $(M_S, \cdot, 1/1)$ is called the *local monoid* with respect to S or *localization of* M at S. The following result from [7, Proposition 2.4.3(iii)] identifies s-ideals of a monoid M that are in 1-1 correspondence with s-ideals of M_S.

Proposition 2.8. *Let M be a monoid and let $S \subseteq M \setminus \{0\}$ be a multiplicatively closed subset of M. Then the proper s-ideals of M_S correspond to the s-ideals of M contained in $M \setminus S$.*

Similar to the above, we can also identify s-irreducible s-ideals of M that are in 1-1 correspondence with s-irreducible s-ideals of M_S, and this will be demonstrated in the next theorem. This result generalizes [2, Theorem 3.1].

Theorem 2.9. *Let M be a monoid and let S be a multiplicatively closed subset of M. For each $I \in \mathcal{I}_s(M_S)$, let $I^c := \{m \in M \mid m/1 \in I\} = I \cap M$ and let $C := \{I^c \mid I \in \mathcal{I}_s(M_S)\}$. Then there is a 1-1 correspondence between $\mathcal{I}_s^+(M_S)$ and $\mathcal{I}_s^+(M)$ contained in C which do not meet S.*

Proof. Let $I \in M_S$. Evidently, $I^c \neq M$. Also, it is clear that $I^c \in C$ and $I^c \cap S = \emptyset$. Suppose $J, K \in \mathcal{I}_s(M)$ such that $J \cap K \subseteq I^c$. This implies

$$J_S \cap K_S = (J \cap K)_S \subseteq I^c_S = I.$$

Since I is s-strongly irreducible , we must have $J_S \subseteq I$ or $K_S \subseteq I$. This subsequently implies that $J \subseteq J^c_S \subseteq I^c$ or $K \subseteq K^c_S \subseteq I^c$, showing that $I^c \in \mathcal{I}_s^+(M)$. For the converse, suppose that $I \in \mathcal{I}_s^+(M)$, $I \cap S = \emptyset$, and $I \in C$. It is clear that $I_S \neq M_S$. Suppose $J, K \in M_S$ such that $J \cap K \subseteq I_S$. This implies

$$J^c \cap K^c = (J \cap K)^c \subseteq I^c_S.$$

Since $I \in C$, we must have $I^c_S = I$. Therefore, $J^c \cap K^c \subseteq I$. Since I is s-strongly irreducible , we have either $J^c \subseteq I$ or $K^c \subseteq I$. Hence, either $J = J^c_S \subseteq I_S$ or $K = K^c_S \subseteq I_S$, which implies that $I_S \in \mathcal{I}^+(M_S)$. □

Corollary 2.10. *Suppose M is a monoid and S is multiplicatively closed subset of M. If $I \in \mathcal{I}_s^+(M)$ and I is a s-primary s-ideal such that $I \cap S = \emptyset$, then $I_S \in \mathcal{I}^+(M_S)$ and a s-primary s-ideal of M_S.*

In terms of local monoids, the next result provides two sufficient conditions for s-primary s-ideals of a monoid to be s-irreducible.

Proposition 2.11. *Let M be a monoid. Then the following are equivalent.*

(i) For the maximal s-ideal $\mathfrak{m}$ of M, every s-primary s-ideal of $M_{\mathfrak{m}}$ is s-irreducible.

(ii) Every s-primary s-ideal of M is s-irreducible.

(iii) For any s-prime s-ideal P of M, every s-primary s-ideal of M_P is s-irreducible.

Proof. (i)⇒(ii): Suppose I is a s-primary s-ideal of M. Obviously, $I \subsetneq \mathfrak{m}$. Then $I_{\mathfrak{m}}$ is a s-primary s-ideal of $M_{\mathfrak{m}}$, and that by assumption implies $I_{\mathfrak{m}} \in \mathcal{I}_s^+(M_{\mathfrak{m}})$. It follows from Theorem 2.9 that $I_{\mathfrak{m}}^c \in \mathcal{I}_s^+(M)$ and as I is s-primary, we must have $I_{\mathfrak{m}}^c = I$. This proves that $I \in \mathcal{I}_s^+(M)$.

(ii)⇒(iii): Suppose that I is a s-primary s-ideal of M_P. Now I^c is a s-primary s-ideal of M with the properties: $I^c \cap (M \setminus P) = \emptyset$ and $I^c \in C$. Moreover, by our assumption, $I^c \in \mathcal{I}_s^+(M)$. Therefore, by Theorem 2.9, we have $I_P^c = I \in \mathcal{I}_s^+(M_P)$.

(iii)⇒(i): Straightforward. □

The next result gives a representation of an s-ideal on a monoid in terms of s-irreducible s-ideals, and it generalizes [19, Corollary 2].

Proposition 2.12. *Every s-ideal I of a monoid M can be represented as follows:*

$$I = \bigcap_{\substack{J \supseteq I \\ J \in \mathcal{I}_s^+(M)}} J.$$

Proof. Let $I \in \mathcal{I}_s(M)$ and consider the set $\mathcal{E} := \{J \in \mathcal{I}_s^+(M) \mid J \supseteq I\}$. Since $M \in \mathcal{I}_s^+(M)$, the set $\mathcal{E}$ is nonempty. It is evident that $I \subseteq \bigcap_{J \in \mathcal{E}} J$. To have the other inclusion, observe that it is sufficient to show the claim: if $0 \neq x \in M$ and if $I \in \mathcal{I}_s(M)$ such that $x \notin I$, then there exists a $J \in \mathcal{I}_s^+(M)$ with the property that $J \supseteq I$ and $x \notin J$. Now this fact follows from a simple application of Zorn's lemma. □

Recall that an r-*laskerian* monoid is a monoid in which every proper r-ideal has a r-primary decomposition. The following result has been lifted from [17].

Proposition 2.13. *Let M be a monoid. Then M is r-laskerian if and only if every r-irreducible r-ideal of M is s-primary.*

Our next result is on minimal decompositions (*cf.* [17, Theorem 7.1]).

Proposition 2.14. *Suppose M is a monoid in which every s-primary s-ideal is s-irreducible. Then every minimal s-primary decomposition of an s-ideal of M is unique.*

Proof. Suppose that $I \in \mathcal{I}_s(M)$ with two minimal s-primary decomposition representations:

$$I = \bigcap_{i=1}^{n} P_i = \bigcap_{i=1}^{m} P_i'.$$

Without loss of generality, assume that $n \leqslant m$. Since $\bigcap_{i=1}^{n} P_i \subseteq P_1'$ and since $P_1' \in \mathcal{I}_s^+(M)$, there exists $j \in \{1, \ldots, n\}$ such that $P_j \subseteq P_1'$. Similarly, the facts: $\bigcap_{i=1}^{m} P_i' \subseteq P_j$ and $P_j \in \mathcal{I}_s^+(M)$ implies there exists $k \in \{1, \ldots, m\}$ such that

$$P_k' \subseteq P_j \subseteq P_1'.$$

Since $\bigcap_{i=1}^{m} P_i'$ is a minimal s-primary decomposition, we must have $P_k' = P_1'$, and so, $k = 1$. Hence, $P_1' = P_j$. Assume that $P_1' = P_1$. Proceeding with induction, we eventually arrive at the conclusions: $P_i' = P_i$ and $n = m$. □

Our next result generalizes [18, Theorem 2.6], and is comparable with [17, Proposition 9.6], which is for r-ideals on r-noetherein, r-local monoids. But first, a lemma.

Lemma 2.15. *Let b and c be elements in a monoid M. Then the following hold.*

(i) If I is an s-ideal of M such that $I \subseteq \langle b\rangle$, then $I = b(I : \langle b\rangle)$.

(ii) $\langle b\rangle \cap \langle c\rangle = b(\langle c\rangle : \langle b\rangle) = c(\langle b\rangle : \langle b\rangle)$.

Proof. (i) Suppose that I is an s-ideal of M such that $I \subseteq \langle b\rangle$. Then $b(I : \langle b\rangle) \subseteq I$, and if $i \in I \subseteq \langle b\rangle$, then $i = rb$ for some $r \in R$, so $r \in (I : \langle b\rangle)$, and hence $i = rb \in b(I : \langle b\rangle)$.

(ii) Since $\langle b\rangle \cap \langle c\rangle \subseteq \langle b\rangle$, it follows that $\langle b\rangle \cap \langle c\rangle = b((\langle b\rangle \cap \langle c\rangle) : \langle b\rangle) = b(\langle c\rangle : \langle b\rangle)$. By symmetry, we also have $\langle b\rangle \cap \langle c\rangle = c(\langle b\rangle : \langle c\rangle)$. □

Theorem 2.16. *Let $(M, \mathfrak{m})$ be an s-local monoid and let I be an s-irreducible $\mathfrak{m}$-primary s-ideal of M. Suppose that $I \subsetneq (I : \mathfrak{m})$. Then the following hold.*

(i) $(I : \mathfrak{m})$ is a s-principal s-ideal.

(ii) $I = (I : \mathfrak{m})\mathfrak{m}$.

(iii) For each s-ideal J of M, we have either $J \subseteq I$ or $(I : \mathfrak{m}) \subseteq J$.

Proof. (i) We have $I \subsetneq (I : \mathfrak{m})$ by hypothesis. so there exists $x \in (I : \mathfrak{m})\setminus I$. If possible, suppose $(I : \mathfrak{m}) \neq \langle x\rangle$. Let $y \in (I : \mathfrak{m}) \setminus \langle x\rangle$. Then, by Lemma 2.15, we have $\langle x\rangle \cap \langle y\rangle = y(\langle x\rangle : \langle y\rangle)$. Since $(\langle x\rangle : \langle y\rangle) \subseteq \mathfrak{m}$ and $y \in (I : \mathfrak{m})$, we further have $y(\langle x\rangle : \langle y\rangle) \subseteq I$. However, I is s-irreducible, so $\langle x\rangle \cap \langle y\rangle \subseteq I$ implies that either $\langle x\rangle \subseteq I$ or $\langle y\rangle \subseteq I$, hence $y \in I$. Therefore, it follows that $(I : \mathfrak{m}) = \langle x\rangle \cup I$. But then $(I : \mathfrak{m}) \subseteq \langle x\rangle$ or $(I : \mathfrak{m}) \subseteq I$, a contradiction.

(ii) Note that by (1), we have $I \subseteq (I : \mathfrak{m}) = \langle x\rangle$. Since $x \in (I : \mathfrak{m}) \setminus I$ implies $(I : \langle x\rangle) = \mathfrak{m}$, as M is s-local with s-maximal s-ideal $\mathfrak{m}$, we have $I = x(I : \langle x\rangle) = x\mathfrak{m}$.

(iii) Suppose J is an s-ideal of M. Without loss of generality, let us assume that $J \not\subseteq I$. Therefore, what remains is to show that $(I : \mathfrak{m}) \subseteq J$; in other words, to show that $x \in J$. If possible suppose that $x \notin J$, then let $j \in J$, so $x \notin \langle j\rangle$. Therefore

$$\langle x\rangle \cap \langle j\rangle = \langle x\rangle(\langle j\rangle : \langle x\rangle) \subseteq x\mathfrak{m} \subseteq I.$$

Since I is s-irreducible and $\langle x\rangle \not\subseteq I$, we must have $\langle j\rangle \subseteq I$, a contradiction. □

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