

# INITIAL AND TERMINAL CONDITIONS FOR DIFFERENTIAL EQUATIONS OF FRACTIONAL DERIVATIVE VIA NON-AUTONOMOUS VARIABLE ORDER

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**Abstract** *This study explores initial and terminal value problems for time-dependent differential equations featuring non-autonomous variable-order fractional operators, integrating both classical integer-order derivatives and fractional-order components. We establish rigorous criteria for the existence and uniqueness of solutions using fixed-point theorem and Banach's contraction principle, focusing on the Riemann-Liouville fractional derivative of variable order ranging between 1 and 2. The analysis extends to Ulam-Hyers-type stability, assessing the system's resilience to perturbations and providing a measure of solution robustness. Theoretical results are validated through numerical examples, demonstrating practical applications under diverse conditions. These examples involve computational simulations that confirm the stability and uniqueness properties derived analytically. The findings contribute to advancing the understanding of variable-order fractional calculus, offering valuable insights for applications in physics, engineering, and control systems where memory-dependent phenomena are prevalent. This work bridges theoretical advancements with practical utility, enhancing methodologies for solving complex differential systems.*

## 1 Introduction

Variable-order fractional operators constitute a rapidly evolving class of mathematical tools that offer enhanced modeling capabilities for complex systems. By allowing the order of differentiation to vary with respect to variables such as time, space, or the state of the system, these operators provide a flexible framework for capturing memory and hereditary effects. Their applicability spans diverse fields including mathematics, physics, engineering, control systems, and optimization. For example, variable-order fractional models have been effectively applied in the analysis of viral infections [1], lymphatic filariasis dynamics [2], and Rift Valley fever [3]. Additional applications include dengue transmission modeling [4], and computational methods for fractional-order systems [5], as well as solute transport in groundwater [6], hepatitis B virus modeling [7], and financial systems [8].

The variable-order fractional calculus of variations provides a rich set of analytical and numerical tools for addressing such applications. However, while constant-order systems have been extensively studied (see [9], [10], [11]), nonlinear differential equations involving variable fractional order remain comparatively underexplored. Some progress has been made through approaches using the Kuratowski measure of non-compactness and related fixed point techniques [12], [13], [14], [15], [16]. More recent developments have introduced generalized forms of fractional derivatives, where the order function is extended from  $\ell(\lambda)$  to  $\ell(\lambda, \nu(\lambda))$ , enhancing

the modeling precision in various settings [17], [18].

A significant contribution in this area was made by Razminia et al., who studied the existence of solutions to a class of variable-order fractional differential equations involving Riemann–Liouville derivatives [19].

$$\begin{cases} \mathcal{D}_{\iota^+}^{\ell(\lambda, \nu(\lambda))} \nu(\lambda) = \mu(\lambda, \nu(\lambda)), \\ \nu(h) = \nu_0, \end{cases}$$

Here,  $\mathcal{D}_{h^+}^{\ell(\lambda, \nu(\lambda))}$  denotes the Riemann-Liouville fractional derivative of variable order  $\ell(\lambda, \nu(\lambda))$ , with  $\mu$  representing a predefined continuous function.

Motivated by this work, we investigate a more general non-autonomous variable-order initial and terminal value problem (NAVOITVP) of the form:

$$\begin{cases} \mathcal{D}_{0^+}^{\ell(\lambda, \nu(\lambda))} \nu(\lambda) + \iota \nu''(\lambda) = \mu(\lambda, \nu(\lambda)), \lambda \in \tilde{\chi}_\tau, \\ \nu(0) = 0, \nu(\rho) = 0, \end{cases} \tag{1.1}$$

where  $\tilde{\chi}_\tau := [0, \rho]$  and  $0 < \rho < +\infty$ ,  $\iota > 0$  and  $\ell : \tilde{\chi}_\tau \times \mathbb{R} \rightarrow (1, 2)$ ,  $\mu : \tilde{\chi}_\tau \times \mathbb{R} \rightarrow \mathbb{R}$  are a continuous functions, and  $\mathcal{D}_{0^+}^{\ell(\lambda, \nu(\lambda))}$  is the R-Liouville fractional derivative of variable-order  $\ell(\lambda, \nu(\lambda))$ .

In this paper, we present a novel approach for analyzing NAVOITVP by establishing new criteria for the existence, uniqueness, and stability of solutions. In contrast to methods that require partitioning the domain or employing piecewise constant approximations, our technique introduces a more streamlined operator-based framework. This allows for a more efficient and theoretically robust analysis. To support the theoretical results, several illustrative examples are included.

## 2 Preliminary

This section introduces key notations, preliminary definitions, and fundamental concepts employed throughout the analysis.

Let  $\mathbb{E} = C(\tilde{\chi}_\tau, \mathbb{R})$  denote the Banach space comprising continuous functions  $\nu$  from  $\tilde{\chi}_\tau$  into  $\mathbb{R}$ , equipped with the supremum norm

$$\|\nu\| = \sup\{|\nu(\lambda)|/\lambda \in \tilde{\chi}_\tau\}.$$

**Definition 2.1.** [20] Consider a continuous mapping  $\ell : \tilde{\chi}_\tau \times \mathbb{R} \rightarrow (1, 2)$  The left-sided Riemann-Liouville fractional integral of variable order  $\ell(\lambda, \nu(\lambda))$  for a function  $\nu(\lambda)$  is formulated as

$$\mathcal{I}_0^{\ell(\lambda, \nu(\lambda))} \nu(\lambda) = \int_0^\lambda \frac{(\lambda - \aleph)^{\ell(\aleph, \nu(\aleph)) - 1}}{\Gamma(\ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph, \lambda > 0, \tag{2.1}$$

**Definition 2.2.** [21] Consider a continuous function  $\ell : \tilde{\chi}_\tau \times \mathbb{R} \rightarrow (1, 2)$  The left-sided Riemann-Liouville fractional derivative of variable order  $\ell(\lambda, \nu(\lambda))$ , applied to a function  $\nu(\lambda)$ , is formulated as

$$\begin{aligned} \mathcal{D}_0^{\ell(\lambda, \nu(\lambda))} \nu(\lambda) &= \left(\frac{d}{dt}\right)^2 \mathcal{I}_0^{2-\ell(\lambda, \nu(\lambda))} \nu(\lambda) \\ &= \left(\frac{d}{dt}\right)^2 \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2-\ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph, \lambda > 0. \end{aligned} \tag{2.2}$$

**Remark 2.3.** [22] For arbitrary functions  $\ell(\lambda, \nu(\lambda))$  and  $v(\lambda, \nu(\lambda))$ , it is observed that the semi-group property does not apply. Specifically, the following inequality holds:

$$\mathcal{I}_{a^+}^{\ell(\lambda, \nu(\lambda))} \mathcal{I}_{a^+}^{v(\lambda, \nu(\lambda))} \nu(\lambda) \neq \mathcal{I}_{a^+}^{\ell(\lambda, \nu(\lambda)) + v(\lambda, \nu(\lambda))} \nu(\lambda).$$

**Lemma 2.4.** [23] Consider a continuous function  $\ell : \tilde{\chi}_\tau \rightarrow (1, 2)$  For any function  $y$  belonging to the space  $C_\delta(\tilde{\chi}_\tau, \mathbb{R})$ , which is defined as

$$\begin{aligned} C_\delta(\tilde{\chi}_\tau, \mathbb{R}) &= \{y(\lambda) \in C(\tilde{\chi}_\tau, \mathbb{R}), \lambda^\delta y(\lambda) \in C(\tilde{\chi}_\tau, \mathbb{R}), (0 \leq \delta < 1)\} \\ &, \text{ the variable-order fractional integral } \mathcal{I}_{0^+}^{\ell(\lambda, \nu(\lambda))} y(\lambda) \text{ exists for all } \lambda \in \tilde{\chi}_\tau. \end{aligned}$$

**Lemma 2.5.** [23] Suppose we have a continuous function  $\ell \in C(\tilde{\chi}_\tau, (1, 2])$  then for any function  $y \in C(\tilde{\chi}_\tau, \mathbb{R})$ , it follows that the variable-order fractional integral  $\mathcal{I}_{0^+}^{\ell(\lambda, \nu(\lambda))} y(\lambda)$  belongs to the space  $C(\tilde{\chi}_\tau, \mathbb{R})$ .

**Theorem 2.6.** ([24])(Ascoli Arzela) Let  $\mathcal{M}$  be a subset of  $C(\tilde{\psi}_\tau, \mathbb{R})$ ,  $\mathcal{M}$  is relatively compact if :

- $\mathcal{M}$  is uniformly bounded.
- $\mathcal{M}$  is equicontinuous.

**Theorem 2.7.** [25]

Let  $\mathfrak{S}$  be a Banach space,  $K \subset \mathfrak{S}$  be a non-empty, closed, convex, and bounded subset, and let  $\wp : K \rightarrow K$  be a completely continuous operator. Then  $\wp$  has at least one fixed point.

**Theorem 2.8.** [25] Let  $(\mathfrak{S}, d)$  a complet metric space. the application  $\wp : \mathfrak{S} \rightarrow \mathfrak{S}$  is contraction with Lipschitz constant  $k$ . so  $\wp$  accept only one fixed point  $y \in \mathfrak{S}$ .

### 3 Existence results

We now present the following hypotheses:

**(Assumption 1)** There are constants  $0 < \eta < 1$  and  $\tilde{\rho} > 0$ , for which the expression  $\lambda^\eta \mu$  maintains continuity across  $\tilde{\chi}_\tau \times \mathbb{R}$  nd for all  $\nu, y \in \mathbb{R}$  and  $\lambda \in \tilde{\chi}_\tau$ , the inequality holds:

$$\lambda^\eta | \mu(\lambda, \nu(\lambda)) - \mu(\lambda, y(\lambda)) | \leq \tilde{\rho} | \nu(\lambda) - y(\lambda) | .$$

**(Assumption 2)** The function  $\ell : \tilde{\chi}_\tau \times \mathbb{R} \rightarrow (1, \ell^*]$  is continuous, with  $1 \leq \ell(\lambda, \nu(\lambda)) \leq \ell^* < 2$  for all  $\lambda \in \tilde{\chi}_\tau$ .

**Remark 3.1.** [26]

- (i) The function  $\Gamma(2-\ell(\lambda, \nu(\lambda)))$  is continuous as it results from the composition of continuous functions. Therefore, we can let  $M_\Gamma = \max | \frac{1}{\Gamma(2-\ell(\lambda, \nu(\lambda)))} |$ .
- (ii) Given the continuity of  $\ell(\lambda, \nu(\lambda))$ , we observe that for  $1 \leq \rho < \infty, \rho^{1-\ell(\lambda, \nu(\lambda))} \leq 1$  and for  $0 \leq \rho \leq 1, \rho^{1-\ell(\lambda, \nu(\lambda))} \leq \rho^{1-\ell^*}$  Consequently, we deduce that  $\rho^{1-\ell(\lambda, \nu(\lambda))} \leq \max(1, \rho^{1-\ell^*}) = \rho^*$ .

**Remark 3.2.** [27] Let  $X$  and  $Y$  be two real numbers. Then, for any positive real numbers  $\aleph$  and  $\beta$ , the following inequality holds:

$$| \aleph X - \beta Y | \leq 2 \max(\aleph, \beta) | X - Y | ,$$

**Lemma 3.3.** [26] Suppose assumption (Assumption 2) holds. Let  $\nu_n, \nu \in C[0, \rho]$ , and assume that  $\nu_n(\lambda) \rightarrow \nu(\lambda), \lambda \in [0, \rho]$  as  $n \rightarrow \infty$ . Then, the following convergence property holds:

$$\int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) d\aleph \rightarrow \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph, \lambda \in [0, \rho],$$

as  $n \rightarrow \infty$ .

To solve the (NAVOITVP) (1.1), we will utilize the following lemma:

**Lemma 3.4.** A function  $\nu \in \mathbb{E}$  is a solution to the NAVOITVP (1.1) if and only if it satisfies the integral equation

$$\begin{aligned} \nu(\lambda) = & \frac{1}{\iota} \left[ - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \\ & + \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\ & + \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\ & \left. - \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right]. \end{aligned}$$

*Proof.* We start by expressing the (NAVOITVP) (1.1) using the definition of the fractional derivative of variable order given by (2.2). This yields:

$$\begin{aligned} & \frac{d^2}{d\lambda^2} \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph + \iota \nu''(\lambda) \\ & = \mu(\lambda, \nu(\lambda)). \end{aligned}$$

Integrating both sides with respect to time gives us:

$$\begin{aligned} & \frac{d}{d\lambda} \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph + \iota \nu'(\lambda) \\ & = \int_0^\lambda \mu(\aleph, \nu(\aleph)) d\aleph + c_1. \end{aligned}$$

Further integration leads to:

$$\begin{aligned} & \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph + \iota \nu(\lambda) \\ & = \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph + c_1 \lambda + c_2. \end{aligned} \quad (3.1)$$

Evaluating the equation 3.1 at  $\lambda = 0$  and  $\lambda = \rho$  allows us to determine that  $c_2 = 0$  and

$$\begin{aligned} c_1 &= \frac{1}{\rho} \left[ \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \\ & \left. - \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right]. \end{aligned}$$

Substituting these values back into the equation yields:

$$\begin{aligned} & \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\ & + \iota \nu(\lambda) = \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\ & + \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\ & - \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph, \end{aligned}$$

Rearranging terms gives us the integral equation:

$$\begin{aligned} \nu(\lambda) &= \frac{1}{\iota} \left[ - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \\ & + \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\ & + \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\ & \left. - \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right]. \end{aligned}$$

Conversely, differentiating both sides of the integral equation (3.4) with respect to time yields:

$$\begin{aligned} & \frac{d}{d\lambda} \left( \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right) + \iota \nu'(\lambda) \\ &= \int_0^\lambda \mu(\aleph, \nu(\aleph)) d\aleph \\ &+ \frac{1}{\rho} \left( \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right) \\ &- \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph. \end{aligned}$$

Differentiating again results in:

$$\begin{aligned} & \frac{d^2}{d\lambda^2} \left( \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right) + \iota \nu''(\lambda) \\ &= \mu(\lambda, \nu(\lambda)), \end{aligned}$$

which corresponds to the original NAVOITVP (1.1). □

*Theorem 2.7 serves as the foundation for our initial result. We present the following theorem:*

**Theorem 3.5.** *Suppose conditions (Assumption 1) and (Assumption 2) are satisfied. If the inequality*

$$\frac{2M_{\Gamma\rho^*\rho}}{2 - \ell^*} + 2\tilde{\rho} \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} < \iota,$$

*holds, then (NAVOITVP) (1.1) admits at least one solution in the space  $\mathbb{E}$ .*

*Proof.* We define an operator

$$C : \mathbb{E} \rightarrow \mathbb{E},$$

as follows:

$$\begin{aligned} C\nu(\lambda) &= \frac{1}{\iota} \left[ - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \\ &+ \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\ &+ \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\ &\left. - \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right]. \end{aligned}$$

Let  $R$  be such that

$$R \geq \frac{\frac{\mu^* \rho^2}{\iota}}{1 - \frac{1}{\iota} \left[ \frac{2M_{\Gamma\rho^*\rho}}{2 - \ell^*} + 2\tilde{\rho} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right]},$$

where

$$\mu^* = \sup_{\lambda \in \tilde{\mathcal{X}}_\tau} |\mu(\lambda, 0)|.$$

Consider the set

$$B_R = \{\nu \in \mathbb{E}, \|\nu\| \leq R\}.$$

Clearly,  $B_R$  is non-empty, bounded, convex, and closed.

We now demonstrate that  $C$  satisfies the conditions outlined in Theorem 2.7. The argument will be presented in several stages.

**Step 1:**  $C(B_R) \subseteq B_R$ . For  $\nu \in B_R$ , we have

$$\begin{aligned}
& |C\nu(\lambda)| \leq \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} |\nu(\aleph)| d\aleph \right. \\
& + \int_0^\lambda (\lambda - \aleph) |\mu(\aleph, \nu(\aleph))| d\aleph \\
& + \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} |\nu(\aleph)| d\aleph \\
& \left. + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) |\mu(\aleph, \nu(\aleph))| d\aleph \right] \\
& \leq \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} |\nu(\aleph)| d\aleph \right. \\
& + \int_0^\lambda (\lambda - \aleph) |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& + \int_0^\lambda (\lambda - \aleph) |\mu(\aleph, 0)| d\aleph \\
& + \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} |\nu(\aleph)| d\aleph \\
& + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& \left. + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) |\mu(\aleph, 0)| d\aleph \right] \\
& \leq \frac{1}{\iota} \left[ M_\Gamma \rho^{1-\ell(\aleph, \nu(\aleph))} \int_0^\lambda \left( \frac{\lambda - \aleph}{\rho} \right)^{1-\ell(\aleph, \nu(\aleph))} |\nu(\aleph)| d\aleph \right. \\
& + \int_0^\lambda (\lambda - \aleph) \aleph^{-\eta} \aleph^\eta |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& + \int_0^\lambda (\lambda - \aleph) |\mu(\aleph, 0)| d\aleph \\
& + M_\Gamma \rho^{1-\ell(\aleph, \nu(\aleph))} \int_0^\rho \left( \frac{\rho - \aleph}{\rho} \right)^{1-\ell(\aleph, \nu(\aleph))} |\nu(\aleph)| d\aleph \\
& + \int_0^\rho \aleph^{-\eta} \aleph^\eta (\rho - \aleph) |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& \left. + \int_0^\rho (\rho - \aleph) |\mu(\aleph, 0)| d\aleph \right] \\
& \leq \frac{1}{\iota} \left[ M_\Gamma \rho^* \int_0^\lambda \left( \frac{\lambda - \aleph}{\rho} \right)^{1-\ell^*} |\nu(\aleph)| d\aleph \right. \\
& + \int_0^\lambda (\lambda - \aleph) \aleph^{-\eta} \aleph^\eta |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& + \int_0^\lambda (\lambda - \aleph) |\mu(\aleph, 0)| d\aleph \\
& + M_\Gamma \rho^* \int_0^\rho \left( \frac{\rho - \aleph}{\rho} \right)^{1-\ell^*} |\nu(\aleph)| d\aleph \\
& + \int_0^\rho \aleph^{-\eta} \aleph^\eta (\rho - \aleph) |\mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0)| d\aleph \\
& \left. + \int_0^\rho (\rho - \aleph) |\mu(\aleph, 0)| d\aleph \right]
\end{aligned}$$



$$\begin{aligned}
& +\tilde{p} \|\nu_n - \nu\| \int_0^\lambda (\lambda - \aleph) \aleph^{-\eta} d\aleph \\
& + \left| \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \\
& +\tilde{p} \|\nu_n - \nu\| \int_0^\rho (\rho - \aleph) \aleph^{-\eta} d\aleph \Big] \\
& \leq \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \\
& +\tilde{p} \|\nu_n - \nu\| \frac{\lambda^{-\eta+2}}{(-\eta+1)(-\eta+2)} \\
& + \left| \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \\
& +\tilde{p} \|\nu_n - \nu\| \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \Big] \\
& \leq \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \\
& +2\tilde{p} \|\nu_n - \nu\| \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \\
& + \left| \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \Big] \\
& \leq \frac{1}{\iota} \left( 2\tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right) \|\nu_n - \nu\| \\
& + \frac{1}{\iota} \left| \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \\
& + \frac{1}{\iota} \left| \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu_n(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu_n(\aleph)))} \nu_n(\aleph) \right. \\
& \left. - \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right|,
\end{aligned}$$

By applying Lemma 3.3, we find that

$$\|C\nu_n(\lambda) - C\nu(\lambda)\| \rightarrow 0, \text{ as } n \rightarrow \infty.$$

This relationship demonstrates that the operator  $C$  is continuous on  $\mathbb{E}$ .

**Step 3:**  $C$  is compact.

In this section, our goal is to demonstrate that  $C(B_R)$  is relatively compact, thereby confirming the compactness of  $C$ . It is evident that  $C(B_R)$  maintains uniform boundedness, as established earlier in Step 1:

$$C(B_R) = \{C(\nu) : \nu \in B_R\} \subset B_R.$$

Consequently, for any  $\nu \in B_R$ , we have  $\|C(\nu)\| \leq R$  indicating the uniform boundedness of  $C(B_R)$ .

Firstly, consider the function  $w(\lambda) = a^\lambda - b^\lambda$ , where  $\lambda \in (-1, 0)$  and  $0 < a < b < 1$ . This function is decreasing because  $\ln a < \ln b < 0$  and  $a^\lambda > b^\lambda > 0$ , leading to

$$\begin{aligned} w'(\lambda) &= a^\lambda \ln a - b^\lambda \ln b < b^\lambda \ln a - b^\lambda \ln b \\ &= b^\lambda (\ln a - \ln b) < 0, \end{aligned}$$

This implies that  $w(\lambda)$  is decreasing. Similarly, for  $\varphi(\aleph) = \left(\frac{\lambda_1 - \aleph}{\rho}\right)^{1 - \ell(\aleph, \nu(\aleph))} - \left(\frac{\lambda_2 - \aleph}{\rho}\right)^{1 - \ell(\aleph, \nu(\aleph))}$  (where  $0 < \frac{\lambda_1 - \aleph}{\rho} < \frac{\lambda_2 - \aleph}{\rho} < 1$ ), we can view  $\varphi(\aleph)$  as analogous to  $w(\aleph)$ , indicating that  $\varphi(\aleph)$  decreases with respect to its exponent  $1 - \ell(\aleph, \nu(\aleph))$ . Therefore, for  $\lambda_1, \lambda_2 \in \tilde{\chi}_\tau$  with  $\lambda_1 < \lambda_2$  and  $\nu \in B_R$ , we have

$$\begin{aligned} |C\nu(\lambda_2) - C\nu(\lambda_1)| &\leq \frac{1}{\iota} \left[ \left| \int_0^{\lambda_2} \frac{(\lambda_2 - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \right. \\ &\quad \left. \left. - \int_0^{\lambda_1} \frac{(\lambda_1 - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \right. \\ &\quad \left. + \left| \int_0^{\lambda_2} (\lambda_2 - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph - \int_0^{\lambda_1} (\lambda_1 - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right| \right. \\ &\quad \left. + \left| \frac{\lambda_2}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \right. \\ &\quad \left. \left. - \frac{\lambda_1}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \right. \\ &\quad \left. + \left| \frac{\lambda_2}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph - \frac{\lambda_1}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right| \right] \\ &\leq \frac{1}{\iota} \left[ \left| \int_0^{\lambda_1} \frac{(\lambda_2 - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) \right. \right. \\ &\quad \left. \left. - \frac{(\lambda_1 - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \right. \\ &\quad \left. + \left| \int_{\lambda_1}^{\lambda_2} \frac{(\lambda_2 - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right| \right. \\ &\quad \left. + \left| \int_0^{\lambda_1} (\lambda_2 - \aleph) \mu(\aleph, \nu(\aleph)) - (\lambda_1 - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right| \right. \\ &\quad \left. + \left| \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right| \right. \\ &\quad \left. + \left| \frac{1}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1 - \ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph [\lambda_2 - \lambda_1] \right| \right. \\ &\quad \left. + \left| \frac{1}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph [\lambda_2 - \lambda_1] \right| \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{l} \left[ \int_0^{\lambda_1} \left| \frac{1}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \right. \right. \\
&\times \left. \left\| (\lambda_2 - \aleph)^{1-\ell(\aleph, \nu(\aleph))} - (\lambda_1 - \aleph)^{1-\ell(\aleph, \nu(\aleph))} \right\| \nu(\aleph) \right] d\aleph \\
&+ \int_{\lambda_1}^{\lambda_2} \frac{(\lambda_2 - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \left| \nu(\aleph) \right| d\aleph \\
&+ \int_0^{\lambda_1} \left[ (\lambda_2 - \aleph) - (\lambda_1 - \aleph) \right] \left| \mu(\aleph, \nu(\aleph)) \right| d\aleph \\
&+ \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \left| \mu(\aleph, \nu(\aleph)) \right| d\aleph \\
&+ \frac{\lambda_2 - \lambda_1}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \left| \nu(\aleph) \right| d\aleph \\
&+ \frac{\lambda_2 - \lambda_1}{\rho} \int_0^\rho (\rho - \aleph) \left| \mu(\aleph, \nu(\aleph)) \right| d\aleph \Big] \\
&\leq \frac{1}{l} \left[ M_\Gamma \left\| \nu \right\| \int_0^{\lambda_1} \left[ (\lambda_1 - \aleph)^{1-\ell(\aleph, \nu(\aleph))} \right. \right. \\
&- \left. \left. (\lambda_2 - \aleph)^{1-\ell(\aleph, \nu(\aleph))} \right] d\aleph \right. \\
&+ M_\Gamma \left\| \nu \right\| \rho^* \int_{\lambda_1}^{\lambda_2} \left( \frac{\lambda_2 - \aleph}{\rho} \right)^{1-\ell^*} d\aleph \\
&+ \int_0^{\lambda_1} \left[ (\lambda_2 - \aleph) - (\lambda_1 - \aleph) \right] \left| \mu(\aleph, \nu(\aleph)) \right. \\
&- \left. \mu(\aleph, 0) + \mu(\aleph, 0) \right| d\aleph \\
&+ \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \left| \mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0) \right. \\
&+ \left. \mu(\aleph, 0) \right| d\aleph \\
&+ \frac{M_\Gamma}{\rho} (\lambda_2 - \lambda_1) \left\| \nu \right\| \rho^* \int_0^\rho \left( \frac{\rho - \aleph}{\rho} \right)^{1-\ell^*} d\aleph \\
&+ \frac{\lambda_2 - \lambda_1}{\rho} \int_0^\rho (\rho - \aleph) \left| \mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0) \right. \\
&+ \left. \mu(\aleph, 0) \right| d\aleph \Big] \\
&\leq \frac{1}{l} \left[ M_\Gamma \left\| \nu \right\| \int_0^{\lambda_1} \rho^{1-\ell(\aleph, \nu(\aleph))} \left[ \left( \frac{\lambda_1 - \aleph}{\rho} \right)^{1-\ell(\aleph, \nu(\aleph))} \right. \right. \\
&- \left. \left. \left( \frac{\lambda_2 - \aleph}{\rho} \right)^{1-\ell(\aleph, \nu(\aleph))} \right] d\aleph \right. \\
&+ \int_0^{\lambda_1} \left[ (\lambda_2 - \aleph) - (\lambda_1 - \aleph) \right] \aleph^{-\eta} \aleph^\eta \\
&\quad \left| \mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0) \right| d\aleph \\
&+ \int_0^{\lambda_1} \left[ (\lambda_2 - \aleph) - (\lambda_1 - \aleph) \right] \left| \mu(\aleph, 0) \right| d\aleph \\
&+ \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \aleph^{-\eta} \aleph^\eta \left| \mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0) \right| d\aleph \\
&+ \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \left| \mu(\aleph, 0) \right| d\aleph + \frac{M_\Gamma \rho^*}{2 - \ell^*} \left\| \nu \right\| (\lambda_2 - \lambda_1) \\
&+ \frac{\lambda_2 - \lambda_1}{\rho} \int_0^\rho (\rho - \aleph) \aleph^{-\eta} \aleph^\eta
\end{aligned}$$

$$\begin{aligned}
 & \times | \mu(\aleph, \nu(\aleph)) - \mu(\aleph, 0) | d\aleph \\
 & + \frac{\lambda_2 - \lambda_1}{\rho} \int_0^\rho (\rho - \aleph) | \mu(\aleph, 0) | d\aleph \Big] \\
 & \leq \frac{1}{\iota} \left[ M_\Gamma \| \nu \| \rho^* \int_0^{\lambda_1} \left[ \left( \frac{\lambda_1 - \aleph}{\rho} \right)^{1-\ell^*} \right. \right. \\
 & \quad \left. \left. - \left( \frac{\lambda_2 - \aleph}{\rho} \right)^{1-\ell^*} \right] d\aleph \right. \\
 & + \tilde{p} \| \nu \| \int_0^{\lambda_1} [(\lambda_2 - \aleph) - (\lambda_1 - \aleph)] \aleph^{-\eta} d\aleph \\
 & + \mu^* [\lambda_1 \lambda_2 - \lambda_1^2] + \tilde{p} \| \nu \| \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \aleph) \aleph^{-\eta} d\aleph \\
 & + \mu^* \left[ \frac{\lambda_2^2}{2} + \frac{\lambda_1^2}{2} - \lambda_1 \lambda_2 \right] + \left[ \frac{M_\Gamma \rho^*}{2 - \ell^*} \| \nu \| (\lambda_2 - \lambda_1) \right] \\
 & + \frac{\lambda_2 - \lambda_1}{\rho} \tilde{p} \| \nu \| \int_0^\rho (\rho - \aleph) \aleph^{-\eta} d\aleph + \mu^* \frac{\lambda_2 - \lambda_1}{\rho} \frac{\rho^2}{2} \Big] \\
 & \leq \frac{1}{\iota} \left[ \left[ \frac{M_\Gamma \| \nu \| \rho^*}{\rho^{1-\ell^*} (2 - \ell^*)} \right] [(\lambda_1)^{2-\ell^*} - (\lambda_2)^{2-\ell^*} \right. \\
 & \quad \left. + 2(\lambda_2 - \lambda_1)^{2-\ell^*} \right] \\
 & + \tilde{p} \| \nu \| \left[ (\lambda_2 - \lambda_1) \frac{\lambda_1^{-\eta+1}}{-\eta+1} \right] + \mu^* \left( \frac{-\lambda_1^2}{2} + \frac{\lambda_2^2}{2} \right) \\
 & + \tilde{p} \| \nu \| \left[ \left( -(\lambda_2 - \lambda_1) \frac{\lambda_1^{-\eta+1}}{-\eta+1} \right) \right. \\
 & \quad \left. + \left( \frac{\lambda_2^{-\eta+2}}{(-\eta+1)(-\eta+2)} - \frac{\lambda_1^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right) \right] \\
 & + \left[ \frac{M_\Gamma \rho^*}{2 - \ell^*} \| \nu \| (\lambda_2 - \lambda_1) \right] \\
 & + \frac{\lambda_2 - \lambda_1}{\rho} \tilde{p} \| \nu \| \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} + \mu^* (\lambda_2 - \lambda_1) \frac{\rho}{2} \Big] \\
 & \leq \frac{1}{\iota} \left[ \left[ \frac{M_\Gamma \| \nu \| \rho^*}{\rho^{1-\ell^*} (2 - \ell^*)} \right] [(\lambda_1)^{2-\ell^*} - (\lambda_2)^{2-\ell^*} + 2(\lambda_2 - \lambda_1)^{2-\ell^*} \right] \\
 & + \mu^* \left( \frac{\lambda_2^2 - \lambda_1^2}{2} \right) + \tilde{p} \| \nu \| \left( \frac{\lambda_2^{-\eta+2} - \lambda_1^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right) \\
 & + \left[ \frac{M_\Gamma \rho^*}{2 - \ell^*} \| \nu \| + \frac{\tilde{p} \| \nu \|}{\rho} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} + \mu^* \frac{\rho}{2} \right] \\
 & \times (\lambda_2 - \lambda_1) \Big]
 \end{aligned}$$

Consequently, as  $\lambda_2 \rightarrow \lambda_1$ , we have

$| C\nu(\lambda_2) - C\nu(\lambda_1) | \rightarrow 0$  indicating that  $C(B_R)$  is equicontinuous.

Combining the results from the previous steps and applying the Arzela-Ascoli theorem, we conclude that  $C$  is a completely continuous operator. Therefore, based on Theorem 2.7, we deduce that  $C$  possesses a fixed point  $\nu$  within  $B_R$ , which serves as a solution to the (NAVOITVP) (1.1). □

#### 4 Uniqueness results

The following result relies on the Banach contraction theorem.

**Theorem 4.1.** *Suppose conditions (Assumption 1) and (Assumption 2) hold. If the inequality*

$$\frac{1}{\iota} \left( 8M_{\Gamma} \rho^* \rho + 2\tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right) < 1, \quad (4.1)$$

is satisfied, then (NAVOITVP) (1.1) has a unique solution in the space  $\mathbb{E}$ .

*Proof.* For any functions  $\nu$  and  $\nu^* \in \mathbb{E}$ , we can express

$$\begin{aligned} & | C\nu(\lambda) - C\nu^*(\lambda) | \\ & \leq \frac{1}{\iota} \left[ \left| \int_0^{\lambda} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) \right. \right. \\ & \quad \left. \left. - \int_0^{\lambda} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu^*(\aleph) d\aleph \right| \right. \\ & \quad \left. + \int_0^{\lambda} (\lambda - \aleph) \aleph^{-\eta} \aleph^{\eta} \right. \\ & \quad \left. \times \text{mid} \mu(\aleph, \nu(\aleph)) - \mu(\aleph, \nu^*(\aleph)) \right| d\aleph \\ & \quad \left. + \left| \int_0^{\rho} \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) \right. \right. \\ & \quad \left. \left. - \int_0^{\rho} \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu^*(\aleph) d\aleph \right| \right. \\ & \quad \left. + \int_0^{\rho} (\rho - \aleph) \aleph^{-\eta} \aleph^{\eta} \right| \mu(\aleph, \nu^*(\aleph)) \\ & \quad \left. - \mu(\aleph, \nu(\aleph)) \right| d\aleph \Big] \\ & \leq \frac{1}{\iota} \left[ 2 \int_0^{\lambda} \left( \sup_{\lambda \in \tilde{\mathcal{X}}_{\tau}} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \right. \right. \\ & \quad \left. \left. + \sup_{\lambda \in \tilde{\mathcal{X}}_{\tau}} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu^*(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu^*(\aleph)))} \right) \right| \nu - \nu^* \Big| d\aleph \\ & \quad + \tilde{p} \|\nu - \nu^*\| \int_0^{\lambda} (\lambda - \aleph) \aleph^{-\eta} d\aleph \\ & \quad + \tilde{p} \|\nu - \nu^*\| \int_0^{\rho} (\rho - \aleph) \aleph^{-\eta} d\aleph \\ & \quad + 2 \int_0^{\rho} \left( \sup \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \right. \\ & \quad \left. + \sup \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu^*(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu^*(\aleph)))} \right) \right| \nu - \nu^* \Big| d\aleph \Big] \\ & \leq \frac{1}{\iota} \left[ 2 \|\nu - \nu^*\| \int_0^{\lambda} (M_{\Gamma} \rho^* + M_{\Gamma} \rho^*) d\aleph \right. \\ & \quad \left. + \tilde{p} \|\nu - \nu^*\| \frac{\lambda^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right. \\ & \quad \left. + 2 \|\nu - \nu^*\| \int_0^{\rho} (M_{\Gamma} \rho^* + M_{\Gamma} \rho^*) d\aleph \right. \\ & \quad \left. + \tilde{p} \|\nu - \nu^*\| \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right] \end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\iota} \left[ 4M_{\Gamma}\rho^* \|\nu - \nu^*\| \int_0^{\lambda} d\aleph \right. \\
 &\quad \left. + 2\tilde{p} \|\nu - \nu^*\| \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} \right. \\
 &\quad \left. + 4M_{\Gamma}\rho^* \|\nu - \nu^*\| \int_0^{\rho} d\aleph \right] \\
 &\leq \frac{1}{\iota} \left[ 4M_{\Gamma}\rho^* \lambda \|\nu - \nu^*\| \right. \\
 &\quad \left. + 2\tilde{p} \|\nu - \nu^*\| \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} + 4M_{\Gamma}\rho^* \rho \|\nu - \nu^*\| \right] \\
 &\leq \frac{1}{\iota} \left[ 4M_{\Gamma}\rho^* \rho \|\nu - \nu^*\| + 2\tilde{p} \|\nu - \nu^*\| \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} \right. \\
 &\quad \left. + 4M_{\Gamma}\rho^* \rho \|\nu - \nu^*\| \right] \\
 &\leq \frac{1}{\iota} \left[ 8M_{\Gamma}\rho^* \rho \|\nu - \nu^*\| + 2\tilde{p} \|\nu - \nu^*\| \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} \right] \\
 &\leq \frac{1}{\iota} \left( 8M_{\Gamma}\rho^* \rho + 2\tilde{p} \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} \right) \|\nu - \nu^*\|.
 \end{aligned}$$

As a result of Equation (4.1), the operator  $C$  becomes a contraction mapping. In accordance with the Banach contraction theorem,  $C$  possesses a unique fixed point, which corresponds to the unique solution of the (NAVOITVP) (1.1).  $\square$

### 5 Ulam-Hyers stability

**Definition 5.1.** [28] In the context of the (NAVOITVP) (1.1), consider the inequality

$$|D_{0^+}^{\ell(\lambda, \vartheta(\lambda))} \vartheta(\lambda) + \aleph \vartheta''(\lambda) - \mu(\lambda, \vartheta(\lambda))| \leq \epsilon, \quad \lambda \in \tilde{\chi}_{\tau}. \tag{5.1}$$

We say that the NAVOITVP (1.1) is Ulam-Hyers stable if there exists a constant  $c_{\mu} > 0$  such that for any  $\epsilon > 0$  and for any solution  $\vartheta \in C(\tilde{\chi}_{\tau}, \mathbb{R})$  of (5.1), there is a solution  $\nu \in C(\tilde{\chi}_{\tau}, \mathbb{R})$  of the NAVOITVP (1.1) satisfying

$$|\vartheta(\lambda) - \nu(\lambda)| \leq c_{\mu} \epsilon, \quad \lambda \in \tilde{\chi}_{\tau}.$$

**Theorem 5.2.** Assume that conditions (Assumption 1), (Assumption 2), and the inequality referenced as (4.1) are satisfied. In this case, the NAVOITVP (see Equation 1.1) exhibits Ulam-Hyers stability.

*Proof.* . Take an arbitrary positive value  $\epsilon > 0$  and let  $\vartheta(\lambda)$  be a function in  $C(\tilde{\chi}_{\tau}, \mathbb{R})$  that fulfills the subsequent inequality:

$$|D_{0^+}^{\ell(\lambda, \vartheta(\lambda))} \vartheta(\lambda) + \aleph \vartheta''(\lambda) - \mu(\lambda, \vartheta(\lambda))| \leq \epsilon, \quad \lambda \in \tilde{\chi}_{\tau}. \tag{5.2}$$

By performing a double integration on each side of this inequality (5.2), we derive the following result.

$$\begin{aligned}
 &\left| \vartheta(\lambda) + \frac{1}{\iota} \left[ \int_0^{\lambda} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) \, d\aleph \right. \right. \\
 &\quad - \int_0^{\lambda} (\lambda - \aleph) \mu(\aleph, \vartheta(\aleph)) \, d\aleph \\
 &\quad - \frac{\lambda}{\rho} \int_0^{\rho} \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) \, d\aleph \\
 &\quad \left. \left. + \frac{\lambda}{\rho} \int_0^{\rho} (\rho - \aleph) \mu(\aleph, \vartheta(\aleph)) \, d\aleph \right] \right| \leq \frac{\epsilon \rho^2}{2}
 \end{aligned}$$

Let  $\lambda \in \tilde{\chi}_\tau$ , then

$$\begin{aligned}
& | \vartheta(\lambda) - \nu(\lambda) | \\
&= \left| \vartheta(\lambda) + \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \right. \\
&\quad - \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\
&\quad - \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\
&\quad \left. \left. + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right] \right| \\
&= \left| \vartheta(\lambda) + \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \right. \\
&\quad - \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \\
&\quad - \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \\
&\quad \left. \left. + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \nu(\aleph)) d\aleph \right] \right| \\
&\quad + \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \right. \\
&\quad - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \\
&\quad \left. \left. + \frac{1}{\iota} \left[ \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \vartheta(\aleph)) d\aleph \right. \right. \right. \\
&\quad - \int_0^\lambda (\lambda - \aleph) \mu(\aleph, r(\aleph)) d\aleph \\
&\quad \left. \left. + \frac{1}{\iota} \left[ \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \right. \right. \right. \\
&\quad - \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \\
&\quad \left. \left. + \frac{1}{\iota} \left[ \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \vartheta(\aleph)) d\aleph \right. \right. \right. \\
&\quad - \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \vartheta(\aleph)) d\aleph \\
&\quad \left. \left. \left. \right] \right] \right] \\
&\leq \left| \vartheta(\lambda) + \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \right. \right. \\
&\quad - \int_0^\lambda (\lambda - \aleph) \mu(\aleph, \vartheta(\aleph)) d\aleph \\
&\quad - \frac{\lambda}{\rho} \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \\
&\quad \left. \left. + \frac{\lambda}{\rho} \int_0^\rho (\rho - \aleph) \mu(\aleph, \vartheta(\aleph)) d\aleph \right] \right| \\
&\quad + \frac{1}{\iota} \left[ \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) d\aleph \right. \\
&\quad - \int_0^\lambda \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \\
&\quad \left. \left. \right] \right|
\end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\iota} \left[ \int_0^\lambda (\lambda - \aleph) | \mu(\aleph, \vartheta(\aleph)) - \mu(\aleph, \nu(\aleph)) | \, d\aleph \right] \\
 & + \frac{1}{\iota} \frac{\lambda}{\rho} \left[ \left| \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \vartheta(\aleph) d\aleph \right. \right. \\
 & \left. \left. - \int_0^\rho \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \nu(\aleph) \, d\aleph \right| \right] \\
 & + \frac{1}{\iota} \frac{\lambda}{\rho} \left[ \int_0^\rho (\rho - \aleph) | \mu(\aleph, \nu(\aleph)) - \mu(\aleph, \vartheta(\aleph)) | \, d\aleph \right] \\
 & \leq \frac{\epsilon \rho^2}{2} + \frac{1}{\iota} 2 \int_0^\lambda \left( \sup_{\lambda \in \tilde{\mathcal{X}}_\tau} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \right. \\
 & \left. + \sup_{\lambda \in \tilde{\mathcal{X}}_\tau} \frac{(\lambda - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \right) | \nu - \vartheta | \, d\aleph \\
 & + \frac{1}{\iota} \int_0^\lambda (\lambda - \aleph) \aleph^{-\eta} \aleph^\eta | \mu(\aleph, \vartheta(\aleph)) - \mu(\aleph, \nu(\aleph)) | \, d\aleph \\
 & + \frac{1}{\iota} 2 \int_0^\rho \left( \sup \frac{(\rho - \aleph)^{1-\ell(\aleph, \vartheta(\aleph))}}{\Gamma(2 - \ell(\aleph, \vartheta(\aleph)))} \right. \\
 & \left. + \sup \frac{(\rho - \aleph)^{1-\ell(\aleph, \nu(\aleph))}}{\Gamma(2 - \ell(\aleph, \nu(\aleph)))} \right) | \vartheta - \nu | \, d\aleph \\
 & + \frac{1}{\iota} \int_0^\rho (\rho - \aleph) \aleph^{-\eta} \aleph^\eta | \mu(\aleph, \nu(\aleph)) - \mu(\aleph, \vartheta(\aleph)) | \, d\aleph \\
 & \leq \frac{\epsilon \rho^2}{2} + \frac{1}{\iota} \left[ 2 \| \vartheta - \nu \| \int_0^\lambda (M_\Gamma \rho^* + M_\Gamma \rho^*) d\aleph \right. \\
 & \left. + \tilde{p} \| \vartheta - \nu \| \frac{\lambda^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right. \\
 & \left. + 2 \| \vartheta - \nu \| \int_0^\rho (M_\Gamma \rho^* + M_\Gamma \rho^*) d\aleph \right. \\
 & \left. + \tilde{p} \| \vartheta - \nu \| \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right] \\
 & \leq \frac{\epsilon \rho^2}{2} + \frac{1}{\iota} \left[ 4 M_\Gamma \rho^* \| \vartheta - \nu \| \right. \\
 & \times \int_0^\lambda d\aleph + 2 \tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \| \vartheta - \nu \| \\
 & \left. + 4 M_\Gamma \rho^* \| \vartheta - \nu \| \int_0^\rho d\aleph \right] \\
 & \leq \frac{\epsilon \rho^2}{2} + \frac{1}{\iota} \left[ 8 M_\Gamma \rho^* \rho + 2 \tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right] \\
 & \times \| \vartheta - \nu \| .
 \end{aligned}$$

Consequently, we have

$$\begin{aligned}
 & \| \vartheta - \nu \| \left( 1 - \frac{1}{\iota} \left[ 8 M_\Gamma \rho^* \rho \right. \right. \\
 & \left. \left. + 2 \tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right] \right) \leq \frac{\epsilon \rho^2}{2} .
 \end{aligned}$$

For each  $\lambda \in \tilde{\chi}_\tau$ , it follows that

$$\begin{aligned} & | \vartheta(\lambda) - \nu(\lambda) | \leq \| \vartheta - \nu \| \\ & \leq \frac{\frac{\rho^2}{2}}{1 - \frac{1}{\iota} \left[ 8M_\Gamma \rho^* \rho + 2\tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right]} \epsilon = c_\mu \epsilon. \end{aligned}$$

Thus, the (NAVOITVP) (1.1) is Ulam-Hyers stable. □

### 6 Numerical examples

**Example 6.1.** Consider the following (NAVOITVP):

$$\begin{cases} D^{\frac{3}{2}+1} \nu(\lambda) + 3\nu''(\lambda) = \frac{\lambda^2 e^{-\lambda+4}}{\lambda+8} + \frac{\nu}{3+\lambda}, \lambda \in [0, \frac{1}{4}], \\ \nu(0) = 0, \nu(\rho) = 0, \end{cases} \tag{6.1}$$

We consider the case where  $[0, \rho] = [0, \frac{1}{4}]$  and  $\ell(\lambda, \nu(\lambda)) = \frac{\lambda}{2} + 1$  a continuous function with  $1 < \ell(\lambda, \nu(\lambda)) < \frac{1}{2} + 1 = \frac{3}{2} = \ell^* < 2$ , Additionally, we have  $\iota = 3$  and  $0 \leq \eta < 1$ , which leads to  $\eta = 0$

$$\begin{aligned} \lambda^\eta | \mu(\lambda, \nu) - \mu(\lambda, y) | &= \lambda^\eta \left| \frac{\exp(-\lambda + 4)}{\lambda + 8} (\lambda^2) \right. \\ & \quad \left. + \frac{1}{3 + \lambda} \nu - \frac{\exp(-\lambda + 4)}{\lambda + 8} (\lambda^2) \right. \\ & \quad \left. - \frac{1}{3 + \lambda} y \right| \\ &= \lambda^\eta \left| \frac{1}{3 + \lambda} \nu - \frac{1}{3 + \lambda} y \right| \\ &\leq \lambda^\eta \frac{1}{3 + \lambda} | \nu - y | \\ &\leq \frac{1}{3} | \nu - y |. \end{aligned}$$

Conditions (Assumption 1) and (Assumption 2) are satisfied with  $\tilde{p} = \frac{1}{3}$ . Furthermore, we evaluate the expression

$$\begin{aligned} & \frac{1}{\iota} \left( 8M_\Gamma \rho^* \rho + 2\tilde{p} \frac{\rho^{-\eta+2}}{(-\eta+1)(-\eta+2)} \right) \\ &= \frac{1}{3} \left[ 8 \frac{1}{4} + 2 \frac{1}{3} \left( \frac{1}{4} \right)^2 \right] = \frac{1}{3} \left[ 2 + \frac{2}{3} \frac{1}{32} \right] = \frac{1}{3} \left[ 2 + \frac{2}{64} \right] \\ &= \frac{130}{192} = 0.67 < 1. \end{aligned}$$

In light of Theorem (4.1), we conclude that the (NAVOITVP) (6.1) has a unique solution. Additionally, by Theorem (5.2), the NAVOITVP (6.1) is Ulam-Hyers stable.

**Example 6.2.** Consider the following (NAVOITVP):

$$\begin{cases} D^{\frac{3}{2}+1} \nu(\lambda) + 4\nu''(\lambda) = \frac{8(\lambda+1)}{3\sqrt{2\pi}} + e^{\sqrt{\lambda+1}} + \frac{\pi}{3} \nu, \lambda \in [0, \frac{1}{4}], \\ \nu(0) = 0, \nu(\rho) = 0, \end{cases} \tag{6.2}$$

Here,  $[0, \rho] = [0, \frac{1}{4}]$  and  $\ell(\lambda, \nu(\lambda)) = \frac{\lambda}{5} + 1$  is a continuous function with  $1 < \ell(\lambda, \nu(\lambda)) <$

$\frac{1}{5} + 1 = \frac{6}{5} = \ell^* < 2$ ,  $\iota = 4$  and  $0 \leq \eta < 1$ , which implies  $\eta = 0$

$$\begin{aligned} \lambda^\eta | \mu(\lambda, \nu) - \mu(\lambda, y) | &= \lambda^\eta \left| \frac{8(\lambda + 1)}{3\sqrt{2}\pi} + (\exp(\sqrt{\lambda + 1})) + \frac{\pi}{3}\nu \right. \\ &\quad \left. - \frac{8(\lambda + 1)}{3\sqrt{2}\pi} - (\exp(\sqrt{\lambda + 1})) - \frac{\pi}{3}y \right| \\ &= \lambda^\eta \left| \frac{\pi}{3}\nu - \frac{\pi}{3}y \right| \\ &\leq \lambda^\eta \frac{\pi}{3} | \nu - y | \\ &\leq \frac{\pi}{3} | \nu - y | . \end{aligned}$$

Thus, conditions (Assumption 1) and (Assumption 2) are satisfied with  $\tilde{\rho} = \frac{\pi}{3}$ . Furthermore, we calculate

$$\begin{aligned} &\frac{1}{\iota} \left( 8M_\Gamma \rho^* \rho + 2\tilde{\rho} \frac{\rho^{-\eta+2}}{(-\eta + 1)(-\eta + 2)} \right) \\ &= \frac{1}{4} \left[ 8 \frac{1}{4} + 2 \frac{\pi}{3} \left( \frac{1}{4} \right)^2 \right] = \frac{1}{4} \left[ 2 + \frac{2\pi}{3} \frac{1}{32} \right] \\ &= \frac{1}{4} \left[ 2 + \frac{2\pi}{64} \right] = \frac{128 + 2\pi}{256} \\ &= 0.52 < 1. \end{aligned}$$

In light of Theorem (4.1), we conclude that the NAVOITVP (6.2) has a unique solution. Additionally, by Theorem (5.2), the NAVOITVP (6.2) is Ulam-Hyers stable.

### 7 Conclusion

*In this research, we aim to derive findings concerning the existence and distinctiveness of solutions to non-autonomous differential equations of variable order, which incorporate both standard and fractional mixed derivatives. This is explored within the framework of the initial and terminal value problem, referred to as NAVOITVP (see Equation 1.1), where the condition  $1 < \ell(\lambda, \nu(\lambda)) < 2$  holds for  $\ell(\lambda, \nu(\lambda))$ . Our approach relies on the Ulam-Hyers stability principle (as outlined in Theorem 5.2) alongside a pair of fixed-point results (presented in Theorems 4.1 and 3.5). Furthermore, we include a variety of numerical illustrations to demonstrate the practical implications of these theoretical outcomes. These efforts contribute to deepening the comprehension of this developing area of study.*

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