

Fractional Pennes bioheat model using Legendre collocation technique

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Abstract *This paper presents numerical results to analyse the effects of heating during thermal treatment under both constant and transient heat flux conditions. The collocation technique is implemented using shifted Legendre polynomials in both spatial and temporal direction as basis functions, and the thermal responses are analyzed with respect to various parameters. The results are validated by comparing them to the analytic solution, which validates the accuracy of the applied technique.*

1 Introduction

Therapies that utilize heat transfer to treat various cancers necessitate a fundamental understanding of how heat is transferred through tissue. We need to determine the precise amount of heat necessary to effectively treat the affected region without damaging healthy tissue. To improve heat-based therapies, researchers have developed various mathematical models to better understand how heat transfers through biological tissue [1, 2, 3, 4].

Based on experimental evidence, the Pennes model [1], is particularly suitable for accurately predicting the actual amount of heat required between blood and tissue, and it remains one of the most widely employed models in thermal therapies. This model is primarily based on four variables: metabolic heat generation, blood perfusion, convection, and thermal conduction.

Fouriers law of heat conduction is basis of this model, which assumes that thermal propagation in living tissue occurs uniformly throughout the tissue and that the speed of heat propagation is infinite. The Pennes equation, initially applied to calculate heat propagation of propagation in the human forearm, is now a widely used model for simulating heat transfer in all living tissue [5].

In recent years, fractional calculus has witnessed remarkable growth, finding applications across diverse scientific and technological fields such as viscosity, bioengineering, medicine, anomalous diffusion, and finance. Fractional-order partial differential equations play a pivotal role in mathematical modeling, simulation, and analysis. Their inherent non-local characteristics make them particularly valuable, as they account for the entire history of a system rather than relying solely on its present state—an advantage that distinguishes them from classical differential equations[6].

Several researchers have investigated the fractional-order Pennes bioheat equation through a variety of analytical and numerical approaches [7, 8, 9, 10, 11, 12]. The Pennes model under constant and transient heat flux conditions has been analyzed using analytical techniques to investigate heat propagation within skin tissue [13]. Liu [14] applied the separation of variables method to study heat variations arising from thermal injury in biological tissue under instantaneous heating conditions. Shih et al. [7] employed an analytical technique to evaluate the

heating responses on semi-infinite living tissue subjected to transient condition. Ahmadikia et al. [5] investigated the effects of heating on skin tissue in both semi-infinite and finite domains by applying the Laplace transform to parabolic and hyperbolic forms of the Pennes bioheat model. Furthermore, Ezzat et al. [10] considered a fractional-order formulation of the Pennes bioheat model to analyze the thermal effects on the skin surface using the Laplace transform.

It is important to note that most fractional-order differential equations do not admit analytical solutions, thereby necessitating the use of computational schemes. Singh [9] investigated the heating impacts on tissues during thermal treatment by applying a fractional backward finite difference method. To analyze the thermal responses of skin tissue under transient and constant heat flux conditions, Damor et al. [11] employed an implicit finite difference scheme. Roohi et al. [12] utilized fractional-order Legendre functions within a Galerkin framework to examine the thermal impacts on tissue. More recently, Verma and Kumar [15] studied the thermal effects due to constant and transient heat flux on skin tissue with both Fourier and non-Fourier models.

In this study, the fractional-order Pennes bioheat model under constant and transient heat flux conditions at the skin surface is considered and solved numerically. Temperature variations are computed with respect to several parameters, including the order of derivatives, tissue length, and heating duration, to assess their effects on skin tissue. The proposed approach employs shifted Legendre polynomials as basis functions in both spatial and temporal coordinates within a collocation framework. To the best of our knowledge, there is limited literature addressing the simultaneous approximation of space and time coordinates using a unified basis and methodology.

1.1 Heat transfer model

We consider, the Pennes bioheat transfer model [1],

$$\rho c \frac{\partial \mathbb{T}}{\partial t} = \frac{\partial^2 \mathbb{T}}{\partial x^2} + W_b c_b (\mathbb{T}_a - \mathbb{T}) + q_{met}, \quad (1.1)$$

where specific heat c , density ρ , temperature \mathbb{T} at time t and distance (x), thermal conductivity (k), artery temperature \mathbb{T}_a , metabolic heat generation q_{met} and blood perfusion rate $W_b = \rho_b w_b$ in skin tissue are respectively.

On writing equation (1.1) in following form in fractional order α ,

$$\rho c \frac{\partial^\alpha \mathbb{T}}{\partial t^\alpha} = \frac{\partial^2 \mathbb{T}}{\partial x^2} + W_b c_b (\mathbb{T}_a - \mathbb{T}) + q_{met}. \quad (1.2)$$

We investigate equation (1.2) under the conditions of sinusoidal and constant heat flux.

1.2 Constant heat flux condition

Initial condition

$$\mathbb{T}(x, 0) = \mathbb{T}_0. \quad (1.3)$$

Boundary conditions

$$-k \frac{\partial \mathbb{T}(L, t)}{\partial x} = 0, \quad (1.4)$$

$$-k \frac{\partial \mathbb{T}(0, t)}{\partial x} = q_0, \quad (1.5)$$

where q_0 is heat flux of the skin's surface.

On using dimensionless conversion [11], we have,

$$\mu = \sqrt{\frac{W_b c_b}{k}} x, \quad \nu = \left(\frac{W_b c_b}{\rho c} \right)^{\frac{1}{\alpha}} t, \quad \mathbb{U} = \left(\frac{\mathbb{T} - \mathbb{T}_0}{q_0} \right) \sqrt{k W_b c_b}, \quad \psi = \frac{q_{met}}{q_0 \sqrt{\frac{W_b c_b}{k}}} \quad (1.6)$$

On using equation (1.6), from equations(1.2) to (1.5), we get,

$$\frac{\partial^\alpha \mathbb{U}}{\partial \nu^\alpha} = \frac{\partial^2 \mathbb{U}}{\partial \mu^2} - \mathbb{U} + \psi \quad (1.7)$$

$$\mathbb{U}(\mu, 0) = 0 \quad (1.8)$$

$$\frac{\partial \mathbb{U} \left(\sqrt{\frac{W_b c_b}{k}} x, \nu \right)}{\partial \mu} = 0 \quad (1.9)$$

$$\frac{\partial \mathbb{U}(0, \nu)}{\partial \mu} = -1 \quad (1.10)$$

1.3 Sinusoidal heat flux condition

On considering cosine heating condition on the skin's surface, we have following form of initial and boundary conditions [11],

$$\mathbb{T}(x, 0) = \mathbb{T}_0. \quad (1.11)$$

Boundary conditions

$$-k \frac{\partial \mathbb{T}(L, t)}{\partial x} = 0, \quad (1.12)$$

$$-k \frac{\partial \mathbb{T}(0, t)}{\partial x} = q_0 \cos(\omega t), \quad (1.13)$$

ω represents the frequency of heating. The dimensionless variables were considered by Damor et al. [11]

$$\mu = \sqrt{\frac{\omega \rho c}{k}} x, \quad \nu = (\omega)^{\frac{1}{\alpha}} t, \quad \mathbb{U} = \left(\frac{\mathbb{T} - \mathbb{T}_a}{q_0} \right) \sqrt{k W_b c_b}, \quad \phi = \frac{q_{met}}{q_0 \sqrt{\frac{W_b c_b}{k}}}, \quad A_1 = \frac{W_b c_b}{\omega \rho c}. \quad (1.14)$$

Now, on using equation(1.14), from equations (1.11) to (1.13), and to equation(1.2), we have,

$$\frac{\partial^\alpha \mathbb{U}}{\partial \nu^\alpha} = \frac{\partial^2 \mathbb{U}}{\partial \mu^2} - A_1 \mathbb{U} + \phi. \quad (1.15)$$

$$\mathbb{U}(\mu, 0) = 0 \quad (1.16)$$

$$\frac{\partial \mathbb{U}(\sqrt{\frac{\omega \rho c}{k}} x, \nu)}{\partial \mu} = 0 \quad (1.17)$$

$$\frac{\partial \mathbb{U}(0, \nu)}{\partial \mu} = -\cos(\nu) \quad (1.18)$$

2 Preliminaries and notations

Definition 2.1. Caputo's derivative of $\mathbb{U}(\mu, \nu)$ is [6],

$$\frac{\partial^\beta \mathbb{U}(\mu, \nu)}{\partial \nu^\beta} = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_a^\nu \frac{1}{(s-\nu)^{1+\beta-n}} \frac{\partial^n \mathbb{U}(\mu, s)}{\partial \nu^n} ds, & n-1 < \alpha < n \\ \frac{\partial^n \mathbb{U}(\mu, \nu)}{\partial \nu^n}, & n = \beta, \end{cases} \quad (2.1)$$

where β is order of derivative.

Like, ordinary differentiation, $\frac{\partial^\beta \mathbb{U}(\mu, \nu)}{\partial \nu^\beta} = 0$, if $\mathbb{U}(\mu, \nu) = \text{constant}$

Definition 2.2. Caputo's derivative of a power function $f(\nu) = \nu^m$, $m \geq 0$ is defined by [6],

$$\frac{\partial^\beta \nu^m}{\partial \nu^\beta} = \begin{cases} 0, & \text{for } m \in \mathbb{N} \text{ and } m < [\beta], \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} \nu^{m-\beta}, & \text{for } m \in \mathbb{N} \text{ and } m \geq [\beta]. \end{cases} \quad (2.2)$$

Definition 2.3. The Legendre polynomials were developed by A. M. Legendre, are defined in $[-1, 1]$ and it satisfies the following recurrence relation [16]

$$L_{i+1}(\mu) = \left(\frac{2i+1}{i+1} \right) \mu L_i(\mu) - \left(\frac{i}{i+1} \right) L_{i-1}(\mu), \quad i = 1, 2, 3, \dots, \quad (2.3)$$

with $L_0(\mu) = 1$ and $L_1(\mu) = \mu$.

On introducing $\mu = (\frac{2p}{h} - 1)$, these polynomials can be defined in interval $[0, h]$. It is called Shifted Legendre polynomials and expressed with the following recurrence relation,

$$L_{i+1}(p) = \frac{(2i+1)(\frac{2p}{h} - 1)}{(i+1)}L_i(p) - \left(\frac{i}{i+1}\right)L_{i-1}(p), \quad i = 1, 2, \dots, \quad (2.4)$$

with $L_0(p) = 1$ and $L_1(t) = (2p - 1)$.

Shifted Legendre polynomial has the following analytical form,

$$L_i(p) = \sum_{k=0}^i \frac{(-1)^{k+i}(k+i)!}{(i-k)!(k!)^2 h^k} p^k, \quad (2.5)$$

and it satisfies the following orthogonality condition

$$\int_0^1 L_i(p)L_j(p)dt = \begin{cases} \frac{h}{(2i+1)}, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \quad (2.6)$$

A function $\mathbb{U}(\mu)$, $\mu \in [0, 1]$ which is L^2 can be described as,

$$\mathbb{U}(\mu) = \sum_{i=0}^{\infty} c_i(\mu)L_i(\mu), \quad (2.7)$$

where $c_i(\mu) = (2i+1) \int_0^1 U(\mu)L_i(\mu)d\mu$, $i = 0, 1, 2, \dots, m$.

For study purposes, we will consider only the first $(m+1)$ terms of equation (2.7)

$$\mathbb{U}(\mu) = \sum_{i=0}^{m+1} c_i(\mu)L_i(\mu). \quad (2.8)$$

Theorem 2.4. [17] Let $\mathbb{U}(\mu)$ express shifted Legendre polynomials as in equation (2.8). Then Caputo's derivative is given as ,

$$\frac{\partial^\beta \mathbb{U}(\mu)}{\partial \nu^\beta} = \sum_{i=\lceil \beta \rceil}^m \sum_{k=\lceil \beta \rceil}^i c_i(\mu) w_{i,k}^\beta \mu^{k-\beta}, \quad (2.9)$$

where,

$$w_{i,k}^\beta = \frac{(-1)^{i+k}(i+k)!}{(i-k)!k!\Gamma(k+1-\beta)}.$$

Theorem 2.5. [17, 18] The error $|E_T(n)| = |D^\beta \mathbb{U}(\mu) - D^\beta \mathbb{U}_n(\mu)|$ for the approximation of $D^\beta \mathbb{U}(\mu)$ by $D^\beta \mathbb{U}_n(\mu)$ has upper bound,

$$|E_T(n)| \leq \sum_{i=m+1}^{\infty} c_i \left(\sum_{i=\lceil \beta \rceil}^i \sum_{j=0}^{k-\lceil \beta \rceil} \theta_{i,j,k} \right), \quad (2.10)$$

where

$$\theta_{i,j,k} = \frac{(-1)^{i+k}(i+k)!(2i+1)}{(i-k)!k!\Gamma(k+1-\beta)} \times \sum_{r=0}^j \frac{(-1)^{(j+r)}(j+r)!}{(j-r)!(r!)^2(k+r+1-\beta)}.$$

Theorem 2.6. [17, 19] The truncation error $|\mathbb{U}(\mu) - \mathbb{U}_N(\mu)|$, where $\mathbb{U}_N(\mu) = \sum_{i=0}^N c_i L_i(\mu)$ is Legendre truncated series of $\mathbb{U}(\mu)$, satisfies following,

$$\|\mathbb{U}(\mu) - \mathbb{U}_N(\mu)\|_{L_w^p(-1,1)} \leq cN^{-m} \sum_{i=\min(m, N+1)}^m \|U^i(\mu)\|_{L_w^p(-1,1)}, \quad (2.11)$$

for $1 \leq p < \infty$, and U which has derivatives of order up to $m \in L_w^p(-1, 1)$ and constant c dependent on m .

As $N \rightarrow \infty$,

$$0 \leq \lim_{N \rightarrow \infty} \left(\|\mathbb{U}(\mu) - \mathbb{U}_N(\mu)\|_{L_w^p(-1,1)} \right) \leq \lim_{N \rightarrow \infty} \leq (cN^{-m} \sum_{i=\min(m, N+1)}^m \|\mathbb{U}^i(\mu)\|_{L_w^p(-1,1)}).$$

Then,

$$\lim_{N \rightarrow \infty} \left(\|\mathbb{U}(\mu) - \mathbb{U}_N(\mu)\|_{L_w^p(-1,1)} \right) = 0. \quad (2.12)$$

3 Technique

The function $\mathbb{U}(\mu, \nu)$ described for $(\mu, \nu) \in [0, 1] \times [0, 1]$. It can be proximated into truncated series in form of a pair of shifted Legendre polynomials as [20],

$$\mathbb{U}(\mu, \nu) \approx \mathbb{U}_{N,n}(\mu, \nu) = \sum_{i=0}^n \sum_{j=0}^N L_i(\nu) c_{ij} L_j(\mu) = P(\mu) C Q(\nu), \quad (3.1)$$

where

$$Q(\nu) = [L_0(\nu), L_1(\nu), L_2(\nu) \dots L_n(\nu)]_{(n+1) \times (n+1)},$$

$$P(\mu) = [L_0(\mu), L_1(\mu), L_2(\mu) \dots L_N(\mu)]_{(N+1) \times (N+1)}^T$$

and

$$C = \begin{bmatrix} c_{00} & c_{01} & \dots & \dots & c_{0N} \\ c_{10} & c_{11} & \dots & \dots & c_{1N} \\ \dots & \dots & \dots & \dots & \dots \\ c_{n0} & c_{n1} & \dots & \dots & c_{nN} \end{bmatrix}_{(n+1) \times (N+1)}.$$

$L_i(\nu)$, $i = 0, 1, \dots, n$ and $L_j(\mu)$, $j = 0, 1, \dots, N$ are shifted Legendre polynomial is time and space, respectively; the subscript T represents the transpose of the matrix. The number of nodes in the spatial and temporal directions are N and n respectively and uniform nodes have been taken on the interval $[0, 1]$. Here, discretization are independent with each other, $\Delta\mu = \frac{1}{N}$ and $\Delta\nu = \frac{1}{n}$ are length of interval.

On using Caputo's differentiation, equation (3.1) reduces to,

$$\frac{\partial^\alpha \mathbb{U}(\mu, \nu)}{\partial \nu^\alpha} = \frac{\partial^\alpha}{\partial \nu^\alpha} P(\mu) C Q(\nu) = P(\mu) C \frac{\partial^\alpha Q(\nu)}{\partial \nu^\alpha}, \quad (3.2)$$

and

$$\frac{\partial^2 \mathbb{U}(\mu, \nu)}{\partial \mu^2} = \frac{\partial^2}{\partial \mu^2} P(\mu) C Q(\nu) = \frac{\partial^2 P(\mu)}{\partial \mu^2} C Q(\nu). \quad (3.3)$$

On differentiating equation (3.1) with respect to ' μ ' then

$$\frac{\partial \mathbb{U}(\mu, \nu)}{\partial \mu} = \frac{\partial}{\partial \mu} P(\mu) C Q(\nu) = \frac{\partial P(\mu)}{\partial \mu} C Q(\nu) \quad (3.4)$$

On further simplification equation (1.2) gives

$$P(\mu) C \frac{\partial^\alpha Q(\nu)}{\partial \nu^\alpha} - \frac{\partial^2 P(\mu)}{\partial \mu^2} C Q(\nu) + P(\mu) C Q(\nu) = \Psi(\mu, \nu) \quad (3.5)$$

Initial and boundary conditions are expressed on using equation (3.1),

$$\mathbb{U}(\mu, 0) = Q(0) C P(\mu) = 0, \quad (3.6)$$

$$\frac{\partial \mathbb{U}(0, \nu)}{\partial \mu} = Q(\nu) C P'(0) = -1 \text{ or } -\cos(\nu), \quad (3.7)$$

$$\frac{\partial(\sqrt{\frac{w_b c_b}{k}} L, \nu)}{\partial \mu} = 0, \quad (3.8)$$

$$\begin{aligned}
Q(0) &= [L_0(0), L_1(0), L_2(0) \dots L_n(0)], \\
P'(0) &= [L_0(0), L_2(0), L_3(0) \dots L_N(0)]^T, \\
P'(\sqrt{\frac{w_b c_b}{k}} L) &= [L_0(\sqrt{\frac{w_b c_b}{k}} L), L_2(\sqrt{\frac{w_b c_b}{k}} L), L_3(\sqrt{\frac{w_b c_b}{k}} L) \dots L_N(\sqrt{\frac{w_b c_b}{k}} L)]^T.
\end{aligned}$$

We simplify equation (3.5), on applying the property of Kronecker product (\otimes) [21] and collocation has been done at nodes (x_k, t_l) , $k = 2, 3 \dots N$, $l = 2, 3, \dots n$, where $N = \frac{1}{\Delta x}$ and $n = \frac{1}{\Delta t}$.

Now, equation (3.5) simplifies to the the system under consideration of $(N - 2)(n - 2) \times Nn$ linear equations

$$\left\{ \frac{\partial^\alpha Q^T}{\partial \nu^\alpha} \otimes P - Q^T \otimes \frac{\partial^2 P}{\partial \mu^2} + Q^T \otimes P \right\} \vec{C} = \vec{\Psi}(\mu, \nu). \quad (3.9)$$

By staking the columns matrix $[C_{ij}]$ and $[\Psi_{ij}]$ on top of one another, \vec{C} and $\vec{\Psi}$ are obtained respectively and equation (3.9) can be expressed as,

$$A_1 \vec{C} = B_1. \quad (3.10)$$

The approximation of the subjected conditions are expressed as,

$$\begin{aligned}
[Q^T(0) \otimes P] \vec{C} &= 0, \\
\left[Q^T \otimes \frac{\partial^2 P(0)}{\partial \mu} \right] \vec{C} &= -1 \text{ or } \cos(\nu), \\
\left[Q^T \otimes \frac{\partial^2 P(\sqrt{\frac{w_b c_b}{k}} L)}{\partial \mu} \right] \vec{C} &= 0.
\end{aligned}$$

These can be written as

$$\begin{aligned}
A_2 \vec{C} &= B_2, \\
A_3 \vec{C} &= B_3, \\
A_4 \vec{C} &= B_4, \\
A_5 \vec{C} &= B_5.
\end{aligned} \quad (3.11)$$

On combining equations (3.10)-(3.11), following system of $(N + 1) \times (n + 1)$ linear equations are obtained

$$AC = B, \quad (3.12)$$

where

$$A = [A_1, A_2, A_3, A_4, A_5]^T$$

and

$$B = [B_1, B_2, B_3, B_4, B_5]^T.$$

The coefficient matrix C is obtained on solving the linear system of equations (3.12), and the approximate solution $U_{N,n}(\mu, \nu)$ is obtained using equation (3.1).

4 Results

In the proposed study the used physical quantities and their numerical values are defined as, $L = 0.02 \text{ m}$, $\omega = 0.05 \text{ sec}^{-1}$, $\mathbb{T}_a = 37^\circ\text{C}$, $q_0 = 5000 \text{ Wm}^{-2}$, $\rho = 1050 \text{ Kg m}^{-3}$, $\rho_b = 1000 \text{ Kg m}^{-3}$, $k = 0.5 \text{ W}^\circ\text{Cm}^{-1}$, $W_b = 0.5 \text{ Kg m}^{-3}$, $q_{met} = 368.1 \text{ Wm}^{-3}$, $c_b = 3770 \text{ J}^\circ\text{CKg}^{-1}$, and $c = 4180 \text{ J}^\circ\text{CKg}^{-1}$.

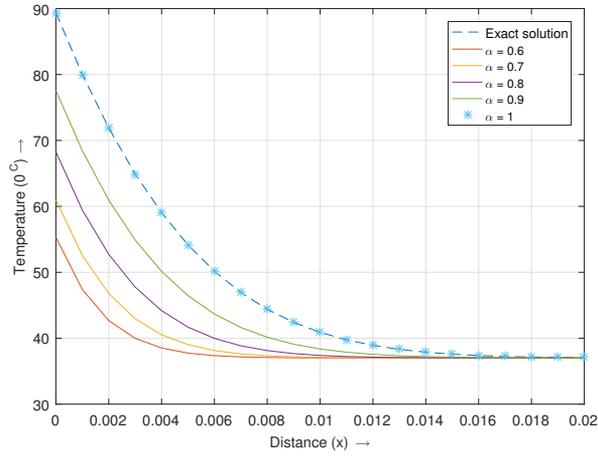


Figure 1. Numerical solution of equation(1.2) corresponding to the distance as $\alpha \rightarrow 1$.

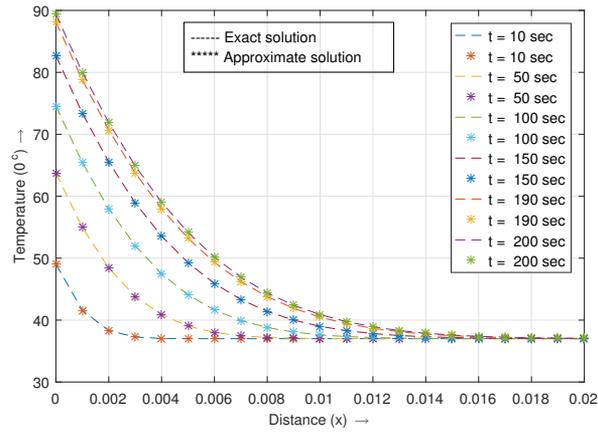


Figure 2. Comparison between the analytic and numerical solution of equation (1.2).

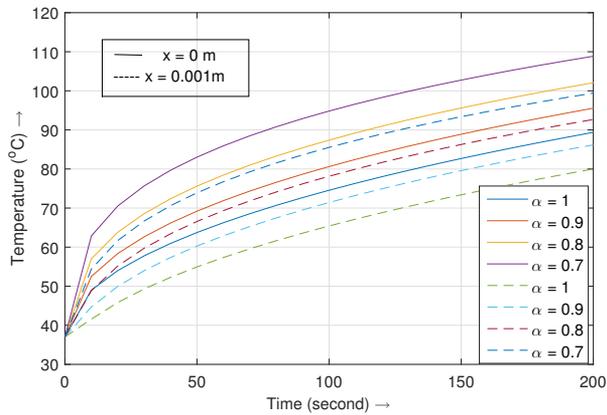


Figure 3. Variation in temperature corresponding to the time of equation(1.2) under the constant heat flux condition.

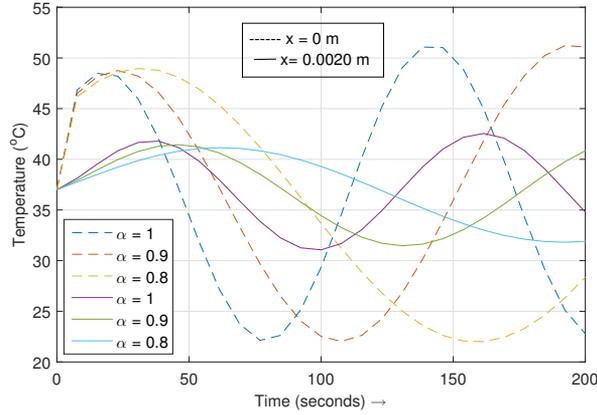


Figure 4. Variation in temperature corresponding to the time of equation(1.2) under the sinusoidal heat flux condition.

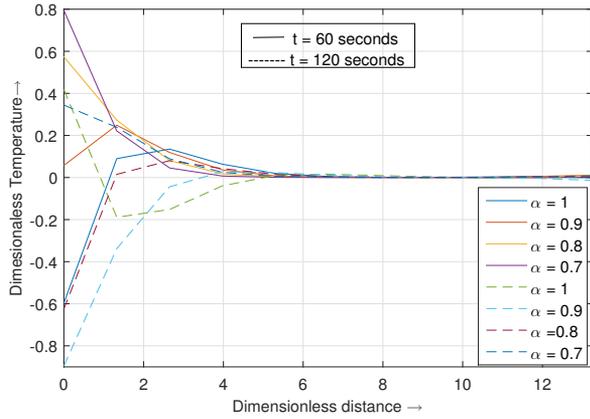


Figure 5. Dimensionless temperature variation of equation(1.2).

4.1 Discussion and interpretation

In this paper, we discuss the impact of temperature variation within the tissue due to various parameters. Fig. (2) illustrates the temperature along the depth of tissue at various times and also a comparison of numerical solution for different values of α with exact solution [?]. It is observed that as $\alpha \rightarrow 1$, the computational results coincide closely with the exact solution. Moreover, the temperature is found to decrease with increasing distance. The temperature variations with constant heat flux conditions are depicted in Fig. (3) at distance $x = 0$ m and $x = 0.001$ m respectively. We can see that temperature and α are related reciprocally. Fig. (4) depicts the variation in temperature with sinusoidal heat flux at distance $x = 0.000$ m and $x = 0.002$ m for different values α . Amplitude increases proportionally with α . The dimensionless temperature corresponding to dimensionless distance at time $t = 60$ seconds and $t = 120$ seconds is shown in Fig. (5). It is obvious that the heating boundary, particularly at $\mu = 0$, significantly affects the temperature profile. As the dimensionless depth exceeds $\mu = 5$, it is clear that the temperature fluctuation is relatively low. The dimensionless temperature is significantly under control due to sinusoidal heat flux of about $\mu = 0$. Furthermore, we observe that the oscillations decrease as the value of α decreases with respect to the depth of the skin tissue.

5 Conclusion

With constant and sinusoidal heat flux conditions, the time-fractional Pennes bioheat equation has been taken into account and the resulting temperature distribution in skin tissue has been examined. It can be observed that temperature variation is minimal in the case of a fractional-order model with constant heat flux. There is a rise in temperature for reducing the value of α . Further, the temperature is reduced as the distance is increasing. At the same time, for sinusoidal heat flux conditions, the oscillations decrease with increasing value of α , and amplitude reduces as the deepness of the tissue increases. We can conclude that the oscillations decrease with tissue depth and lower values of α . The results obtained from the Legendre collocation approach could be useful for representing the temperature response in the fractional-order Pennes bioheat equation for sinusoidal and constant heat flux. The obtained results may be beneficial for the experimental model to estimate the appropriate value of α . We expect that the results derived from the proposed technique may help suggest the actual amount of heat required for real-life problems.

Declaration

Data Availability Statement

No datasets were generated or analyzed in this study.

Conflict of interest

The author (s) did not disclose any kind of potential conflicts of interest.

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