

MULTIPLICATIVE DERIVATIONS ACTING ON SOME SUBSETS OF 3-PRIME NEAR-RINGS

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Abstract *The aim of this article is to discuss the commutativity of a 3-prime near-ring \mathcal{N} which admits a multiplicative derivation d satisfying certain conditions on the Jordan ideals and on the semi-group ideals of \mathcal{N} . In addition, we present an example proving the necessity of the hypothesis of 3-prime imposed on the different results.*

1 Introduction

A left near-ring \mathcal{N} is a triplet $(\mathcal{N}, +, \cdot)$, where “+” and “ \cdot ” are two binary operations such that (i) $(\mathcal{N}, +)$ is a group (not necessarily abelian), (ii) (\mathcal{N}, \cdot) is a semigroup, and (iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for every $x, y, z \in \mathcal{N}$. Similarly, if instead of (iii), \mathcal{N} satisfies the right distributive law, then \mathcal{N} is said to be a right near-ring. Throughout this article, \mathcal{N} represents a left near-ring. We denote by $\mathcal{Z}(\mathcal{N}) = \{x \in \mathcal{N} \mid xy = yx \text{ for all } y \in \mathcal{N}\}$ the multiplicative center of \mathcal{N} . A near-ring \mathcal{N} is known as zero-symmetric if $0 \cdot x = 0$ for every $x \in \mathcal{N}$ (recall that left distributivity yields that $x \cdot 0 = 0$), and usually \mathcal{N} will be 3-prime, if for $x, y \in \mathcal{N}$, $x\mathcal{N}y = \{0\}$ implies $x = 0$ or $y = 0$, and \mathcal{N} is said to be 2-torsion free if $x \in \mathcal{N}$ and $2x = 0$ implies $x = 0$. For any pair of elements $x, y \in \mathcal{N}$, $[x, y] = xy - yx$ and $x \circ y = xy + yx$ will denote the well-known Lie product and Jordan product, respectively. The Jordan ideal \mathcal{J} of \mathcal{N} is an additive subgroup that has the property $j \circ n \in \mathcal{J}$ and $n \circ j \in \mathcal{J}$ for all $j \in \mathcal{J}$, $n \in \mathcal{N}$ (for more details, see [14]). A nonempty subset \mathcal{I} of \mathcal{N} will be called a semigroup right ideal (resp. semigroup left ideal) if $\mathcal{N}\mathcal{I} \subseteq \mathcal{I}$ (resp. $\mathcal{I}\mathcal{N} \subseteq \mathcal{I}$) and if \mathcal{I} is both a semi group right ideal and a semi group left ideal, it will be called a semigroup ideal. A derivation d on \mathcal{N} is an additive endomorphism of \mathcal{N} with the property that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$, or equivalently as noted in [29], that $d(xy) = xd(y) + d(x)y$ for any $x, y \in \mathcal{N}$. Since E. C. Posner published his paper [27] in 1957, many authors have studied the properties of derivations of prime and semiprime rings. The study of the structure of 3-prime near-ring acting derivations was introduced by H. E. Bell et al. [3] in 1987. After this date, several researchers developed and generalized this kind of study on prime rings and 3-prime near-rings in several senses using other additive applications, namely generalized derivations, semiderivation, generalized semiderivations and (α, β) -derivations (see for example [1, 5, 6, 7, 8, 9, 10, 11, 12, 13, 16, 18, 20, 21, 22, 23, 24, 25, 26, 28]). A mapping $d : \mathcal{N} \rightarrow \mathcal{N}$, not necessarily additive, is known as a multiplicative derivation on \mathcal{N} if $d(xy) = xd(y) + d(x)y$ for every $x, y \in \mathcal{N}$. The concept of multiplicative derivation first appeared in the work of Daif [15]. Clearly, every derivation is a multiplicative derivation, but the converse is not true. Recently, some results concerning commutativity of prime rings with derivations were proved for multiplicative derivations. It is therefore natural to ask what can be said about the commutativity of 3-prime near-rings in which the mapping being considered is not additive?.

2 Preliminary results

We introduce the following lemmas which are important to develop the proof of our theorems.

Lemma 2.1. *Let \mathcal{N} be a 3-prime near-ring and \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} .*

- (i) [4, Lemma 1.4(i)] *If $x, y \in \mathcal{N}$ and $x\mathcal{I}y = \{0\}$, then $x = 0$ or $y = 0$.*
- (ii) [4, Lemma 1.3(i)] *If $x \in \mathcal{N}$ and $x\mathcal{I} = \{0\}$ or $\mathcal{I}x = \{0\}$, then $x = 0$.*

Lemma 2.2. [2, Lemma 4] *Let \mathcal{N} be a near-ring and $d : \mathcal{N} \rightarrow \mathcal{N}$ is a multiplicative derivation of \mathcal{N} . Then,*

$$(xd(y) + d(x)y)z = xd(y)z + d(x)yz \text{ for all } x, y, z \in \mathcal{N}.$$

Lemma 2.3. *Let \mathcal{N} be a 3-prime near-ring and \mathcal{J} a nonzero Jordan ideal of \mathcal{N} .*

- (i) [14, Lemma 3] *If \mathcal{N} is 2-torsion free and $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$, then \mathcal{N} is a commutative ring.*
- (ii) *If $x\mathcal{J} = \{0\}$, then $x = 0$.*

For the proof of (ii), we can reason as in the case of the proof of [14, Lemma 1] with appropriate changes.

Lemma 2.4. [2, Lemma 5] *Let \mathcal{N} be a 3-prime near-ring. If \mathcal{N} admits a nonzero multiplicative derivation d such that $ad(\mathcal{N}) = \{0\}$ or $d(\mathcal{N})a = \{0\}$, then $a = 0$.*

Lemma 2.5. *Let \mathcal{N} be a 3-prime near-ring.*

- (i) [4, Lemma 1.2(iii)] *If $z \in \mathcal{Z}(\mathcal{N}) \setminus \{0\}$ and $xz \in \mathcal{Z}(\mathcal{N})$, then $x \in \mathcal{Z}(\mathcal{N})$.*
- (ii) [4, Lemma 1.5] *If $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$, then \mathcal{N} is a commutative ring.*

Lemma 2.6. [19, Lemma 2.1] *A near-ring \mathcal{N} admits a multiplicative derivation if and only if it is zero-symmetric.*

Lemma 2.7. [17, Theorem 1] *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a nonzero multiplicative derivation d , then the following assertions are equivalent*

- i) $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$,
- ii) \mathcal{N} is a commutative ring.

3 Main results

We begin with the following theorem which is essential to prove some results in this work.

Theorem 3.1. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a multiplicative derivation d , then the following properties hold true:*

- (i) *If $d(\mathcal{J}^2) = \{0\}$, then $d(\mathcal{J}) = \{0\}$.*
- (ii) *If $d(\mathcal{J}) = \{0\}$, then $d(\mathcal{Z}(\mathcal{N})) = \{0\}$.*

Proof. (i) Suppose that

$$d(ij) = 0 \text{ for all } i, j \in \mathcal{J}. \quad (3.1)$$

In particular, taking $j = j \circ j$ in (3.1), we get $d(i(j \circ j)) = 0$ for all $i, j \in \mathcal{J}$, which implies that $id(j(j + j)) + 2d(i)j^2 = 2d(i)j^2 = 0$ for all $i, j \in \mathcal{J}$. By 2-torsion freeness of \mathcal{N} , we find that

$$d(i)j^2 = 0 \text{ for all } i, j \in \mathcal{J}. \quad (3.2)$$

On the other hand, putting $x \circ j$ instead of j in (3.1), by the property defining of d , we get

$$id(x \circ j) + d(i)(x \circ j) = 0 \text{ for all } i, j \in \mathcal{J}, x \in \mathcal{N}. \quad (3.3)$$

Substituting jx for x in (3.3), we find that $id(j(x \circ j)) + d(i)j(x \circ j) = 0$ for all $i, j \in \mathcal{J}, x \in \mathcal{N}$. By (3.1), the first term of the sum is zero and hence $d(i)jxj + d(i)j^2x = 0$ for all $i, j \in \mathcal{J}, x \in \mathcal{N}$. Using (3.2) together Lemma (2.6), the preceding relation reduces to $d(i)j\mathcal{N}j = \{0\}$ for all $i, j \in \mathcal{J}$. By 3-primeness of \mathcal{N} , we can see that

$$d(i)j = 0 \text{ for all } i, j \in \mathcal{J}. \tag{3.4}$$

Replacing j by $j \circ x$, where $x \in \mathcal{N}$, in (3.4) and using it again, we arrive at $d(i)xj = 0$ for all $i, j \in \mathcal{J}, x \in \mathcal{N}$. Which leads to $d(i)\mathcal{N}j = \{0\}$ for all $i, j \in \mathcal{J}$. As $\mathcal{J} \neq \{0\}$ and \mathcal{N} is 3-prime, the previous relation assures that $d(\mathcal{J}) = \{0\}$.

(ii) Assume that

$$d(i) = 0 \text{ for all } i \in \mathcal{J}. \tag{3.5}$$

Taking $z \circ i$ instead of i in (3.5) where $z \in \mathcal{Z}(\mathcal{N})$ and applying it again, we find that $d(z)(i+i) = 0$ for all $i \in \mathcal{J}$. By 2-torsion freeness of \mathcal{N} , it follows that $d(z)i = 0$ for all $i \in \mathcal{J}$. Taking $i = x \circ i$ in the last equation and in view of the 3-primeness of \mathcal{N} and the fact that \mathcal{J} is nonzero, we arrive at the conclusion $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. The proof is complete. \square

Theorem 3.2. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If d is a multiplicative derivation on \mathcal{N} satisfying $[d(x), d(j)] = 0$ for all $j \in \mathcal{J}$ and $x \in \mathcal{I}$, then \mathcal{N} is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.*

Proof. Let d be a multiplicative derivation of \mathcal{N} verifying the following condition

$$d(x)d(j) = d(j)d(x) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.6}$$

Dividing the proof into cases which exhaust all the possibilities.

Case 1: Suppose that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$.

Let $z \in \mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$. Replacing x by zx in (3.6) and using Lemma 2.2, we arrive at

$$zd(x)d(j) + d(z)xd(j) = d(j)zd(x) + d(j)d(z)x \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

Because of (3.6), the last equation reduces to

$$d(z)xd(j) = d(j)d(z)x \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.7}$$

Taking xt instead of x in (3.7), where $t \in \mathcal{N}$, and using it again, we find that $d(z)xt d(j) = d(z)xd(j)t$ for all $j \in \mathcal{J}, x \in \mathcal{I}, t \in \mathcal{N}$, that is $d(z)\mathcal{I}[t, d(j)] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$. As $d(z) \neq 0$ and in view of Lemma 2.1(i), we obtain $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$. Thus, \mathcal{N} is a commutative ring by Lemma 2.7.

Case 2: Assume that $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. In this case, putting $d(j)x$ instead of x in (3.6) and applying Lemma 2.2, we deduce that

$$d(j)d(x)d(j) + d^2(j)xd(j) = d(j)d(j)d(x) + d(j)d^2(j)x \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

After simplifying, we obtain $d^2(j)xd(j) = d(j)d^2(j)x$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Taking xt instead of x , where $t \in \mathcal{N}$, in the previous equation, we obtain $d^2(j)\mathcal{I}[d(j), t] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$. In virtue of Lemma 2.1(i), for any $j \in \mathcal{J}$, we have either $d^2(j) = 0$ or $d(j) \in \mathcal{Z}(\mathcal{N})$. But, the both conditions give $d^2(\mathcal{J}) = \{0\}$. \square

Theorem 3.3. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a multiplicative derivation d satisfies any one of the following properties:*

(i) $[d(j), d(x)] = [d(j), x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,

(ii) $[d(j), d(x)] = -[d(j), x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,

then \mathcal{N} is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.

Proof. (i) Assume that \mathcal{N} admits a multiplicative derivation d satisfies

$$[d(j), d(x)] = [d(j), x] \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.8}$$

We divide the proof into two steps:

Step 1: If $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. Putting zx instead of x in (3.8), where $z \in \mathcal{Z}(\mathcal{N})$ and $d(z) \neq 0$, and invoking Lemma 2.2, we obtain for all $j \in \mathcal{J}, x \in \mathcal{I}$

$$d(j)zd(x) + d(j)d(z)x - d(z)xd(j) - zd(x)d(j) = zd(j)d(x) - zd(x)d(j).$$

It follows that $d(j)d(z)x = d(z)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Applying the same arguments as used in case 1 in the proof of Theorem 3.2, we get \mathcal{N} is a commutative ring.

Step 2: If $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. In this case, replacing x by $d(j)x$ in (3.8) and using Lemma 2.2, we find that for all $j \in \mathcal{J}, x \in \mathcal{I}$,

$$[d(j), d(j)x] = [d(j), d(d(j)x)]$$

$$d(j)[d(j), x] = d(j)d(d(j)x) - d(d(j)x)d(j)$$

$$d(j)[d(j), d(x)] = d(j)d(j)d(x) + d(j)d^2(j)x - (d(j)d(x)d(j) + d^2(j)xd(j))$$

$$d(j)d(j)d(x) - d(j)d(x)d(j) = d(j)d(j)d(x) + d(j)d^2(j)x - d^2(j)xd(j) - d(j)d(x)d(j).$$

Simplifying the last equation, we obtain $d(j)d^2(j)x = d^2(j)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Applying the same arguments as used in the second case of the proof of Theorem 3.2, we obtain $d^2(\mathcal{J}) = \{0\}$.

(ii) Using similar arguments, we get the required result. □

Theorem 3.4. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a multiplicative derivation d which satisfies any one of the following assertions:*

(i) $[j, d(x)] = [d(j), x]$ for all $j \in \mathcal{J}, x \in \mathcal{N}$,

(ii) $[j, d(x)] = -[d(j), x]$ for all $j \in \mathcal{J}, x \in \mathcal{N}$,

then \mathcal{N} is a commutative ring or $d^2(\mathcal{J}) = \{0\}$.

Proof. (i) We separate the proof into two cases, by assumption we have

$$[j, d(x)] = [d(j), x] \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}. \tag{3.9}$$

Suppose that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. Putting zx instead of x by in (3.9), where $z \in \mathcal{Z}(\mathcal{N})$ and $d(z) \neq 0$, we find that $jzd(x) + jd(z)x - d(z)xj - zd(x)j = zjd(x) - zd(x)j$ for all $j \in \mathcal{J}, x \in \mathcal{N}$ which assures that

$$jd(z)x = d(z)xj \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}. \tag{3.10}$$

Taking xt instead of x , where $t \in \mathcal{N}$, in the last equation and reasoning as above, we arrive at $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ and hence \mathcal{N} is a commutative ring by Lemma 2.3(i).

Assume that $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. In this case, substituting $d(j)x$ for x in (3.9), we find that

$$jd(j)d(x) + jd^2(j)x - d^2(j)xj - d(j)d(x)j = d(j)jd(x) - d(j)d(x)j \text{ for all } j \in \mathcal{J}, x \in \mathcal{N}. \tag{3.11}$$

In particular, taking $x = j$ in (3.9) and using 2-torsion freeness of \mathcal{N} , we get $jd(j) = d(j)j$ for all $j \in \mathcal{J}$. So, (3.11) reduces to $jd^2(j)x = d^2(j)xj$ for all $j \in \mathcal{J}, x \in \mathcal{N}$. Replacing x by xt , where $t \in \mathcal{N}$, in the preceding equation and using it, we arrive at $d^2(j)xjt = d^2(j)xtj$ for all $j \in \mathcal{J}, x, t \in \mathcal{N}$. The last equation gives $d^2(j)\mathcal{N}[j, t] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$. Taking into account the 3-primeness of \mathcal{N} , the latter relation shows that

$$d^2(j) = 0 \text{ or } j \in \mathcal{Z}(\mathcal{N}) \text{ for all } j \in \mathcal{J}. \tag{3.12}$$

Let $j_0 \in \mathcal{Z}(\mathcal{N}) \cap \mathcal{J}$, (3.9) gives $d(j_0)x = xd(j_0)$ for all $x \in \mathcal{N}$ which means that $d(j_0) \in \mathcal{Z}(\mathcal{N})$ and therefore $d^2(j_0) = 0$. Consequently, (3.12) yields $d^2(\mathcal{J}) = \{0\}$.

(ii) Using similar techniques as above, we can prove that \mathcal{N} is a commutative ring or $d^2(\mathcal{J}) = \{0\}$. □

Theorem 3.5. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a multiplicative derivation d verifies any one of the following conditions:*

- (i) $[j, d(x)] = [j, x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
 - (ii) $[j, d(x)] = -[j, x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
- then \mathcal{N} is a commutative ring or $d(\mathcal{J}) = \{0\}$.

Proof. (i) By the given hypothesis, we have

$$[j, d(x)] = [j, x] \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.13}$$

The proof of the theorem is divided into two parts. In the first part, we assume that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. Choosing $z \in \mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$ and replacing x by zx in (3.13), in virtue of Lemma 2.2, we can see that

$$jzd(x) + jd(z)x - d(z)xj - zd(x)j = zjd(x) - zd(x)j \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

The previous relation shows that $jd(z)x = d(z)xj$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. The rest of the proof is similar to the steps used after equation (3.10) in the proof of Theorem 3.4, we conclude that \mathcal{N} is a commutative ring.

In the second part, we suppose that $d(\mathcal{Z}(\mathcal{N})) = \{0\}$. Substituting jx for x in (3.13), we get

$$jjd(x) + jd(j)x - d(j)xj - jd(x)j = jjd(x) - jd(x)j \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

After simplifying, we have $jd(j)x = d(j)xj$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Putting xt instead of x , where $t \in \mathcal{N}$, in the preceding relation and using it again, we arrive at $d(j)xjt = d(j)xtj$ for all $j \in \mathcal{J}, x \in \mathcal{I}, t \in \mathcal{N}$, which implies that $d(j)\mathcal{I}[j, t] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$. By Lemma 2.1(i), we conclude that $d(j) = 0$ or $j \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$, the both conditions give $d(\mathcal{J}) = \{0\}$.

(ii) We can using similar approach as above with suitable changes. □

Theorem 3.6. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . If \mathcal{N} admits a multiplicative derivation d such that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and d satisfies any one of the following conditions:*

- (i) $d(jx) = d(x)d(j)$ for all $x \in \mathcal{I}, j \in \mathcal{J}$,
- (ii) $[d(j), d(x)] = [j, x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
- (iii) $[d(j), d(x)] = -[j, x]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,

then \mathcal{N} is a commutative ring.

Proof. (i) Let d be a multiplicative derivation of \mathcal{N} which satisfies

$$d(jx) = d(x)d(j) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.14}$$

Replacing x by jx in (3.14) and using Lemma 2.2, we obtain $d(j)jx = d(j)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Taking $x = xt$, where $t \in \mathcal{N}$, in the last relation and using it again, we find that $d(j)xd(j)t = d(j)xt d(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}, t \in \mathcal{N}$. The previous equation yields $d(j)\mathcal{I}[d(j), t] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$. By Lemma 2.1(i), we infer that $d(j) = 0$ or $d(j) \in \mathcal{Z}(\mathcal{N})$ for all $j \in \mathcal{J}$ which reduces to $d(\mathcal{J}) \subseteq \mathcal{Z}(\mathcal{N})$. So that, \mathcal{N} is a commutative ring by Lemma 2.7.

(ii) Suppose that

$$[d(j), d(x)] = [j, x] \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.15}$$

Let z an element of $\mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$. Substituting zx for x in (3.15) and using Lemma 2.2, we obtain

$$d(j)zd(x) + d(j)d(z)x - d(z)xd(j) - zd(x)d(j) = zd(j)d(x) - zd(x)d(j) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

After simplifying, we find that $d(j)d(z)x = d(z)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Now, using the same arguments as used in the case 1 of the proof of Theorem 3.2, we get \mathcal{N} is a commutative ring. (iii) Using the same reasoning as used in (ii), we obtain the desired result. □

Theorem 3.7. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . There is no nonzero multiplicative derivation d of \mathcal{N} verifying any one of the following assertions:*

- (i) $d(x)d(j) = \pm[x, j]$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
- (ii) $d(x)d(j) = \pm(x \circ j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$.

Proof. (i) Assume that \mathcal{N} admits a nonzero multiplicative derivation d such that

$$d(x)d(j) = \pm[x, j] \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.16}$$

Replacing x by jx , where $x \in \mathcal{I}$, in (3.16) and using Lemma 2.2, we obtain $d(j)\mathcal{I}d(j) = \{0\}$ for all $j \in \mathcal{J}$. Applying Lemma 2.1(i), we find that

$$d(j) = 0 \text{ for all } j \in \mathcal{J},$$

and therefore our hypothesis gives $xj = jx$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Substituting xt for x , where $t \in \mathcal{N}$, in the last equation and using it again, we infer that $\mathcal{I}[t, j] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$, the Lemma 2.1(ii) gives $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ and Lemma 2.3(i) shows that \mathcal{N} is a commutative ring. Now, returning to our hypothesis, we get $d(x)d(j) = 0$ for all $x \in \mathcal{I}, j \in \mathcal{J}$. Putting $x = xy$, where $y \in \mathcal{N}$, in the previous relation and using the 3-primeness of \mathcal{N} , we find that $d(x) = 0$ or $d(j) = 0$ for all $x \in \mathcal{I}, j \in \mathcal{J}$, the both cases show $d = 0$ which leads to a contradiction.

(ii) Suppose that \mathcal{N} has a nonzero multiplicative derivation d satisfies

$$d(x)d(j) = \pm(x \circ j) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.17}$$

Putting $x = jx$, where $x \in \mathcal{I}$, in (3.17) and invoking Lemma 2.2, we get $d(j)\mathcal{I}d(j) = \{0\}$ for all $j \in \mathcal{J}$. By Lemma 2.1(i), we infer that

$$d(j) = 0 \text{ for all } j \in \mathcal{J}.$$

In this case, (3.17) becomes $jx = x(-j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Substituting xt for x , where $t \in \mathcal{N}$, in the last equation and using it again and putting $-j$ instead of j , we deduce $\mathcal{I}[j, t] = \{0\}$ for all $j \in \mathcal{J}, t \in \mathcal{N}$, which means $\mathcal{J} \subseteq \mathcal{Z}(\mathcal{N})$ by Lemma 2.1(ii) and Lemma 2.3(i) shows that \mathcal{N} is a commutative ring. Our assumption gives $d(x)d(j) = 2xj$ for all $x \in \mathcal{I}, j \in \mathcal{J}$. Taking $x = yx$, where $y \in \mathcal{I}$, in the preceding relation and using Lemma 2.1(ii) we find that $d(x)d(j) = 0$ for all $x \in \mathcal{I}, j \in \mathcal{J}$. Using the same technique as above, we lead to a contradiction. \square

In the last part of this section, we combine Jordan's products and Lie's product. Our goal is to study this question and see if there exists a multiplicative derivation d of \mathcal{N} that satisfies the proposed identities.

Theorem 3.8. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . There is no nonzero multiplicative derivation d of \mathcal{N} such that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ for which*

- (i) $[d(j), d(x)] = \pm(d(j) \circ x)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
- or*
- (ii) $[d(j), d(x)] = \pm(j \circ x)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$.

Proof. (i) Suppose that \mathcal{N} has a multiplicative derivation d such that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and

$$[d(j), d(x)] = \pm(d(j) \circ x) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.18}$$

Since $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$, choosing $z \in \mathcal{Z}(\mathcal{N})$ verifying $d(z) \neq 0$ and replacing x by zx in (3.18), in view of Lemma 2.2, we obtain for all $j \in \mathcal{J}, x \in \mathcal{I}$

$$d(j)zd(x) + d(j)d(z)x - d(z)xd(j) - zd(x)d(j) = zd(j)d(x) - zd(x)d(j).$$

It follows that $d(j)d(z)x = d(z)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$ which shows that \mathcal{N} is a commutative ring by the first part of the proof of Theorem 3.2. In this case, by 2-torsion freeness of \mathcal{N} ,

(3.18) gives $\mathcal{I}d(j) = \{0\}$ for all $j \in \mathcal{J}$ and thus $d(\mathcal{J}) = \{0\}$ by Lemma 2.1(ii). Hence, $d(\mathcal{Z}(\mathcal{N})) = \{0\}$ by Theorem 3.1(ii), which is contrary to our assumption that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$. (ii) Assume there does exist such a nonzero multiplicative derivation d verifying $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and

$$[d(j), d(x)] = \pm(j \circ x) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.19}$$

Let $z \in \mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$, putting $x = zx$ in (3.19) and applying Lemma 2.2, we get for all $j \in \mathcal{J}, x \in \mathcal{I}$

$$d(j)zd(x) + d(j)d(z)x - d(z)xd(j) - zd(x)d(j) = zd(j)d(x) - zd(x)d(j).$$

It follows that $d(j)d(z)x = d(z)xd(j)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. Note that the same techniques as used the case 1 of the proof of Theorem 3.2 are still valid in the present situation, and hence \mathcal{N} is a commutative ring. Now, using (3.19) and 2-torsion freeness of \mathcal{N} , we find that $\mathcal{I}j = \{0\}$ for all $j \in \mathcal{J}$. Thereby, in view of Lemma 2.1(ii) we conclude that $\mathcal{J} = \{0\}$, leading to a contradiction. \square

Theorem 3.9. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring, \mathcal{I} be a nonzero semigroup ideal of \mathcal{N} and \mathcal{J} be a nonzero Jordan ideal of \mathcal{N} . There is no nonzero multiplicative derivation d of \mathcal{N} such that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ for which*

(i) $[j, d(x)] = \pm(d(j) \circ x)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$,
or

(ii) $[j, d(x)] = \pm(j \circ x)$ for all $j \in \mathcal{J}, x \in \mathcal{I}$.

Proof. (i) Assume that \mathcal{N} admits a multiplicative derivation d verifies $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and

$$[j, d(x)] = \pm(d(j) \circ x) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.20}$$

Substituting zx for x by in (3.20), where $z \in \mathcal{Z}(\mathcal{N})$ and $d(z) \neq 0$ and using Lemma 2.2, we find that

$$jzd(x) + jd(z)x - d(z)xj - zd(x)j = zjd(x) - zd(x)j \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

Simplifying the previous expression, we obtain $jd(z)x = d(z)xj$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. As this relation is similar to equation (3.10), even if $x \in \mathcal{I}$, the result of (3.10) remains true, hence \mathcal{N} is a commutative ring. By 2-torsion freeness of \mathcal{N} , equation (3.20) reduces to $\mathcal{I}d(j) = \{0\}$ for all $j \in \mathcal{J}$, it follows that $d(\mathcal{J}) = \{0\}$, this means that $d(\mathcal{Z}(\mathcal{N})) = \{0\}$ by Theorem 3.1(ii) which contradicts our original assumption that $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$.

(ii) Suppose that \mathcal{N} has a multiplicative derivation d satisfies $d(\mathcal{Z}(\mathcal{N})) \neq \{0\}$ and

$$[j, d(x)] = \pm(j \circ x) \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}. \tag{3.21}$$

Let $z \in \mathcal{Z}(\mathcal{N})$ such that $d(z) \neq 0$. Taking zx instead of x in (3.21) and invoking Lemma 2.2, we obtain

$$jzd(x) + jd(z)x - d(z)xj - zd(x)j = zjd(x) - zd(x)j \text{ for all } j \in \mathcal{J}, x \in \mathcal{I}.$$

It follows that $jd(z)x = d(z)xj$ for all $j \in \mathcal{J}, x \in \mathcal{I}$. By applying the same arguments as those used after equation (3.10) in the proof of Theorem 3.4, we get \mathcal{N} is a commutative ring. In this case, (3.21) becomes $2xj = 0$ for all $x \in \mathcal{I}, j \in \mathcal{J}$ and in virtue of 2-torsion freeness of \mathcal{N} , we can see that $\mathcal{I}j = \{0\}$ for all $j \in \mathcal{J}$ and therefore $\mathcal{J} = \{0\}$ by Lemma 2.1(ii), a contradiction. \square

The following example shows the necessity of \mathcal{N} being 3-prime in the hypotheses of our theorems.

Example 3.10. Let \mathcal{S} be a zero-symmetric 2-torsion free noncommutative unitary left near-ring. Consider $\mathcal{N}, \mathcal{J}, \mathcal{I}$ and $d_1, d_2 : \mathcal{N} \rightarrow \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} \mid 0, x, y \in \mathcal{S} \right\}, \quad \mathcal{J} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & 0 & 0 \end{pmatrix} \mid 0, x \in \mathcal{S} \right\},$$

$$\mathcal{I} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & x & 0 \end{pmatrix} \mid 0, x \in \mathcal{S} \right\}, \quad d_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x^3 & 0 & 0 \end{pmatrix}$$

$$\text{and } d_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & y^2 & 0 \end{pmatrix}.$$

It is obvious that \mathcal{N} is a 2-torsion free near-ring which is not 3-prime, \mathcal{I} is a nonzero semigroup ideal of \mathcal{N} , \mathcal{J} is a nonzero Jordan ideal of \mathcal{N} and d_1, d_2 are nonzero multiplicative derivations of \mathcal{N} such that $d_1(\mathcal{Z}(\mathcal{N})) \neq \{0\}$, $d_1^2(\mathcal{J}) \neq \{0\}$ and d_1, d_2 satisfy the following properties:

1. $d_1(\mathcal{J}^2) = \{0\}$ and $d_1(\mathcal{J}) \neq \{0\}$,
2. $d_2(\mathcal{J}) = \{0\}$ and $d_2(\mathcal{Z}(\mathcal{N})) \neq \{0\}$
3. $[d_1(A), d_1(J)] = 0$,
4. $d_1(A)d_1(J) = \pm[A, J]$,
5. $[d_1(J), d_1(A)] = \pm[d(J), A]$,
6. $[J, d_1(B)] = \pm[d_1(J), B]$,
7. $[J, d_1(A)] = \pm[J, A]$,
8. $d_1(JA) = d_1(A)d_1(J)$,
9. $[d_1(J), d_1(A)] = \pm[J, A]$,
10. $[d_1(J), d_1(A)] = \pm(d_1(J) \circ A)$,
11. $[J, d_1(A)] = \pm(d_1(J) \circ A)$,
12. $[J, d_1(A)] = \pm(J \circ A)$,
13. $[d_1(J), d_1(A)] = \pm(J \circ A)$

for all $A \in \mathcal{I}, J \in \mathcal{J}, B \in \mathcal{N}$. But neither $d_1(\mathcal{J}) = \{0\}$ nor $d_1^2(\mathcal{J}) = \{0\}$ nor \mathcal{N} is a commutative ring.

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