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# AN IMPROVEMENT OF THE ESTIMATES OF THE MODULUS OF THE HANKEL DETERMINANTS OF SECOND AND THIRD ORDER FOR THE CLASS $\mathcal{S}$ OF UNIVALENT FUNCTIONS

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**Abstract** Using some properties of the Grunsky coefficients we improve earlier results for upper bounds of the Hankel determinants of the second and third order for the class $\mathcal{S}$ of univalent functions.

## 1 Introduction and preliminaries

Let $\mathcal{A}$ be the class of functions $f$ analytic in the open unit disc $\mathbb{D} = \{z : |z| < 1\}$ and normalized such that $f(0) = f'(0) - 1 = 0$, i.e., of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, and let its subclass $\mathcal{S}$ consist of univalent functions in the unit disc $\mathbb{D}$. Further, let $\mathcal{S}^\star$ and $\mathcal{K}$ denote the subclasses of $\mathcal{A}$ which are starlike and convex in $\mathbb{D}$, respectively, and let $\mathcal{U}$ denote the set of all $f \in \mathcal{A}$ satisfying

$$\left| \left(\frac{z}{f(z)}\right)^2 f'(z) - 1 \right| < 1 \qquad (z \in \mathbb{D}).$$

(see [6, 7]).

In the study of the class of univalent functions and its subclasses, a significant topic is finding upper estimates (preferably sharp) of the Hankel determinant, especially of the second and third order, for a function $f$ from $\mathcal{A}$ is defined by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2 \tag{1.1}$$

and

$$H_3(1) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \tag{1.2}$$

respectively.

Hankel determinants are used for studies in the theory of singularities (see [2]), as well as in the study of power series with integral coefficients. The upper bound of their modulus is of special interest in the theory of univalent functions and for some subclasses of the class $\mathcal{S}$ the sharp estimation of $|H_2(2)|$ are known. For example, for the classes $\mathcal{S}^\star$ and $\mathcal{U}$ we have that

$|H_2(2)| \le 1$ (see [3], [8]), while $|H_2(2)| \le \frac{1}{8}$ for the class $\mathcal{K}$ ([3]). The sharp estimate of $H_3(1)$ seems to be more challenging problem and quite few are known. A review on this can be found in [12], while new non-sharp upper bounds for different classes and conjectures about the sharp ones are given in [10]. Other related results are given in [4, 11].

In their paper [9] the authors gave the next upper bound of $|H_2(2)|$ and $|H_3(1)|$ for the class $\mathcal{S}$:

**Theorem 1.1.** *For the class $\mathcal{S}$ we have*

$$|H_2(2)| \le A, \quad \textit{where} \quad 1 \le A \le \frac{11}{3} = 3,666\ldots$$

*and*

$$|H_3(1)| \le B, \quad \textit{where} \quad \frac{4}{9} \le B \le \frac{32+\sqrt{285}}{15} = 3.258796\cdots$$

In this paper we improve these results by proving:

**Theorem 1.2.** *For the class $\mathcal{S}$ we have the next estimations:*

$(i)$ $|H_2(2)| \le 1.3614\ldots;$

$(ii)$ $|H_3(1)| \le 1.6787\ldots.$

The proof of this theorem will make use mainly the notations and results given in the book of N.A. Lebedev ([5]).

For an univalent function $f$ from $\mathcal{S}$ we have

$$\log \frac{f(t)-f(z)}{t-z} = \sum_{p,q=0}^{\infty} \omega_{p,q} t^p z^q,$$

where $\omega_{p,q}$ are the so-called Grunsky's coefficients such that $\omega_{p,q} = \omega_{q,p}$. This coefficients satisfy the Grunsky's inequality ([1, 5]):

$$\sum_{q=1}^{\infty} q \left| \sum_{p=1}^{\infty} \omega_{p,q} x_p \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_p|^2}{p}, \tag{1.3}$$

where $x_p$ are arbitrary complex numbers such that last series converges.

Next, it is well-known that if

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots \tag{1.4}$$

belongs to $\mathcal{S}$, then also does

$$f_2(z) = \sqrt{f(z^2)} = z + c_3 + c_5 z^5 + \ldots. \tag{1.5}$$

Then, for the function $f_2$ the appropriate Grunsky's coefficients are of the form $\omega^{(2)}_{2p-1,2q-1}$ and the inequality (1.3) appears to be

$$\sum_{q=1}^{\infty} (2q-1) \left| \sum_{p=1}^{\infty} \omega^{(2)}_{2p-1,2q-1} x_{2p-1} \right|^2 \le \sum_{p=1}^{\infty} \frac{|x_{2p-1}|^2}{2p-1}. \tag{1.6}$$

Finally, from [5, p.57] we have that the coefficients $a_2, a_3, a_4$ of $f$ can be expressed by Grunsky's coefficients $\omega^{(2)}_{2p-1,2q-1}$ of $f_2$ given by (1.5) as:

$$\begin{aligned} a_2 &= 2\omega_{11}, \\ a_3 &= 2\omega_{13} + 3\omega_{11}^2, \\ a_4 &= 2\omega_{33} + 8\omega_{11}\omega_{13} + \frac{10}{3}\omega_{11}^3 \\ a_5 &= 2\omega_{35} + 8\omega_{11}\omega_{33} + 5\omega_{13}^2 + 18\omega_{11}^2\omega_{13} + \frac{7}{3}\omega_{11}^4 \\ 0 &= 3\omega_{15} - 3\omega_{11}\omega_{13} + \omega_{11}^3 - 3\omega_{33} \\ 0 &= \omega_{17} - \omega_{35} - \omega_{11}\omega_{33} - \omega_{13}^2 + \frac{1}{3}\omega_{11}^4. \end{aligned} \tag{1.7}$$

Here and in the rest of the paper, for simplicity of the expressions, we omit upper index "(2)" in $\omega^{(2)}_{2p-1,2q-1}$.

We note that in the book [5] there exists a typing mistake for the coefficient $a_5$. Namely, instead of the therm $5\omega_{13}^2$, there is $5\omega_{15}^2$.

Also, from (1.6) for $x_{2p-1} = 0$, $p = 3, 4, \dots$ we have

$$\begin{aligned} &|\omega_{11}x_1 + \omega_{31}x_3|^2 + 3|\omega_{13}x_1 + \omega_{33}x_3|^2 \\ &+5|\omega_{15}x_1 + \omega_{35}x_3|^2 + 7|\omega_{17}x_1 + \omega_{37}x_3|^2 \le |x_1|^2 + \frac{|x_3|^2}{3}. \end{aligned} \tag{1.8}$$

From (1.8), for $x_1 = 1$ and $x_3 = 0$, since $\omega_{31} = \omega_{13}$, we have the next inequalities

$$|\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 + 7|\omega_{15}|^2 \le 1,$$

and further

$$\begin{aligned} |\omega_{11}|^2 &\le 1, \\ |\omega_{11}|^2 + 3|\omega_{13}|^2 &\le 1, \\ |\omega_{11}|^2 + 3|\omega_{13}|^2 + 5|\omega_{15}|^2 &\le 1. \end{aligned}$$

This leads to:

$$\begin{aligned} |\omega_{11}| &\le 1, \\ |\omega_{13}| &\le \frac{1}{\sqrt{3}}\sqrt{1 - |\omega_{11}|^2}, \\ |\omega_{15}| &\le \frac{1}{\sqrt{5}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}, \\ |\omega_{17}| &\le \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2}. \end{aligned} \tag{1.9}$$

We note that we can get the first inequality from (1.9) using the fact

$$|a_2| = |2\omega_{11}| \le 2 \quad \Rightarrow \quad |\omega_{11}| \le 1$$

(see (1.7)).

## 2 Proof of Theorem 2

Proof of part $(i)$. Using the definition of $H_2(2)$ given by (1.1) and relations (1.7), we have

$$H_2(2) = 4\omega_{11}\omega_{33} - \frac{7}{3}\omega_{11}^4 - 4\omega_{13}^2 + 4\omega_{11}^2\omega_{13}.$$

Next, from the fifth relation in (1.7) we obtain

$$\omega_{33} = \omega_{15} - \omega_{11}\omega_{13} + \frac{1}{3}\omega_{11}^3, \tag{2.1}$$

and after combining the two previous relations we have

$$H_2(2) = 4\omega_{11}\omega_{15} - \omega_{11}^4 - 4\omega_{13}^2,$$

i.e.,

$$|H_2(2)| \le 4|\omega_{11}||\omega_{15}| + |\omega_{11}|^4 + 4|\omega_{13}|^2.$$

Applying (1.9) gives

$$|H_2(2)| \le \frac{4}{\sqrt{5}}|\omega_{11}|\sqrt{1-|\omega_{11}|^2-3|\omega_{13}|^2} + |\omega_{11}|^4 + 4|\omega_{13}|^2 := F_1(|\omega_{11}|,|\omega_{13}|), \tag{2.2}$$

where

$$F_1(x,y) = \frac{4}{\sqrt{5}}x\sqrt{1-x^2-3y^2} + x^4 + 4y^2. \tag{2.3}$$

Now, we will find the maximum of the function $F_1$ on its domain

$$D_1 := \left\{0 \le x \le 1,\ 0 \le y \le \frac{1}{\sqrt{3}}\sqrt{1-x^2}\right\}.$$

Numerically we can verify that the system of equations $\partial F_1/\partial x(x,y) = 0$ and $\partial F_1/\partial y(x,y) = 0$ has only one real solution $(x_1,y_1) = \left(\sqrt{\frac{11}{30}}, \frac{1}{30}\sqrt{\frac{281}{2}}\right) = (0.60553\ldots, 0.395109\ldots)$ in the interior of $D_1$ such that then

$$F_1(x_1,y_1) = 1.19889\ldots.$$

Further, let see that maximum values of $F_1$ on the boundary of the domain $D_1$.

1) For $y=0$, from (2.3) we have

$$F_1(x,0) = \frac{4}{\sqrt{5}}x\sqrt{1-x^2} + x^4, \quad 0 \le x \le 1.$$

Using the first derivative test we can conclude that the function $F_1(x,0)$ has its maximum at the point $x_0 = 0.9181\ldots$ which satisfies the equation $5x^8 - 5x^6 + 4x^4 - 4x^2 + 1 = 0$ and

$$F_1(x_0,0) = 1.3614\ldots.$$

2) For $x=0$, since $0 \le y \le \frac{1}{\sqrt{3}}$, we have

$$F_1(0,y) = 4y^2 \le \frac{4}{3} = 1.333\ldots.$$

3) Finally, for $0 \le x \le 1$:

$$F_1\left(x, \frac{1}{\sqrt{3}}\sqrt{1-x^2}\right) = \frac{1}{3}(3x^4 - 4x^2 + 4) \le \frac{4}{3}.$$

From all the previous facts and (2.2) we conclude that $|H_2(2)| \le 1.3614\ldots$.

Proof of part $(ii)$ From the six relation in (1.7) and the relation (2.1), after simple calculations, we get

$$\omega_{35} = \omega_{17} - \omega_{11}\omega_{15} + \omega_{11}^2\omega_{13} - \omega_{13}^2. \tag{2.4}$$

Now, using the relations (1.7), (2.1) and (2.4), we obtain

$$\begin{aligned} a_4 &= 2(\omega_{15} + 3\omega_{11}\omega_{13} + 2\omega_{11}^3) \\ a_5 &= 2\omega_{17} + 6\omega_{11}\omega_{15} + 12\omega_{11}^2\omega_{13} + 3\omega_{13}^2 + 5\omega_{11}^4. \end{aligned} \tag{2.5}$$

Further, from the definition of $H_3(1)$ given by (1.2), the relation (1.7) for $a_2$ and $a_3$, and (2.5) for $a_4$ and $a_5$, after some calculations we have

$$H_3(1) = 2\omega_{17}(2\omega_{13} - \omega_{11}^2) + 4\omega_{11}\omega_{13}\omega_{15} + 2\omega_{11}^3\omega_{15} - 3\omega_{11}^2\omega_{13}^2 - 2\omega_{13}^3 - 4\omega_{15}^2.$$

So,

$$\begin{aligned} |H_3(1)| \le\ & 2|\omega_{17}||2\omega_{13} - \omega_{11}^2| + 4|\omega_{11}||\omega_{13}||\omega_{15}| + 2|\omega_{11}|^3|\omega_{15}| \\ & + 3|\omega_{11}|^2|\omega_{13}|^2 + 2|\omega_{13}|^3 + 4|\omega_{15}|^2. \end{aligned} \tag{2.6}$$

We start analysing the above inequality.

Since for the functions from the class $\mathcal{S}$, $|a_3 - a_2^2| \le 1$ (see [1]), and since from (1.7),

$$|2\omega_{13} - \omega_{11}^2| = |a_3 - a_2^2|,$$

we receive

$$|2\omega_{13} - \omega_{11}^2| \le 1.$$

Using this and the estimate

$$|\omega_{17}| \le \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2 - 5|\omega_{15}|^2} \le \frac{1}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}$$

given in (1.9), for the first term in (2.6), we have

$$2|\omega_{17}||2\omega_{13} - \omega_{11}^2| \le \frac{2}{\sqrt{7}}\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2}. \tag{2.7}$$

Using the estimate for $|\omega_{15}|$ given in (1.9) and the estimate in (2.7), inequality (2.6) reduces to

$$\begin{aligned} |H_3(1)| \le & \left(\frac{2}{\sqrt{7}} + 4|\omega_{11}||\omega_{13}| + 2|\omega_{11}|^3\right)\sqrt{1 - |\omega_{11}|^2 - 3|\omega_{13}|^2} \\ & + \frac{4}{5} - \frac{4}{5}|\omega_{11}|^2 - \frac{12}{5}|\omega_{13}|^2 + 3|\omega_{11}|^2|\omega_{13}|^2 + 2|\omega_{13}|^3 \\ :=& F_2(|\omega_{11}|, |\omega_{13}|), \end{aligned}$$

where

$$F_2(x,y) = \left(\frac{2}{\sqrt{7}} + 4xy + 2x^3\right)\sqrt{1 - x^2 - 3y^2} + \frac{4}{5} - \frac{4}{5}x^2 - \frac{12}{5}y^2 + 3x^2y^2 + 2y^3 \tag{2.8}$$

and $(x,y) \in D_1 = \left\{0 \le x \le 1,\ 0 \le y \le \frac{1}{\sqrt{3}}\sqrt{1 - x^2}\right\}$.

Numerical calculation give that the system of equations $\partial F_2/\partial x(x,y) = 0$ and $\partial F_2/\partial y(x,y) = 0$ has only two real solutions in the interior of $D_1$, that are $(x_2, y_2) = (0.583\ldots, 0.206\ldots)$ and $(x_3, y_3) = (0.0131\ldots, 0.00748\ldots)$ such that

$$F_2(x_2, y_2) = 1.6787\ldots \quad \text{and} \quad F_2(x_3, y_3) = 1.5559\ldots.$$

Now, we consider the maximum values of the function $F_2(x,y)$ on the boundary of $D_1$.

1) The relation (2.8) for $y=0$ gives
$$F_2(x,0)=\left(\frac{2}{\sqrt{7}}+2x^3\right)\sqrt{1-x^2}+\frac{4}{5}-\frac{4}{5}x^2,\quad 0\le x\le 1.$$
Since $F_2(0,0)=\frac{2}{\sqrt{7}}+\frac{4}{5}=1.5559\ldots$, $F_2(1,0)=0$ and $\partial F_2/\partial x(x,0)<0$ when $0<x<1$, we conclude that
$$F_2(x,0)\le F_2(0,0)=\frac{2}{\sqrt{7}}+\frac{4}{5}=1.555928\ldots,\quad 0\le x\le 1$$
.

2) From (2.8) for $x=0$ we receive
$$F_2(0,y)=\frac{2}{\sqrt{7}}\sqrt{1-3y^2}+\frac{4}{5}-\frac{12}{5}y^2+2y^3,\quad 0\le y\le\frac{1}{\sqrt{3}}.$$
Since $F_2(0,0)=\frac{2}{\sqrt{7}}+\frac{4}{5}=1.555928\ldots$, $F_2(0,\frac{1}{\sqrt{3}})=\frac{1}{4}$ and
$$\partial F_2/\partial y(0,y)=-\frac{6}{\sqrt{7}}\frac{1}{\sqrt{1-3y^2}}-6y(\frac{4}{5}-y)\le 0$$
when $0\le y\le\frac{1}{\sqrt{3}}$, we get
$$F_2(0,y)\le F_2(0,0)=\frac{2}{\sqrt{7}}+\frac{4}{5}=1.5559\ldots,\quad 0\le y\le\frac{1}{\sqrt{3}}.$$

3) At the end, for $0\le x\le 1$,
$$F_2(x,\frac{1}{\sqrt{3}}\sqrt{1-x^2})=\frac{2}{3\sqrt{3}}(1-x^2)^{\frac{3}{2}}+x^2(1-x^2).$$
The last function has its maximum $\frac{7}{16}$ for $x=\frac{1}{2}$. So
$$F_2\left(x,\frac{1}{\sqrt{3}}\sqrt{1-x^2}\right)\le\frac{7}{16}=0.4375.$$

Finally, using all the previous facts we conclude that
$$|H_3(1)|\le 1.6787\ldots.$$

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