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NON-AUTO BÄCKLUND TRANSFORMATION APPROACH TO OBTAIN EXACT SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATION WITH TIME DEPENDENT COEFFICIENTS

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Abstract *The paper focuses on obtaining explicit exact solutions for the generalized Burgers Fisher equation with time dependent coefficients. The generalized Burgers Fisher (GBF) equation is a partial differential equation that combines features of both the Burgers equation and the Fisher equation. The time dependent coefficients in the equation add complexity to the problem, making it challenging to find exact solutions. The Clarkson - Kruskal direct technique and known solutions of the constant-coefficient GBF equation are combined in our method to find the solutions of time-dependent GBF equation. The resulting framework produces a range of explicit solutions, such as parabolic wave profiles with curved structures, singular solutions with discontinuities or blow-up behavior, kink- profile solutions reflecting sharp transitions, and regular smooth solutions*

1 Introduction

Nonlinear partial differential equations (PDEs) are important for understanding a variety of physical phenomena, including fluid dynamics and population dynamics. Among these equations, the generalized Burgers- Fisher (GBF) equation stands out as a fundamental model that describes the dynamics of nonlinear waves in diverse contexts. The GBF equation can be understood as a combination of the classical Fisher and Burgers equations.
The Burgers equation [1]

$$w_t + a\, ww_x = b\, w_{xx},$$

where $a\, ww_x$ *represents nonlinear convection and* $b\, w_{xx}$ *represents diffusion named after Dutch mathematician Jan Burgers, is a fundamental nonlinear PDE that arises in fluid dynamics, acoustics, and other fields.It describes the evolution of a one-dimensional fluid flow with viscosity and nonlinearity.*
The Fisher equation [2]

$$w_t = D\, w_{xx} + r\, w(1-w),$$

where $D\, w_{xx}$ *represents diffusion and* $r\, w(1-w)$ *represents logistic growth, proposed by biologist and statistician Ronald A. Fisher, is a reaction-diffusion equation that models the spread of a population in space. It is widely used in mathematical biology to study the dynamics of species migration and population growth.*
The generalized Burgers- Fisher (GBF) equation [3]

$$w_t + a\, w^m w_x = b\, w_{xx} + c\, w(1-w^n),$$

combines these two equations to create a versatile model that accounts for both nonlinear advection (as in the Burgers equation) and diffusion - reaction (growth) processes (as in the Fisher equation).
Numerous nonlinear phenomena in biological and physical systems can only be simulated using the time-dependent coefficients of the GBF. Growth rates, diffusion rates, and convection intensity are all examples of variables that can change over time due to outside influences and changes in the environment. Thus, adding time-dependent coefficients gives the model more realism.
The time dependent GBF equation can be written as:

$$w_t + q_1(t)\, ww_x + q_2(t)\, w_{xx} + q_3(t)\, (w - w^2) = 0, \tag{1.1}$$

where w is the dependent variable, t is time, x is space, w_t represents the partial derivative of w concerning time, w_x represents the partial derivative of w concerning space, and w_{xx} is the second spatial derivative. The parameters $q_1(t), q_2(t)$ and $q_3(t)$ control the advection, diffusion and reaction terms respectively. This equation is particularly interesting due to its ability to describe a wide range of phenomena, from shock wave formation in fluid dynamics to the spread of a population with logistic growth. The nonlinear advection term ww_x leads to the formation of shock waves, while the reaction diffusion term $w(1 - w)$ governs the population dynamics, introducing nonlinearity and complexity.
Equation (1.1) has wide range of applications in the field of traffic flow, population dynamics, fluid mechanics etc.

- *In traffic flow modeling, time-dependent coefficients represent changing road or driving conditions, and the equation reflects the combined effects of nonlinear advection (which represents vehicle interactions) and diffusion (which models density smoothing).*
- *The Fisher-type reaction term simulates growth and competition in biological systems and population dynamics, while time-dependent parameters take seasonal influences or environmental changes into consideration.*
- *In fluid mechanics, the Burgers component depicts wave propagation and viscous flow; adding time-dependent factors makes modeling fluids with changeable properties more realistic.*

In literature, many methods [4, 5, 6] such as Hirota's bilinear method [7], Lie group method [8, 9, 10, 11, 12, 13, 14, 15, 16], F expansion method [17], Clarkson and Kruskal's (CK) direct method [18], Jacobi elliptic function method [19], Bäcklund transformations [20, 21, 22, 23, 24] etc have been proposed by researchers to find explicit exact solutions [25, 26, 27] of PDEs.
A Bäcklund transformation is a method for generating new solutions to differential equations from previously known ones. It provides a set of relations that connect one solution to another rather than resolving the equation directly. There are two types of Bäcklund transformations. An auto- Bäcklund transformation connects two solutions of the same equation. This allows us to build more complicated solutions step by step from simple ones. A non - auto Bäcklund transformation connects solutions of two different equations. This is useful because sometimes a difficult nonlinear equation can be linked to a simpler or linear one. For example, the Burgers equation relates to the heat equation through the Cole - Hopf transformation. In this way, Bäcklund transformations are important for finding exact solutions and for revealing hidden connections between different equations. In this study, we investigate exact solutions to the time-dependent generalized Burgers Fisher equation by using non auto- Bäcklund approach which combines solutions from the constant coefficient generalized Burgers Fisher equation with the powerful direct method proposed by Clarkson and Kruskal.
The work in this paper is organized as follows : Section (2) provides the outline of direct method by Clarkson and kruskal. In section (3), explicit exact solutions of eq. (1.1) has been obtained. Section (4) results and discussions has been provided. Section (5) provides the concluding remarks.

2 Direct method approch

To find symmetry reductions of nonlinear PDEs without group theoretic techniques, Clarkson and Kruskal (CK) introduced a new method called the direct method [18]. This method assumes a specific reduced form of the dependent variable using new similarity variables. It substitutes this form into the original PDE and derives compatibility conditions that define the reduction. This method has been widely used to obtain exact solutions of nonlinear partial differential equations, particularly those with variable or time-dependent coefficients, and offers a useful substitute for Lie symmetry analysis.
Consider a partial differential equation of the form

$$F(x,t,w,w_x,w_t,w_{xt},\dots)=0, \qquad w=w(x,t). \tag{2.1}$$

Instead of using infinitesimal symmetries like Lie group approaches, the Clarkson-Kruskal method employs a direct similarity ansatz of the form:

$$w(x,t)=A(x,t)+B(x,t)\,W(\eta), \qquad \eta=\phi(x,t), \tag{2.2}$$

where $A(x,t)$, $B(x,t)$, and $\phi(x,t)$ are unknown functions, and $W(\eta)$ is to be determined.
In next step, evaluate the derivatives

$$w_x = A_x + B_xU + BW'\phi_x, \tag{2.3}$$
$$w_t = A_t + B_tU + BW'\phi_t, \tag{2.4}$$
$$w_{xx} = A_{xx} + B_{xx}W + 2B_xW'\phi_x + BW''(\phi_x)^2 + BW'\phi_{xx}, \tag{2.5}$$

where $W'=\frac{dW}{d\eta}$, $W''=\frac{d^2W}{d\eta^2}$.
To generate the system of overdetermined equations, substitute eq. (2.2) into eq. (2.1) and use the system with constant coefficients. Compare the coefficients of different derivatives of dependent variables. Solving the system of determining equations yields the values of $A(x,t), B(x,t)$ and $\phi(x,t)$ as well as the time - dependent coefficients. Finally, reconstruct the solution of the PDE as

$$w(x,t)=A(x,t)+B(x,t)\,W(\phi(x,t)).$$

This method is very powerful and its applications can be seen in Refs. [1, 20, 28, 29, 30]. When we are dealing with variable coefficient PDEs, this method is quite difficult to use as it involves nonlinear equations of high degree of complexity but with the help of softwares like Maple it becomes easier.

3 Explicit exact solutions of GBF equation

The Burgers Fisher equation has been investigated for improved differential transform method [31], laplace decomposition method [32], extended cubic B–spline finite element method [33], etc. Now we elucidate the procedure to find solutions of GBF equation by using non-auto-Bäcklund transformation between the GBF equation and the classical ones.
We seek reductions of the GBF equation (1.1) in the form

$$w(x,t)=A(x,t)+B(x,t)\,W(\varsigma(x,t),\tau(x,t)), \tag{3.1}$$

where $A(x,t), B(x,t), \varsigma(x,t), \tau(x,t)$ are to be determined and $W(\varsigma,\tau)$ is solution of constant coefficient generalized Burgers Fisher equation [34]

$$W_\tau + WW_\varsigma + \delta\, W_{\varsigma\varsigma} + W(1-W)=0. \tag{3.2}$$

Substitute (3.1) into equation (1.1) and then eliminating W_τ by using equation (3.2), we obtain

$$\begin{aligned}
&\left[q_1(t)B^2\tau_x+2q_2(t)B\tau_x^2\right]W^3-\left[q_1(t)B^2\tau_x+5q_2(t)B\tau_x^2\right]W^2W_\varsigma+q_2(t)B\tau_x^2W_{\varsigma\varsigma}\\
&+\left[B\tau_t+q_1(t)AB\tau_x\ +q_1(t)B\left(-B\tau_x+B_x\right)+q_2(t)\left(2B_x\tau_x-3B\tau_x^2+B\tau_{xx}\right)\right.\\
&\left.-q_3(t)B^2\right]W^2+\left[2q_2(t)B\tau_x^2\right]WW_\varsigma^2+\left[-B\tau_t-q_1(t)AB\tau_x+q_1(t)B^2\varsigma_x\right.\\
&\left.-2q_2(t)B_x\tau_x+4q_2(t)B\tau_x\varsigma_x+3q_2(t)B\tau_x^2-q_2(t)B\tau_{xx}\right]WW_\varsigma\\
&-\left[q_1(t)B^2\delta\tau_x+q_2(t)B\tau_x\varsigma_x+\left(4\delta\tau_x-\varsigma_x\right)\tau_x\right]WW_{\varsigma\varsigma}+2q_2(t)\delta B\tau_x^2\,WW_{\varsigma\varsigma\varsigma}\\
&+\left[B_t-B\tau_t+q_1(t)A\left(B_x-B\tau_x\right)+q_1(t)BA_x+q_2(t)B_{xx}-2q_2(t)B_x\tau_x\right.\\
&\left.+q_2(t)B\left(\tau_x^2-\tau_{xx}\right)+q_3(t)\left(B-2AB\right)\right]W-q_2(t)B\left[\tau_x\varsigma_x+\left(2\delta\tau_x-\varsigma_x\right)\tau_x\right]W_\varsigma^2\\
&+4q_2(t)B\delta\tau_x^2W_\varsigma W_{\varsigma\varsigma}+\left[B\varsigma_t+q_1(t)AB\varsigma_x+2q_2(t)B_x\varsigma_x-2q_2(t)B\tau_x\varsigma_x\right.\\
&\left.+q_2(t)B\varsigma_{xx}\right]W_\varsigma-\left[B\delta\tau_t+q_1(t)AB\delta\tau_x+2q_2(t)\delta B_x\tau_x-2\delta B_x\tau_x^2-q_2(t)B\varsigma_x^2\right.\\
&\left.+q_2(t)B\delta\tau_{xx}\right]W_{\varsigma\varsigma}-2q_2(t)B\delta\varsigma_x\tau_xW_{\varsigma\varsigma\varsigma}+q_2(t)B\ \delta^2\tau_x^2\ W_{\varsigma\varsigma\varsigma\varsigma}+A_t\\
&+q_1(t)AA_x+q_2(t)A_{xx}+q_3(t)(A-A^2)=0.
\end{aligned}\tag{3.3}$$

Equating the coefficients of various powers of W of (3.3) , we obtain the following set of determining equations :

$$\begin{aligned}
&\tau_x=0,\ q_1(t)AA_x+q_2(t)A_{xx}+q_3(t)(A-A^2)=0,\\
&q_1(t)BB_x-q_3(t)B^2+B\tau_t=0,\\
&B\varsigma_t+q_1(t)AB\varsigma_x+2q_2(t)B_x\varsigma_x+q_2(t)B\varsigma_{xx}=0,\\
&-B\tau_t+q_1(t)B^2\varsigma_x=0,\\
&B_t-B\tau_t+q_1(t)AB_x+q_1(t)BA_x+q_2(t)B_{xx}+q_3(t)\left(B-2AB\right)=0,\\
&B\delta\tau_t-q_2(t)B\varsigma_x^2=0.
\end{aligned}\tag{3.4}$$

By using Maple to solve the determining equations, we obtain

$$\begin{aligned}
&\tau=\ln\left(\frac{1}{2}c_1+\Omega(t)+\sqrt{c_1\Omega(t)+\Omega(t)^2+1}\right)+c_2,\\
&\varsigma=\frac{x}{c_3}-\frac{1}{2}\ln\Omega(t)+\frac{1}{2}\ln\left(\frac{1}{2}c_1+\Omega(t)+\sqrt{c_1\Omega(t)+\Omega(t)^2+1}\right)\\
&\quad+\frac{1}{2}\operatorname{arctanh}\left(\frac{c_1\Omega(t)+2}{2\sqrt{c_1\Omega(t)+\Omega(t)^2+1}}\right)+c_4,\\
&A=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1},\ B=\Omega(t),\\
&q_1(t)=\frac{\left(\frac{d}{dt}\Omega(t)c_3\right)}{\Omega(t)\sqrt{c_1\Omega(t)+\Omega(t)^2+1}},\ q_2(t)=\frac{\left(\frac{d}{dt}\Omega(t)c_3^2\right)}{\sqrt{c_1\Omega(t)+\Omega(t)^2+1}},\\
&q_3(t)=\frac{\frac{d}{dt}\Omega(t)}{\sqrt{c_1\Omega(t)+\Omega(t)^2+1}},
\end{aligned}\tag{3.5}$$

where c_1,c_2,c_3,c_4 are arbitrary constants and $\Omega(t)$ is arbitrary function of t.
The compatibility between the equations (1.1) and (3.2) gives the following theorem:

Theorem 3.1. *If $W=W(\varsigma,\tau)$ is solution of equation (3.2), then $w(x,t)=A(x,t)+B(x,t)\,W(\varsigma,\tau)$ will be solution of (1.1), where ς,τ,A,B are declared by (3.5).*

Some exact solutions of equation (3.2) discussed in [34] are listed below:
For arbitrary δ:

$$W(\varsigma,\tau)=\frac{1}{1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}},\tag{3.6a}$$

$$W(\varsigma,\tau)=\frac{1}{2}\left(\frac{\varsigma}{4\delta}-\frac{4\delta+1}{\delta}\tau+k_1\right)+\frac{1}{2}. \tag{3.6b}$$

For $\delta=\frac{1}{2}$*:*

$$W(\varsigma,\tau)=-k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right), \tag{3.7a}$$

$$W(\varsigma,\tau)=k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right), \tag{3.7b}$$

$$W(\varsigma,\tau)=k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1+k_2\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)-k_2}, \tag{3.7c}$$

$$W(\varsigma,\tau)=k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1-k_2\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)-k_2}. \tag{3.7d}$$

For $\delta=-\frac{1}{2}$*:*

$$W(\varsigma,\tau)=1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right), \tag{3.8a}$$

$$W(\varsigma,\tau)=1-k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\tanh\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right), \tag{3.8b}$$

$$W(\varsigma,\tau)=1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1+k_2\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{k_2-\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}, \tag{3.8c}$$

$$W(\varsigma,\tau)=1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1-k_2\tanh\left(-e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{k_2-\tanh\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}, \tag{3.8d}$$

where $\mu=\frac{1+4\delta}{2}$.
Case-(i): *By using the solution that corresponds to any* δ*, the general solutions of equation* (1.1) *are as follows:*

$$w_1(x,t)=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[\frac{1}{1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}}\right], \tag{3.9a}$$

$$w_2(x,t)=\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[\frac{1}{2}\left(\frac{\varsigma}{4\delta}-\frac{4\delta+1}{\delta}\tau+k_1\right)\right] \tag{3.9b}$$

Case(ii): *By using the solution that corresponds to* $\delta=\frac{1}{2}$*, the general solutions of equation* (1.1) *are as follows:*

$$w_3(x,t)=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[-k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)\right], \tag{3.10a}$$

$$w_4(x,t)=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)\right], \tag{3.10b}$$

$$w_5(x,t)=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1+k_2\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{\tan\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)-k_2}\right], \tag{3.10c}$$

$$w_6(x,t)=-\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[k_1e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1-k_2\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{\tanh\left(e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)-k_2}\right] \tag{3.10d}$$

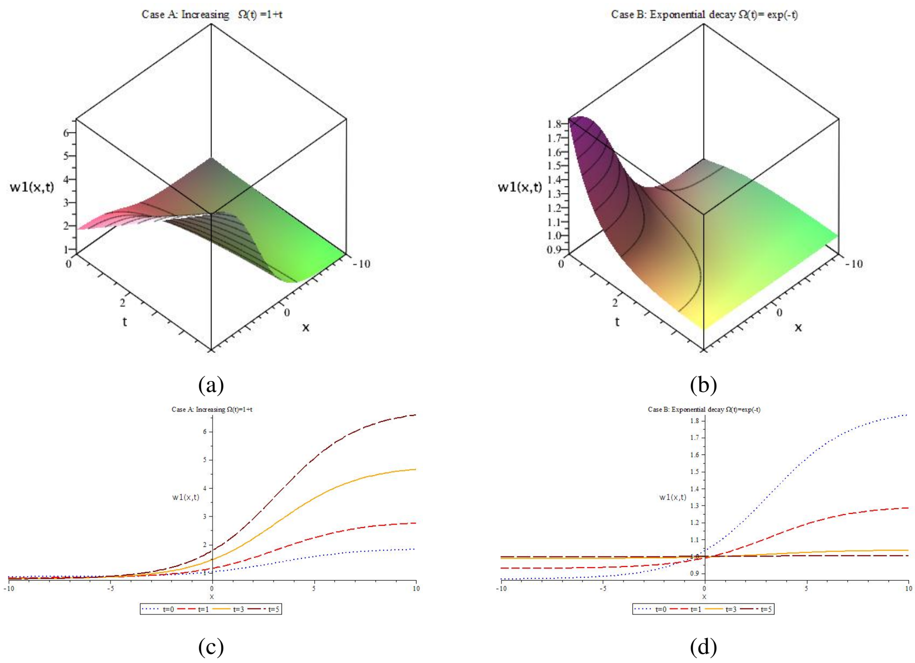


Figure 1: (a,b) represents 3D profile and (c,d) represents 2D wave effects of regular solution $w_1(x,t)$. Panels (a,c) shows unbounded growth with $\Omega(t) = 1+t$ and panels (b,d) shows damped behavior with $\Omega(t) = e^{-t}$.

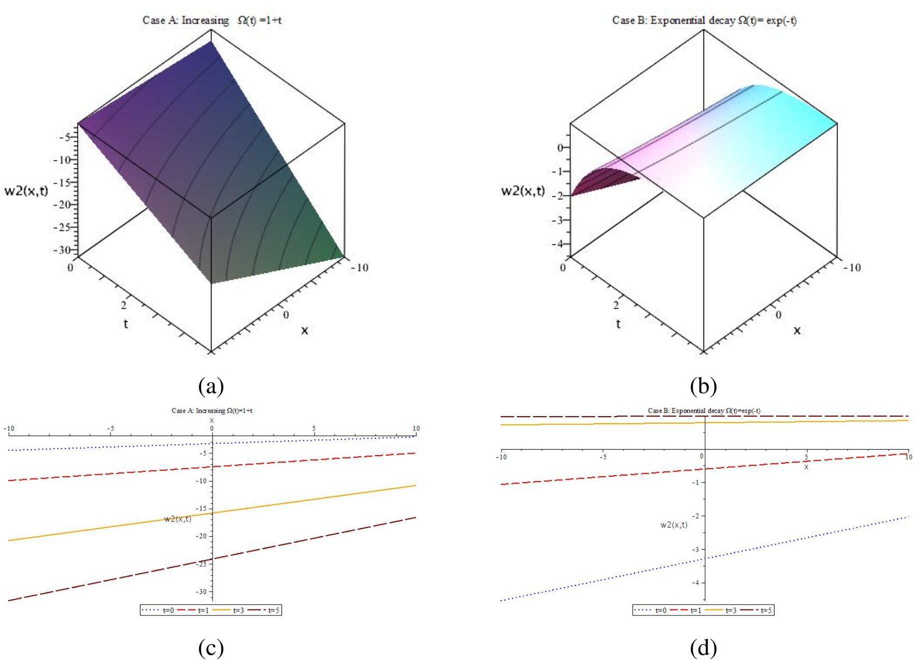


Figure 2: (a,b) represents 3D profile and (c,d) represents 2D wave effects of travelling-wave solution $w_2(x,t)$. Panels (a,c) shows unbounded growth with $\Omega(t) = 1+t$ and panels (b,d) shows damped behavior with $\Omega(t) = e^{-t}$.

Case-(iii): *By using the solution that corresponds to $\delta = -\frac{1}{2}$, the general solutions of equation (1.1) are as follows:*

$$w_7(x,t) = -\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[1+k_1 e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)\right], \tag{3.11a}$$

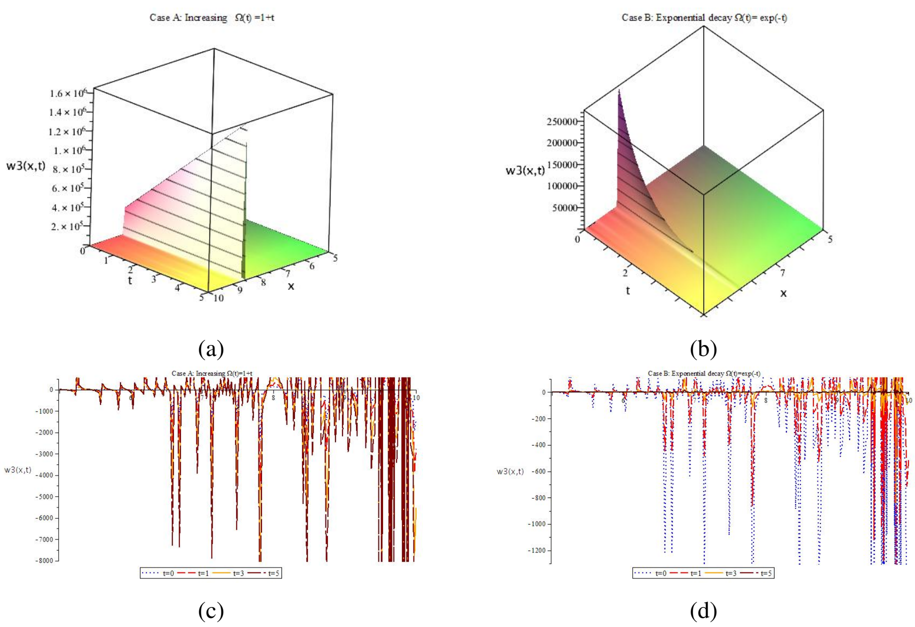


Figure 3: Profiles of solutions $w_3(x,t)$: Growing oscillatory solutions (a,c) are obtained under $\Omega(t) = 1 + t$, and decaying oscillatory solutions (b,d) are obtained under $\Omega(t) = e^{-t}$.

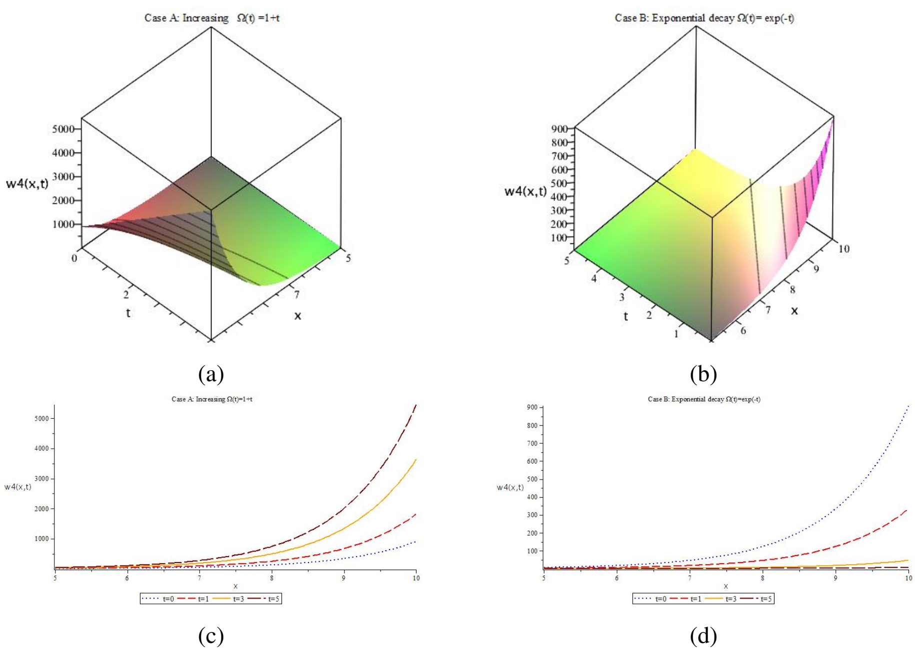


Figure 4: Parabolic-type solutions $w_4(x,t)$: (a,c) $\Omega(t) = 1 + t$ results in unbounded oscillatory growth, whereas (b,d) $\Omega(t) = e^{-t}$ creates damped oscillations.

$$w_8(x,t) = -\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[1-k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\tanh\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)\right], \tag{3.11b}$$

$$w_9(x,t) = -\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[1+k_1e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1+k_2\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{k_2-\tan\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}\right], \tag{3.11c}$$

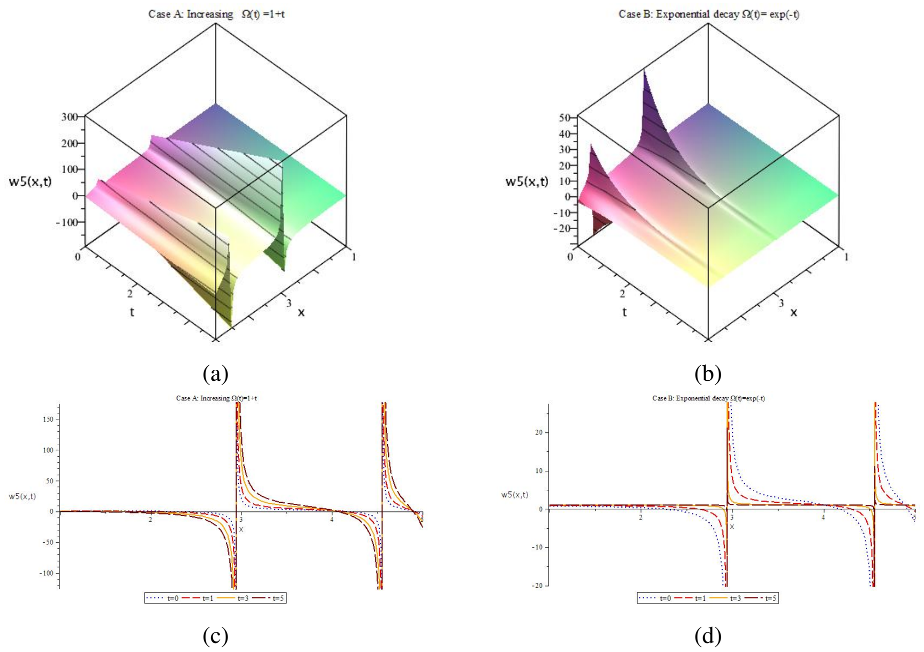


Figure 5: Singular and kink-type solutions $w_5(x,t)$: singular solution with unbounded oscillatory growth (a,c) are obtained under $\Omega(t) = 1 + t$, whereas kink-type solution with small, damped oscillations (b,d) are obtained under $\Omega(t) = e^{-t}$.

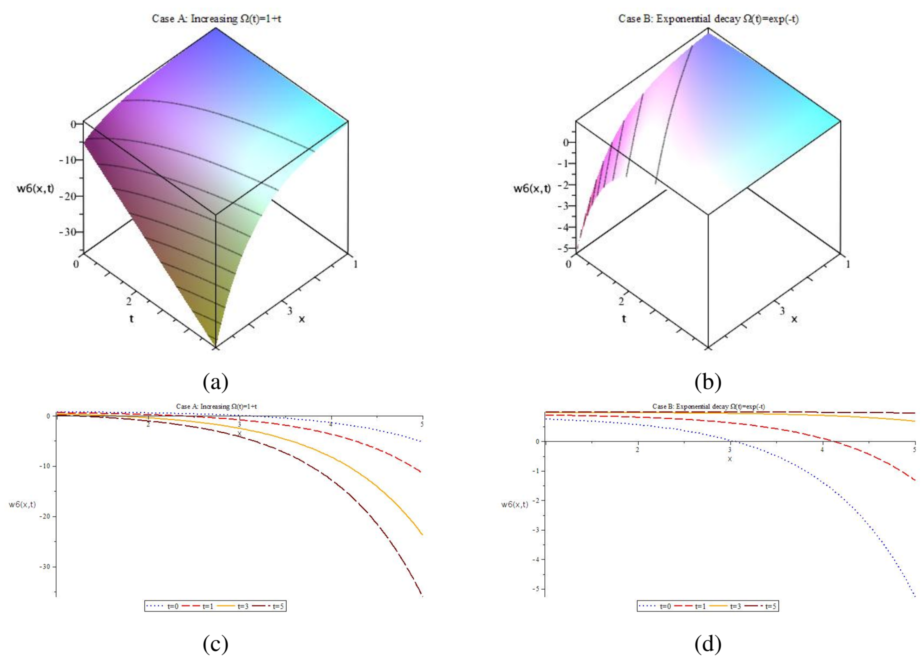


Figure 6: Singular and kink-type solutions, $w_6(x,t)$: For $\Omega(t) = 1 + t$, (a,c) results in a singular solution with unbounded oscillatory growth, whereas (b,d) $\Omega(t) = e^{-t}$ gives a kink-type solution with small, damped oscillations.

$$w_{10}(x,t) = -\frac{1}{2}\Omega(t)+\frac{1}{2}+\frac{1}{2}\sqrt{c_1\Omega(t)+\Omega(t)^2+1}+\Omega(t)\left[1+k_1 e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\frac{1-k_2\tanh\left(-e^{\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}{k_2-\tanh\left(e^{-\frac{1}{2\delta}(\varsigma-\mu\tau)}\right)}\right]. \tag{3.11d}$$

Here k_1, k_2 are arbitrary constants and $A, B, \varsigma, \tau, q_1(t), q_2(t)$, and $q_3(t)$ are given in (3.5).

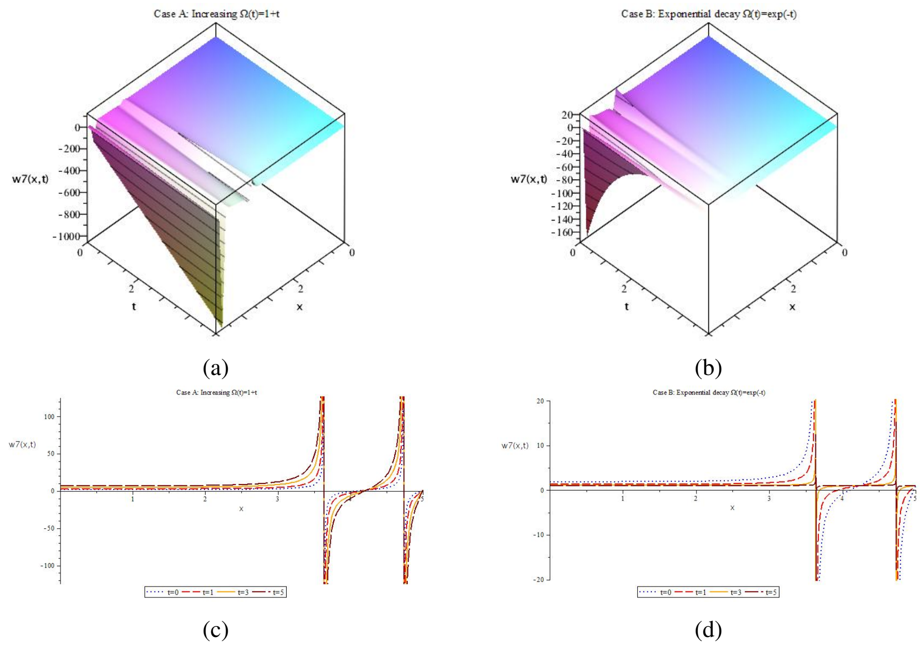


Figure 7: The solutions $w_7(x,t)$ in (a,c) with $\Omega(t) = 1 + t$ and (b,d) with $\Omega(t) = e^{-t}$ exhibit singular blow-up behavior with unbounded growth.

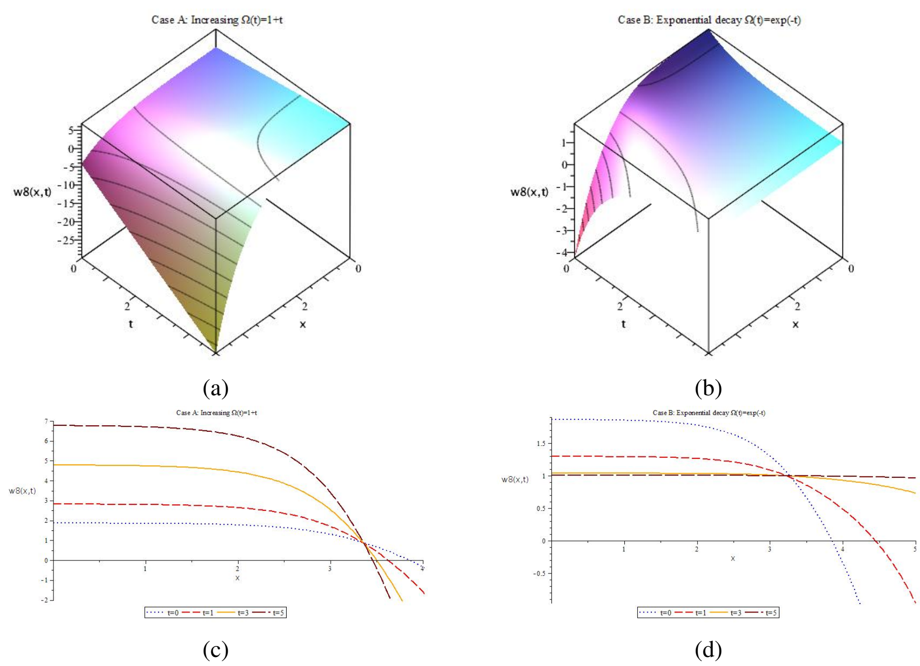
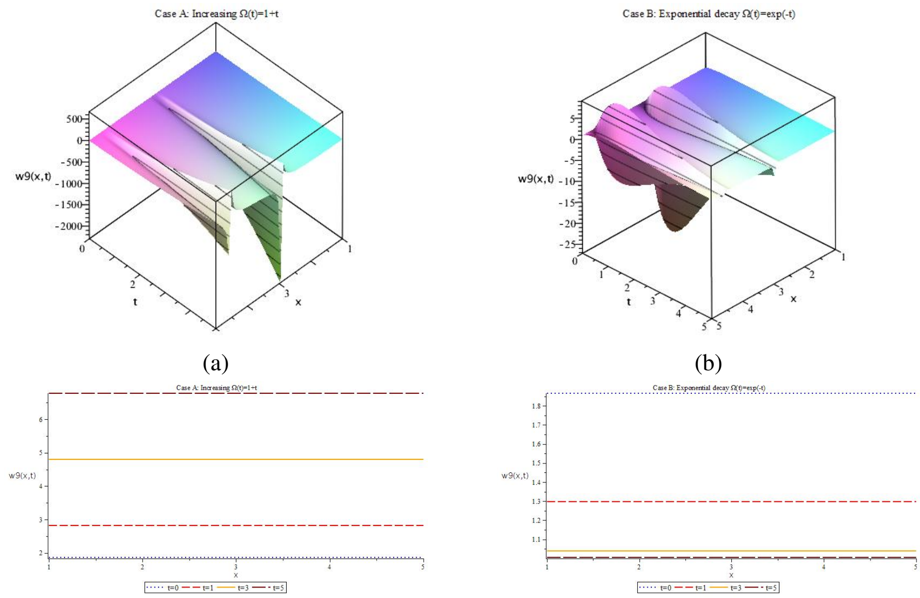


Figure 8: Kink-type wave profiles of solutions $w_8 x, t$: (a,c) represents singular blow-up behaviour for $\Omega(t) = 1 + t$, and (b,d) represents regular bounded behaviour with damped oscillations for $\Omega(t) = e^{-t}$.

4 Results and Discussions

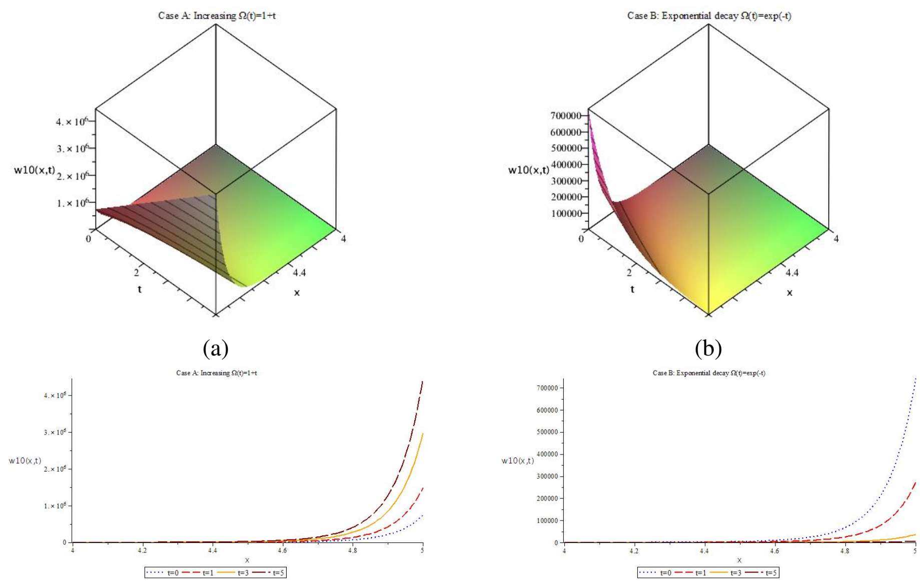
Three dimensional plots of obtained solutions are discussed in this section. We derive various kinds of exact solutions each with distinct physical interpretations. Solutions with kink profiles exhibit sudden changes between two steady states. These are frequently observed in traffic flow models. They also show up in population dynamics when species densities rapidly shift between two points of equilibrium. Situations in which the solution includes discontinuities or becomes



(a) (b)

(c) (d)

Figure 9: Profiles of solutions $w_9(x,t)$: (a,c) show singular behavior with unbounded growth for $\Omega(t) = 1 + t$, whereas (b,d) show regular behavior with bounded, damped oscillations for $\Omega(t) = e^{-t}$.



(a) (b)

(c) (d)

Figure 10: Regular travelling-wave solutions $w_{10}(x,t)$: (a,c) unbounded growth for $\Omega(t) = 1+t$, and (b,d) damped decay for $\Omega(t) = e^{-t}$.

unbounded are referred to as singular solutions. Events such as unexpected shock formation in fluid flow or blow-up behavior in chemical reactions are reflected by singular solutions. Stable wave propagation is modeled by regular solutions, which remain continuous and smooth without sudden shifts. These can show the continuous increase of a biological population or the slow spread of toxins in the environment. Equations (3.9a)*,* (3.9b)*,* (3.10a) *to* (3.10d) *and* (3.11a) *to* (3.11d) *provide solutions of time dependent generalized Burgers Fisher equation* (1.1)*.The plotting of the figures is carried in software Maple for randomly selected parameter values of functions and constants. The study of the graphical behavior of solutions is as follows.*

The time-varying coefficients $\Omega(t)$ explicitly determine the behavior of the wave in the long term. For $\Omega(t) = 1 + t$, the growing coefficient strengthens diffusion/forcing, resulting in normal but unbounded growth of the solution. For $\Omega(t) = e^{-t}$, the decreasing coefficient reduces the effect over time, resulting in normal bounded reduction. Therefore, time-dependent diffusion terms control whether wave propagation builds up or weakens with time.

5 Conclusion remarks

In this study, we have combined the direct method with constant-coefficient Burgers-Fisher equation to find exact solutions of the time-dependent generalized Burgers-Fisher equation. The approach's efficiency is illustrated by the derived solutions, which are shown in equations (3.9a), (3.9b), (3.10a) -(3.10d), and (3.11a)-(3.11d), together with their graphical representations in figures 1 - 10. The solutions are categorized as singular, regular, traveling wave, and kink-type forms, each of which reflects unique wave structures and propagation properties. The outcomes demonstrate that the solutions effectively capture essential wave patterns. A time-dependent diffusion term is crutial: more diffusion disperses the wave and adds dissipation, whereas less diffusion preserves sharp fronts and allows stable propagation. Future research may address higher-dimensional forms of the equation, extend the method to other nonlinear evolution equations with variable coefficients, and validate the analytical results through numerical simulations to study stability and long-term dynamics.

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