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INTRODUCTION TO HOM-POLYGROUPS

E.O. Adeleke[1], A.A.A. Agboola[2*], B. Davvaz[3] and M.A. Ibrahim[4]

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* Corresponding author.

Abstract A Hom-group is a nonassociative generalization of a group while a polygroup is a generalization of a group, and, a special case of a hypergroup. In this paper, we introduce the notion of a Hom-polygroup as a generalization of a polygroup. We present several foundational properties of Hom-polygroups with examples. We study homomorphisms of Hom-polygroups, and show that the kernel of a Hom-polygroup homomorphism is normal. We also study the quotient structure of Hom-polygroups, and present the first, second and third isomorphism theorems of Hom-polygroups.

1 Introduction

A Hom-group is a nonassociative generalization of a group introduced by Laurent-Gengouxa et al. in [19] which was greatly expanded by Hassanzadeh in [16, 17] and Liang et al. in [20]. Agboola et al. introduced the concept of neutrosophic Hom-group in [1] and they provided more properties of Hom-groups in [2]. The theory of hyperstructure was born by Marty in 1934 at the 8$^{\text{th}}$ Congress of Scandinavian Mathematicians. In [21], the concept of hypergroup was first introduced as a generalization of the classical group. Since the introduction of hypergroups by Marty in 1934, many researchers have expanded the concept and made several generalizations. Some of the researchers who have made significant contributions to the development of hypergroups include Ameri and Zahedi [3], Corsini [7], Corsini and Leoreanu [8], Davvaz and Leoreanu [13], Vougiokls [22, 23, 24] and a host of other researchers. A polygroup is a special class of a hypergroup introduced by Comer [5, 6]. The concept was later developed by Davvaz [9, 10, 11, 12, 14], Ghadiri and Waphare [15] and many more scholars. In the present paper, we introduce the notion of a Hom-polygroup as a generalization of a Hom-group. We present the foundational properties of Hom-polygroups. We study homomorphisms of Hom-polygroups, and show that the kernel of a Hom-polygroup homomorphism is normal. We also study the quotient structure of Hom-polygroups, and present the first, second and third isomorphism theorems of Hom-polygroups.

2 Preliminaries

In this section, we will recall some basic definitions, notations, examples and results relating to Hom-groups as presented by Adeleke et al. in [1], Agboola et al. in [2], Basdouri et al. in [4], Hassanzadeh in [16] & [17], Jiang et al. in [18] and Liang et al. in [20]; and those concerning hypergroups and polygroups as can be found respectively in Davvaz & Leoreanu-Fotea [13] and Davvaz in [14] which we are going to use in the sequel.

2.1 Hom-groups

Definition 2.1. Let G be a nonempty set, $* : G \times G \to G$ a binary operation on G, $\alpha : G \to G$ a bijective set map and $1 \in G$ a distinguished element. The quadruple $(G, *, \alpha, 1)$ is called a Hom-group if the following conditions hold:

(i) The product map α satisfies the Hom-associativity property

$$\alpha(g) * (h * k) = (g * h) * \alpha(k) \quad \forall g, h, k \in G.$$

(ii) The product map α is multiplicative that is

$$\alpha(g * h) = \alpha(g) * \alpha(h) \quad \forall g, h \in G.$$

(iii) The element $1 \in G$ called the unit element satisfies the Hom-unitary conditions

$$g * 1 = 1 * g = \alpha(g) \quad \forall g \in G.$$

(iv) For every element $g \in G$, there exits an element $g^{-1} \in G$ such that

$$g * g^{-1} = g^{-1} * g = 1.$$

(v) For any $g \in G$, there exists $k \in \mathbb{N}$ satisfying the Hom-invertibility condition

$$\alpha^k(g * g^{-1}) = \alpha^k(g^{-1} * g) = 1.$$

The smallest such k is denoted as the invertibility index of g. If the invertibility index of $g \in G$ is k, then the invertibility index of $\alpha(g)$ is $k - 1$.

If only conditions (i) and (ii) are satisfied, G is called a Hom-semigroup. A Hom-semigroup with condition (iii) is called a Hom-monoid and a Hom-monoid with condition (iv) is called a Hom-group.

For simplicity except otherwise, $g * h$ will sometimes be written as gh.

Example 2.2. Let $\mathbb{C}$ be the set of complex numbers and let $* : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be the binary operation on $\mathbb{C}$ defined by $z_1 * z_2 = \overline{z_1 + z_2}$ $\forall z_1, z_2 \in \mathbb{C}$. Let $\alpha : \mathbb{C} \to \mathbb{C}$ be a mapping defined by $\alpha(z) = \overline{z}$ $\forall z \in \mathbb{C}$. Then $(\mathbb{C}, *, \alpha, 0)$ is an abelian Hom-group.

Proposition 2.3. *Let G be a Hom-group. The unit element of G is unique.*

Proposition 2.4. *Let G be a Hom-group. Then $\forall g, h \in G$:*

(i) $\alpha(1) = 1$;

(ii) g^{-1} *is unique;*

(iii) $(g^{-1})^{-1} = g$;

(iv) $(gh)^{-1} = h^{-1}g^{-1}$;

(v) $(\alpha(g))^{-1} = \alpha(g^{-1})$.

Proposition 2.5. *If the elements g, h, k in a Hom-group G satisfy $gh = gk$ or $hg = kg$, then $h = k$.*

Proposition 2.6. *Let G be a Hom-group. Then $\forall g, h, k \in G$:*

(i) $\alpha^{-1}(gh) = \alpha^{-1}(g)\alpha^{-1}(h)$;

(ii) $(\alpha^{-1}(g)h)k = g(h\alpha^{-1}(k))$.

Proposition 2.7. *Let (G, μ) be a group and let $\alpha : G \to G$ be a group automorphism. Then $(G, \alpha \circ \mu, \alpha)$ is a Hom-group.*

Definition 2.8. Let H be a nonempty subset of a Hom-group (G, α) that is closed under the binary operation in G. H is said to be a Hom-subgroup of G if (H, α) is itself is a Hom-group under the binary operation inherited from G and we write $H \leq G$.

Definition 2.9. Let H be a Hom-subgroup of a Hom-group G and let $g \in G$. The sets gH an Hg are defined respectively by $\{gh : h \in H\}$ and $\{hg : h \in H\}$. gH is called a Hom-left coset of H in G while Hg is called a Hom-right coset of H in G. The set of all distinct Hom-left cosets of H in G is denoted by G/H.

Generally, $gH \neq Hg$ except if G is abelian. However, if $gH = Hg \; \forall g \in G$, then H is called a Hom-normal subgroup of G and we write $H \lhd G$.

Proposition 2.10. *Let H be a Hom-subgroup of a finite Hom-group G. For all $g, h \in G$, the following statements are equivalent:*

(i) $gH = hH$;

(ii) $gH \cap hH \neq \emptyset$;

(iii) $g^{-1}h \in H$;

(iv) $\alpha(h) \in gH$;

(v) $\alpha(g)H = \alpha(h)H$.

Proposition 2.11. *Let H be a Hom-subgroup of a Hom-group G. For all $g \in G$, the following statements are equivalent:*

(i) $gH = Hg$;

(ii) for $h \in H$, $(gh)\alpha(g^{-1}) \in H$;

(iii) $(gH)\alpha(g^{-1}) \subseteq H$;

(iv) $(gH)\alpha(g^{-1}) = H$;

(v) $\alpha(g)H = H\alpha(g)$.

Proposition 2.12. *Let H be a Hom-normal subgroup of a Hom-group $(G, *, \alpha)$. Then $(G/H, \odot, \beta)$ is a Hom-group where $\odot$ is defined $\forall aH, bH \in G/H$ by $aH \odot bH = a * bH$ and β is defined $\forall aH \in G/H$ by $\beta(aH) = \alpha(a)H$.*

Definition 2.13. Let (G, α) and (H, β) be two Hom-groups. The map $\phi : G \to H$ is called a Hom-group homomorphism if ϕ satisfies the following two conditions:

(i) $\forall g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$;

(ii) for each $g \in G$, $\beta(\phi(g)) = \phi(\alpha(g))$.

In addition, if ϕ is a bijection, then we call ϕ an isomorphism and we write $G \cong H$.

Definition 2.14. The map $\phi : (G, \alpha) \to (H, \beta)$ is called a weak Hom-group homomorphism if $\phi(1_G) = 1_H$ and $\beta \circ \alpha(gk) = (\phi \circ \alpha(g))(\phi \circ \alpha(k)) \; \forall g, k \in G$.

2.2 Hypergroups and Polygroups

Definition 2.15. Let H be a non-empty set and $\star : H \times H \to \mathcal{P}^*(H)$ be a hyperoperation. The couple $(H, \star)$ is called a hypergroupoid. For any two non-empty subsets A and B of H and $x \in H$, we define

$$A \star B = \bigcup_{a \in A, b \in B} a \star b, A \star x = A \star \{x\} \text{ and } x \star B = \{x\} \star B.$$

Definition 2.16. A hypergroupoid $(H, \star)$ is called a *semihypergroup* if $\forall a, b, c \in H$, we have $(a \star b) \star c = a \star (b \star c)$, which means that

$$\bigcup_{u \in a \star b} u \star c = \bigcup_{v \in b \star c} a \star v.$$

A hypergroupoid $(H, \star)$ is called a *quasihypergroup* if $\forall a \in H$, we have $a \star H = H \star a = H$. This condition is also called the reproduction axiom.

Definition 2.17. A hypergroupoid $(H, \star)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

Example 2.18. Let H be a normal subgroup of a group $(G, *)$. Define the set $G/H = \{xH : x \in G\}$. Then $(G/H, \circledast)$ is a hypergroup where $\forall xH, yH \in G/H$, $xH \circledast yH = \{zH : z \in x * y\}$.

Definition 2.19. A nonempty subset K of a hypergroup $(H, *)$ is called a subhypergroup if it is a hypergroup.

If K is a subhypergroup of a hypergroup $(H, *)$, then $\forall x \in K$, $x * K = K * x = K$.

Definition 2.20. Let $(H, \star)$ and $(K, \circ)$ be two hypergroups. A map $\phi : H \to K$, is called

(i) an inclusion homomorphism if $\forall x, y \in H$, we have $\phi(x \star y) \subseteq \phi(x) \circ \phi(y)$;

(ii) a good homomorphism if $\forall x, y \in H$, we have $\phi(x \star y) = \phi(x) \circ \phi(y)$;

(iii) a good isomorphism if it is a bijective good homomorphism and we write $H \cong K$.

Definition 2.21. Let $(H, *)$ be a hypergroup. H is called a canonical hypergroup if the following conditions hold:

(i) it is commutative;

(ii) there exists $e \in H$ such that $x * e = e * x = x$, $\forall x \in H$;

(iii) for all $x \in H$, there exists a unique $x^{-1} \in H$ such that $e \in (x * x^{-1}) \cap (x^{-1} * x)$;

(iv) for all $x, y, z \in H$, if $x \in y * z$, then $y \in x * z^{-1}$ and $z \in y^{-1} * x$.

Example 2.22. Let $C(n) = \{e_0, e_1, \cdots, e_{k(n)}\}$, where

$$k(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is an even natural number,} \\ \frac{n-1}{2}, & \text{if } n \text{ is an odd natural number.} \end{cases}$$

For all $e_s, e_t \in C(n)$, define $e_s * e_t = \{e_p, e_v\}$, where $p = \min\{s + t, n - (s + t)\}$, $v = \mid s - t \mid$. Then $(C(n), *)$ is a canonical hypergroup.

Definition 2.23. A polygroup is a system $\mathcal{P} = (P, *, e, ^{-1})$ where $e \in P$ is a unit element, $^{-1} : P \to P$ is a unitary operation, $* : P \times P \to \mathcal{P}(P)$ is a hyperoperation, and the following axioms hold $\forall x, y, z \in P$:

(i) $x * (y * z) = (x * y) * z$;

(ii) $x * e = e * x = x$;

(iii) $x \in y * z$ implies that $y \in x * z^{-1}$ and $z \in y^{-1} * x$.

The following results are consequences of the axioms of a polygroup $(P, *, e, ^{-1})$.

(i) $e^{-1} = e$;

(ii) $\forall x \in P$, $x^{-1} = x$;

(iii) $\forall x, y \in P$, $(x * y)^{-1} = y^{-1} * x^{-1}$; and

(iv) $\forall x \in P$, $e \in (x * x^{-1}) \cap (x^{-1} * x)$;
where $A^{-1} = \{a^{-1} : a \in A\}$.

Definition 2.24. A polygroup P in which every element has order 2 that is $x^{-1} = x$ $\forall x \in P$ is called a symmetric polygroup. It can easily be shown that a symmetric polygroup P is commutative.

Example 2.25. Let H be a subgroup of a group $(G, *)$. Define a system $G//H = (\{HgH : g \in G\}, \circledast, H, ^{-1})$ where $(HgH)^{-1} = Hg^{-1}H$ and

$$(Hg_1H) \circledast (Hg_2H) = \{Hg_1hg_2H : h \in H\}.$$

Then $(G//H, \circledast, H, ^{-1})$ is a polygroup.

Definition 2.26. Let P be a polygroup. A nonempty subset K of P is called a subpolygroup of P written $K \leq P$ if:

(i) $x, y \in K$ implies that $xy \in K, \forall x, y \in K$;

(ii) $x \in K$ implies that $x^{-1} \in K, \forall x \in K$.

Definition 2.27. Let N be a subpolygroup of a polygroup P. N is said to be normal in P written $N \lhd P$ if $xNx^{-1} \subseteq N, \forall x \in N$.

Definition 2.28. Let $(P, *, e_P, ^{-1})$ and $(Q, \star, e_Q, ^{-1})$ be polygroups. Let $\phi : P \to Q$ be a mapping such that $\phi(e_P) = e_Q$. Then, ϕ is called:

(i) an inclusion homomorphism if $\phi(x * y) \subseteq \phi(x) \star \phi(y), \forall x, y \in P$;

(ii) a strong or good homomorphism if $\phi(x * y) = \phi(x) \star \phi(y), \forall x, y \in P$;

(iii) a strong or good isomorphism if ϕ is a bijective strong or good homomorphism and we write $P \cong Q$.

Definition 2.29. Let $\phi : P \to Q$ be a strong polygroup homomorphism.

(i) The kernel of ϕ denoted by Kerϕ is defined by Ker$\phi = \{x \in P : \phi(x) = e_Q\}$.

(ii) The image of ϕ denoted by Imϕ is defined by Im$\phi = \{y \in Q : y = \phi(x), \text{ for some } x \in P\}$.

3 Hom-polygroups

In this section, we present the notion of a Hom-polygroup as a generalization of a Hom-group. We present several foundational properties of Hom-polygroups with examples. We study homomorphisms of Hom-polygroups, and show that the kernel of a Hom-polygroup homomorphism is normal. We also study the quotient structure of Hom-polygroups, and present the first, second and third isomorphism theorems of Hom-polygroups.

Definition 3.1. A Hom-Polygroup is a quintuplet $(P, e, *, ^{-1}, \alpha)$ consisting of a nonempty set P together with a unit element e, a hyperoperation $*$, a unitary operation $^{-1} : P \to P$, and a bijective map $\alpha : P \to P$ such that the following axioms are satisfied:

(i) for all $x, y, z \in P$,
$$\alpha(x) * (y * z) = (x * y) * \alpha(z),$$

(ii) the map α is multiplicative, i.e., for all $x, y \in P$,
$$\alpha(x * y) = \alpha(x) * \alpha(y),$$

(iii) for all $x \in P$,
$$x * e = e * x = \{\alpha(x)\} := \alpha(x), \alpha(e) = \alpha^{-1}(e) = e,$$

(iv) for all $x, y, z \in P$,
$$x \in y * z \;\Rightarrow y \in x * z^{-1} \text{ and } z \in y^{-1} * z,$$

(v) for any $x \in P$ there exists $n \in \mathbb{N}$ satisfying the condition
$$e \in \alpha^n(x * x^{-1}) = \alpha^n(x^{-1} * x).$$

The smallest such n is denoted as the invertibility index of x.

Lemma 3.2. *Let $(P, e, *, ^{-1}, \alpha)$ be a Hom-polygroup. Then*

(i) $e^{-1} = e$,

(ii) $e \in (\alpha(x) * \alpha(x^{-1})) \cap (\alpha(x^{-1}) * \alpha(x)) \; \forall x \in P$.

Proof. (i) By definition,

$$\begin{aligned} e * e &= \alpha(e) = e \\ \therefore e^{-1} &= e. \end{aligned}$$

(ii) Consider

$$\begin{aligned} (\alpha(x) * \alpha(x^{-1})) \cap (\alpha(x^{-1}) * \alpha(x)) &= \alpha(xx^{-1}) \cap (\alpha(x^{-1}x) \\ &= \{e\} \cap \{e\} \\ &= \{e\} \\ \therefore e &\in (\alpha(x) * \alpha(x^{-1})) \cap (\alpha(x^{-1}) * \alpha(x)). \end{aligned}$$

□

Proposition 3.3. *Let $(P, e, *, ^{-1}, \alpha)$ be a Hom-polygroup. Then*

(i) $(\alpha(x))^{-1} = \alpha(x^{-1})\ \forall x \in P$*;*

(ii) $\alpha^{-1}(xy) = \alpha^{-1}(x)\alpha^{-1}(y)\ \forall x, y \in P$*.*

Proof. (i) By definition,

$$\begin{aligned} e &\in \alpha(x * x^{-1}) \\ &= \alpha(x)\alpha(x^{-1}) \\ \therefore (\alpha(x))^{-1} &= \alpha(x^{-1}). \end{aligned}$$

(ii) Consider

$$\begin{aligned} xy &= \alpha(\alpha^{-1}(xy)) \\ &= \alpha(\alpha^{-1}(\alpha\alpha^{-1}(x)\alpha\alpha^{-1}(y)) \\ &= \alpha(\alpha^{-1}\alpha(\alpha^{-1}(x)\alpha^{-1}(y)) \\ &= \alpha(\alpha^{-1}(x)\alpha^{-1}(y)) \quad \text{[since } \alpha \text{ is invertible, we have]} \\ \alpha^{-1}(xy) &= \alpha^{-1}(x)\alpha^{-1}(y). \end{aligned}$$

□

Proposition 3.4. *Let $(P, e, *, ^{-1}, \alpha)$ be a Hom-polygroup. Then for each $x \in P$, the inverse element x^{-1} is unique.*

Proof. Suppose that a and b are two left inverses of x. Then $e \in \alpha(ax) = \alpha(a)\alpha(x)$ and $e \in \alpha(bx) = \alpha(b)\alpha(x)$. Now,

$$\begin{aligned} e &\in \alpha(a)\alpha(x) \\ \Rightarrow \alpha(a) &\in e(\alpha(x))^{-1} = e\alpha(x^{-1}) = \alpha^2(x^{-1}) \quad \text{[since } \alpha \text{ is invertible, we have]} \\ a &\in \alpha(x^{-1}). \quad \text{[Similarly we have]} \\ b &\in \alpha(x^{-1}) \\ \therefore a &= b. \end{aligned}$$

The same can be done for the right inverses of x. Hence the proof. □

Proposition 3.5. *Let $(P, e, *, ^{-1}, \alpha)$ be a Hom-polygroup. Then*

(i) $(\alpha^{-1}(x))^{-1} = \alpha^{-1}(x^{-1})\ \forall x \in P$*;*

(ii) $(\alpha^{-1}(x^{-1}))^{-1} = \alpha^{-1}(x)\ \forall x \in P$*;*

(iii) $(\alpha(x)\alpha^{-1}(x^{-1}))^{-1} = \alpha^{-1}(x)\alpha(x^{-1}) = \alpha^{-1}(x)\alpha^2(x^{-1}) = x\alpha(x^{-1})\ \forall x \in P$*.*

Proof. (i) It is sufficient to show that the inverse of $\alpha^{-1}(x)$ is $\alpha^{-1}(x^{-1})$. To this end, consider

$$\begin{aligned}\alpha^{-1}(x)\alpha^{-1}(x^{-1}) &= \alpha^{-1}(xx^{-1}) = \{e\} \quad \text{and also,}\\ \alpha^{-1}(x^{-1})\alpha^{-1}(x) &= \alpha^{-1}(x^{-1}x) = \{e\}.\end{aligned}$$

The required result follows from the uniqueness of the inverse element in P.
(ii) It suffices to show that the inverse of $\alpha^{-1}(x^{-1})$ is $\alpha^{-1}(x)$. To this end, consider

$$\begin{aligned}\alpha^{-1}(x^{-1})\alpha^{-1}(x) &= \alpha^{-1}(x^{-1}x) = \{e\} \quad \text{and also,}\\ \alpha^{-1}(x)\alpha^{-1}(x^{-1}) &= \alpha^{-1}(xx^{-1}) = \{e\}.\end{aligned}$$

The required result follows from the uniqueness of the inverse element in P.
(iii)

$$\begin{aligned}(\alpha(x)\alpha^{-1}(x^{-1}))^{-1} &= (\alpha^{-1}(x^{-1}))^{-1}(\alpha(x))^{-1}\\ &= \alpha^{-1}(x)\alpha(x^{-1}). \qquad (3.1)\\ &= \alpha(\alpha^{-1}(xx^{-1})[\alpha^{-1}(x)\alpha(x^{-1})]\\ &= [\alpha^{-1}(xx^{-1})\alpha^{-1}(x)]\alpha^2(x^{-1})\\ &= \alpha^{-1}(x)\alpha^2(x^{-1}). \qquad (3.2)\end{aligned}$$

From (3.1), we have

$$\begin{aligned}(\alpha(x)\alpha^{-1}(x^{-1}))^{-1} &= \alpha^{-1}(x)\alpha(x^{-1})\\ &= [\alpha^{-1}(x)\alpha(x^{-1})]\alpha(\alpha^{-1}(xx^{-1})\\ &= \alpha(\alpha^{-1}(x))[\alpha(x^{-1})\alpha^{-1}(xx^{-1})]\\ &= x\alpha(x^{-1}). \qquad (3.3)\end{aligned}$$

The required results follow from (3.1), (3.2) and (3.3). □

Proposition 3.6. *Let $(P, e, *, ^{-1}, \alpha)$ be a Hom-polygroup. Then*

(i) $(x * y)^{-1} = y^{-1} * x^{-1}\ \forall x, y \in P$;

(ii) $x \in y * z \Rightarrow x^{-1} \in z^{-1} * y^{-1}\ \forall x, y, z \in P$.

Proof. (i) Consider

$$\begin{aligned}(xy)(y^{-1}x^{-1}) &= \alpha\alpha^{-1}(xy)(y^{-1}x^{-1})\\ &= [\alpha^{-1}(xy)y^{-1}]\alpha(x^{-1})\\ &= [(\alpha^{-1}(x)\alpha^{-1}(y))\alpha\alpha^{-1}(y^{-1})]\alpha(x^{-1})\\ &= [x(\alpha^{-1}(y)\alpha^{-1}(y^{-1}))]\alpha(x^{-1})\\ &= [x(\alpha^{-1}(yy^{-1})]\alpha(x^{-1})\\ &\subseteq (xe)\alpha(x^{-1})\\ &= \alpha(x)\alpha(x^{-1})\\ &= \alpha(xx^{-1})\\ &= \{e\}.\end{aligned}$$

Also consider

$$\begin{aligned}(y^{-1}x^{-1})(xy) &= (y^{-1}x^{-1})\alpha\alpha^{-1}(xy)\\ &= \alpha^{-1}(y^{-1})[x^{-1}(\alpha^{-1}(x)\alpha^{-1}(y)]\\ &= \alpha^{-1}(y^{-1})[\alpha\alpha^{-1}(x^{-1})(\alpha^{-1}(x)\alpha^{-1}(y))]\\ &= \alpha^{-1}(y^{-1})[(\alpha^{-1}(x^{-1})\alpha^{-1}(x))y]\\ &= \alpha^{-1}(y^{-1})[\alpha^{-1}(x^{-1}x)y]\\ &\subseteq \alpha(y^{-1})(ey)\\ &= \alpha(y^{-1})\alpha(y)\\ &= \alpha(y^{-1}y)\\ &= \{e\}.\end{aligned}$$

Accordingly, $(x * y)^{-1} = y^{-1} * x^{-1}$.
(ii) Follows directly from (i). □

Example 3.7. Let $P = \{e, a\}$, $\alpha : P \to P$ be a mapping defined by $\alpha(x) = x\ \forall x \in P$ and let ${}^{-1} : P \to P$. Let $*$ be a hyperoperation defined in the Cayley table bellow.

$*$	e	a	b
e	e	a	b
a	a	a	$\{e, a, b\}$
b	b	$\{e, a, b\}$	b

${}^{-1}$ is defined by

${}^{-1}$	e	a	b
	e	b	a

It can easily be checked that $(P, e, *, {}^{-1}, \alpha)$ is a Hom-polygroup.

Definition 3.8. Let $(P, e, *, {}^{-1}, \alpha)$ be a Hom-polygroup. P is called a Hom-symmetric polygroup if every element of P has oder 2. That is $x^{-1} = x\ \forall x \in P$.

Proposition 3.9. *Let (P, α) be a Hom-symmetric polygroup. Then P is commutative.*

Proof. Suppose that P is a Hom-symmetric polygroup. Let $x, y \in P$ be arbitrary. Then $x^{-1} = x$ and $y^{-1} = y$. Now,

$$\begin{aligned}xy &= x^{-1}y^{-1}\\ &= (yx)^{-1} \quad \text{[from Proposition 3.6 (i)]}\\ &= yx.\end{aligned}$$

Hence, P is commutative. □

Proposition 3.10. *Let (P, α) be a Hom-polygroup and let $x, y \in P$. P is commutative if and only if $(xy)^{-1} = x^{-1}y^{-1}$.*

Proof. Suppose that $(xy)^{-1} = x^{-1}y^{-1}$. Then by Proposition 3.6 (i), we have

$$\begin{aligned}
y^{-1}x^{-1} &= x^{-1}y^{-1} \\
\Rightarrow \alpha^2(y^{-1}x^{-1}) &= \alpha^2(x^{-1}y^{-1}) \\
\Rightarrow \alpha^2(y^{-1})\alpha^2(x^{-1}) &= \alpha^2(x^{-1})\alpha^2(y^{-1}) \quad \text{[premultiply both sides by } \alpha^3(y) \text{ to have]} \\
e\alpha^3(x^{-1}) &= \alpha^2(yx^{-1})\alpha^3(y^{-1}) \quad \text{[postmultiply both sides by } \alpha^4(y) \text{ to have]} \\
e\alpha^3(x^{-1}y) &= \alpha^3(yx^{-1})e \\
\Rightarrow \alpha^4(x^{-1}y) &= \alpha^4(yx^{-1}) \\
\Rightarrow \alpha^4(x^{-1})\alpha^4(y) &= \alpha^4(y)\alpha^4(x^{-1}) \quad \text{[premultiply both sides by } \alpha^5(x) \text{ to have]} \\
e\alpha^5(y) &= [\alpha^4(x)\alpha^4(y)]\alpha^5(x^{-1}) \quad \text{[postmultiply both sides by } \alpha^6(x) \text{ to have]} \\
e\alpha^5(yx) &= \alpha^5(xy)e \\
\Rightarrow \alpha^6(yx) &= \alpha^6(xy) \quad \text{[since } \alpha \text{ is invertible, we should have]} \\
yx &= xy
\end{aligned}$$

and therefore P is commutative.

Conversely, suppose that P is commutative. Then

$$\begin{aligned}
(xy)^{-1} &= (yx)^{-1} \quad \text{[using Proposition 3.6 (i), we have]} \\
&= x^{-1}y^{-1}.
\end{aligned}$$

The proof is complete. □

Proposition 3.11. *Let (P, α) be a Hom-polygroup and let $x, y \in P$. P is commutative if and only if $(xy)^2 = x^2y^2$*

Proof. Suppose that $(xy)^2 = x^2y^2$. Then

$$\begin{aligned}
(xy)(xy) &= (xx)(yy) \\
\Rightarrow (xy)\alpha\alpha^{-1}(xy) &= (xx)\alpha\alpha^{-1}(yy) \\
\Rightarrow \alpha(x)[y\alpha^{-1}(xy)] &= \alpha(x)[x\alpha^{-1}(yy)] \quad \text{[premultiply both sides by } \alpha^2(x^{-1}) \text{ to have]} \\
e\alpha(y\alpha^{-1}(xy) &= e\alpha(x\alpha^{-1}(yy)) \\
\Rightarrow \alpha^2(y\alpha^{-1}(xy) &= \alpha^2(x\alpha^{-1}(yy)) \\
\Rightarrow \alpha^2(y)\alpha(xy) &= \alpha^2(x)\alpha(yy)) \\
\Rightarrow \alpha^2(y)[\alpha(x)\alpha(y)] &= \alpha^2(x)[\alpha(y)\alpha(y)] \\
\Rightarrow [\alpha(y)\alpha(x)]\alpha^2(y) &= [\alpha(x)\alpha(y)]\alpha^2(y) \quad \text{[postmultiply both sides by } \alpha^3(y^{-1}) \text{ to have]} \\
\alpha^2(yx)e &= \alpha^2(gh)e \\
\Rightarrow \alpha^3(yx) &= \alpha^3(xy)) \quad \text{[since } \alpha \text{ is invertible, we should have]} \\
yx &= xy
\end{aligned}$$

and therefore P is commutative.

Conversely, suppose that P is commutative. Then

$$\begin{aligned}(xy)^2 &= (xy)(xy)\\ &= (xy)\alpha\alpha^{-1}(xy)\\ &= \alpha(x)[y\alpha^{-1}(xy)]\\ &= \alpha(x)[\alpha\alpha^{-1}(y)(\alpha^{-1}(x)\alpha^{-1}(y)]\\ &= \alpha(x)[(\alpha^{-1}(y)(\alpha^{-1}(x))y]\\ &= \alpha(x)[(\alpha^{-1}(yx)y]\\ &= \alpha(x)[(\alpha^{-1}(xy)\alpha\alpha^{-1}(y)]\\ &= \alpha(x)[x(\alpha^{-1}(y)\alpha^{-1}(y))]\\ &= \alpha(x)[x\alpha^{-1}(y^2)]\\ &= x^2y^2.\end{aligned}$$

The proof is complete. □

Corollary 3.12. *Let (P, α) be a Hom-polygroup such that $x^2 = e\ \forall x \in G$. Then P is a Hom-symmetric polygroup. More generally, if P is a Boolean Hom-polygroup, then P is a commutative Hom-polygroup.*

Definition 3.13. Let H be a nonempty subset of a Hom-polygroup $(P, e, *, ^{-1}, \alpha)$. H is said to be a Hom-subpolygroup of P if H is also is a Hom-polygroup with the same operations and unit element.

We write $H \leq P$ if H is a Hom-subpolygroup of a Hom-polygroup P.

Proposition 3.14. *A nonempty subset H of a Hom-polygroup P is a Hom-subpolygroup if and only if the following conditions hold:*

(i) $x, y \in H$ implies that $xy \subseteq H\ \forall x, y \in H$;

(ii) $x \in H$ implies that $x^{-1} \in H\ \forall x \in H$.

Proposition 3.15. *A nonempty subset H of a Hom-polygroup P is a Hom-subpolygroup if and only if*

$$x, y \in H \ \text{ implies that } \ xy^{-1} \subseteq H \ \ \forall x, y \in H.$$

Definition 3.16. Let H and K be Hom-subpolygroups of a Hom-polygroup (P, α). The product HK is defined by

$$HK = \{hk : h \in H \ \text{ and } \ k \in K\}.$$

Definition 3.17. Let H be a Hom-subpolygroup of a Hom-polygroup (P, α) and let $x \in P$. xH (resp. Hx) the Hom-left coset of H in P (resp. the Hom-right coset of H in P) are defined by $xH = \{xh : h \in H\}$ (resp. $Hx = \{hx : h \in H\}$). The set of all Hom-left cosets of H in P is denoted by P/H.

Definition 3.18. Let N be a Hom-subpolygroup of a Hom-polygroup (P, α). N is called a Hom-normal subpolygroup of P if $xN = Nx\ \forall x \in P$. We write $N \lhd P$ if N is Hom-normal in P.

Proposition 3.19. *Let H and K be Hom-subpolygroups of a Hom-polygroup P. Then $H \cap K \leq P$.*

Proof. Suppose that $H, K \leq P$. Then $H \neq \emptyset$, $K \neq \emptyset$ and $\therefore H \cap K \neq \emptyset$. Let $x, y \in H \cap K$ be arbitrary. Then $x, y \in H$ and $x, y \in K$. Since $H \leq P$ and $K \leq P$, we should then have $xy^{-1} \subseteq H$ and $xy^{-1} \subseteq K$ from which we have $xy^{-1} \subseteq H \cap K$. Accordingly, $H \cap K \leq P$. □

Proposition 3.20. *Let H be a Hom-subpolygroup of a Hom-polygroup (P, α). For all $x, y \in P$, the following statements are equivalent:*

(i) $xH = yH$;

(ii) $xH \cap yH \neq \emptyset$;

(iii) $x^{-1}y \subseteq H$;

(iv) $\alpha(y) \subseteq xH$;

(v) $\alpha(x)H = \alpha(y)H$.

Proof. See [20]. □

Proposition 3.21. *Let H be a Hom-subpolygroup of a Hom-polygroup (P, α). For all $x \in P$, the following statements are equivalent:*

(i) $xH = Hx$;

(ii) for $y \in H$, $(xy)\alpha(x^{-1}) \subseteq H$;

(iii) $(xH)\alpha(x^{-1}) \subseteq H$;

(iv) $(xH)\alpha(x^{-1}) = H$;

(v) $\alpha(x)H = H\alpha(x)$.

Proof. See [20]. □

Proposition 3.22. *Let H and N be Hom-subpolygroups of a Hom-polygroup (P, α) such that $N \lhd P$. Then*

(i) $xN = Nx$, $\forall x \in P$;

(ii) $(xN)(yN) = (xy)N$, $\forall x, y \in P$;

(iii) $xN = yN$, $\forall y \in xN$;

(iv) $H \cap N \lhd H$.

(v) $HN = NH$ *is a Hom-subpolygroup of* P.

(vi) $N \lhd NH$.

Proof. (i) Suppose that $N \lhd P$. Then by Proposition 3.21 (ii), we have $(xm)\alpha(x^{-1}) \subseteq N$ $\forall m \in N$. This means that $\exists n \in N$ such that $(xm)\alpha(x^{-1}) = n$. Postmultiply both sides by $\alpha^2(x)$, we have

$$\begin{aligned} n\alpha^2(x) &= [(xm)\alpha(x^{-1})]\alpha^2(x) \\ &= \alpha(xm)e \\ &= \alpha^2(xm) \\ &= \alpha^2(x)\alpha^2(m) \end{aligned}$$

Since we can find $y \in P$ such that $y = \alpha^2(x)$ and find $t \in N$ such that $t = \alpha^2(m)$, it then follows that $yt = ny$ that is $yN = Ny$ and we are done.
(ii) By (i), $(xN)(yN) = (Nx)(yN)$. Then there exit $m, n \in N$ such that $(xn)(ym) = (nx)(ym)$. Now,

$$\begin{aligned} (xn)(ym) &= (nx)(ym) = \alpha\alpha^{-1}(nx)\alpha\alpha^{-1}(ym) \\ &= \alpha(n)[x\alpha^{-1}(ym)] \\ &= \alpha(n)[x(\alpha^{-1}(y)\alpha^{-1}(m))] \\ &= \alpha(n)[\alpha\alpha^{-1}(x)(\alpha^{-1}(y)\alpha^{-1}(m))] \\ &= \alpha(n)[(\alpha^{-1}(x)\alpha^{-1}(y))m] \\ &= \alpha(n)[(\alpha^{-1}(xy)m] \\ &= [n\alpha^{-1}(xy)]\alpha(m) \quad \text{[by (i), we have]} \\ &= [\alpha^{-1}(xy)n]\alpha(m) \\ &= (xy)(nm). \end{aligned}$$

Since $nm \in N$, the required result follows.
(iii) Suppose that $y \in xN$. Then $\exists m \in N$ such that $y = xm$. Now, let $z \in yN$. Then $\exists n \in N$ such that $z = yn = (xm)n$. Consider the following:

$$\begin{aligned} z &= (xm)n \\ &= \alpha\alpha^{-1}(xm)(ne) \\ &= [\alpha^{-1}(xm)n]\alpha(e) \\ &= [\alpha^{-1}(x)\alpha^{-1}(m)]n \\ &= [\alpha^{-1}(x)\alpha^{-1}(m)]\alpha\alpha^{-1}(n) \\ &= x(\alpha^{-1}(m)\alpha^{-1}(n)) \\ &= x\alpha^{-1}(mn). \end{aligned}$$

Since $\alpha^{-1}(mn) \in N$ it follows that $z \in xN$ and therefore, $xN \subseteq yN$.

Conversely, suppose that $y \in xN$. Then $\exists m \in N$ such that $y = xm$ and this implies that $x = ym^{-1}$. Now, let $z \in xN$. Then $\exists n \in N$ such that $z = xn = (ym^{-1})n$. Consider the following:

$$\begin{aligned} z &= (ym^{-1})n \\ &= \alpha\alpha^{-1}(ym^{-1})(ne) \\ &= [\alpha^{-1}(ym^{-1})n]\alpha(e) \\ &= [\alpha^{-1}(y)\alpha^{-1}(m^{-1})]n \\ &= [\alpha^{-1}(y)\alpha^{-1}(m^{-1})]\alpha\alpha^{-1}(n) \\ &= y(\alpha^{-1}(m^{-1})\alpha^{-1}(n)) \\ &= y\alpha^{-1}(m^{-1}n). \end{aligned}$$

Since $\alpha^{-1}(m^{-1}n) \in N$ it follows that $z \in yN$ and therefore, $yN \subseteq xN$. Hence, $xN = yN$.
(iv) That $H \cap N \leq H$ is clear. For Hom-normality, let $x \in H \cap N$ and $h \in H$. Now consider

$$\begin{aligned} (hx)\alpha(h^{-1}) &= \alpha(h)(xh^{-1}) \\ &\subseteq H \\ \therefore H \cap N &\lhd H. \end{aligned}$$

(v) See [20].
(vi) Let $n \in N$ and $x \in NH$. Since $N \lhd P$, it follows that

$$\begin{aligned} (xn)\alpha(x^{-1}) &\subseteq N \subseteq NH \\ \therefore (xn)\alpha(x^{-1}) &\subseteq NH \\ \therefore N &\lhd NH. \end{aligned}$$

□

Definition 3.23. Let (P, α) be a Hom-polygroup and let $N \leq P$. The relation $x \equiv y(\text{mod}N)$ is defined if and only if $(xy^{-1}) \cap N \neq \emptyset \; \forall x, y \in P$. This relation is defined by xN_Py.

Proposition 3.24. *The relation N_P is an equivalence relation.*

Proof. For reflexive and symmetric properties of N_P, see [13] Lemma 3.3.6 page 90. For transitive property, suppose that xN_Py and yN_Pz. Then $(xy^{-1}) \cap N \neq \emptyset$ and $(yz^{-1}) \cap N \neq \emptyset$. Now let $p \in (xy^{-1}) \cap N$ and $q \in (yz^{-1}) \cap N$. Then $p \in xy^{-1}$ and $p \in N$, $q \in yz^{-1}$ and $q \in N$ from we obtain $x \in py$ and $z^{-1} \in y^{-1}q$. Now consider the following:

$$\begin{aligned} xz^{-1} &\subseteq (py)(y^{-1}q) \\ &= (py)\alpha\alpha^{-1}(y^{-1}q) \\ &= \alpha(p)[y\alpha^{-1}(y^{-1}q)] \\ &= \alpha(p)[\alpha\alpha^{-1}(y)(\alpha^{-1}(y^{-1})\alpha^{-1}(q)] \\ &= \alpha(p)[(\alpha^{-1}(y)\alpha^{-1}(y^{-1}))q] \\ &= \alpha(p)[\alpha^{-1}(yy^{-1})q] \\ &= \alpha(p)[eq] \\ &= (pe)\alpha(q) \\ &= p\alpha(q) \subseteq N. \end{aligned}$$

We have just shown that $(xz^{-1}) \cap N \neq \emptyset$ and therefore xN_Pz. Since the relation N_P is reflexive, symmetric and transitive, it follows that it is an equivalence relation. The proof is complete. □

Definition 3.25. The equivalence class of any element $x \in P$ is denoted by $[x]N_P$ and we let $[P : N] = \{[x]N_P : x \in P\}$. We define a hyperoperation $\circledast$ on $[P : N]$ by

$$[x]N_P \circledast [y]N_P = \{[z]N_P : z \in [x]N_P * [y]N_P\}.$$

Lemma 3.26. *Let (P, α) be a Hom-polygroup and let $N \lhd P$. Then $[x]N_P = xN \ \forall x \in P$.*

Proof. Suppose that $y \in [x]N_P$. Then xN_Py meaning that $(x^{-1}y) \cap N \neq \emptyset$ which implies that $x^{-1}y \subseteq N$ so that $y \in xN$. Thus, $[x]N_P \subseteq xN$. Similarly, it can be shown that $xN \subseteq [x]N_P$ and therefore $[x]N_P = xN$.

We therefore conclude that $[P : N] = P/N$. □

Lemma 3.27. *Let (P, α) be a Hom-polygroup and let $N \lhd P$. Then $\forall x, y \in P$, $(xy)N = zN$ $\forall z \in xy$.*

Proof. Suppose that $z \in xy$. Then $\alpha(z) \in \alpha(xy)$ and $zN \subseteq (xy)N$. On the other hand, suppose that $p \in (xy)N$. Then $\exists n \in N$ such that $p \in (xy)n$. Now, consider the following:

$$\begin{aligned} p &\in (xy)n \\ &= (xy)\alpha\alpha^{-1}(n) \\ &= \alpha(x)[y\alpha^{-1}(n)] \\ \therefore \alpha(x) &\in p[y\alpha^{-1}(n)]^{-1} \\ &= p[\alpha(n^{-1})y^{-1}] \quad \text{[postmultiply both sides by } \alpha(y)\text{, we have]} \\ \alpha(x)\alpha(y) &\subseteq p[\alpha(n^{-1})y^{-1}]\alpha(y) \\ &= p[\alpha^2(n^{-1})e] \\ &= p\alpha^3(n^{-1}) \\ &\subseteq pN \quad \text{[from which we obtain]} \\ p &\in \alpha(x)\alpha(y)N = \alpha(xy)N = \alpha(z)N = zN \\ \therefore (xy)N &\subseteq zN \\ \therefore (xy)N &= zN \quad \forall z \in xy. \end{aligned}$$

□

Lemma 3.28. *Let (P, α) be a Hom-polygroup and let $N \lhd P$. Then $\forall x, y \in P$,*

$$[([x]N_P)([y]N_P)]N_P = ([x]N_P)([y]N_P).$$

Proof. By Lemma 3.26, we have that

$$([x]N_P)([y]N_P)]N_P = [(xN)(yN)]N_P = ((xN)(yN))N = ((xy)N)N.$$

By Lemma 3.27, we have that $\forall z \in xy$, $(xy)N = zN$ from which we have that $((xy)N)N = (zN)N$. Now, we claim that $(zN)N = zN$. For the proof of the claim, $\exists n \in N$ such that

$$\begin{aligned}
(zN)N &= (zn)n \\
&= (zn)\alpha\alpha^{-1}(n) \\
&= \alpha(z)(n\alpha^{-1}(n)) \\
&\subseteq \alpha(z)N \\
&= zN \\
\therefore ((xy)N)N &= (xy)N \\
&= (xN)(yN) \\
&= ([x]N_P)([y]N_P) \text{ that is} \\
([x]N_P)([y]N_P)]N_P &= ([x]N_P)([y]N_P).
\end{aligned}$$

The proof is complete. □

Proposition 3.29. *The system* $([P : N], [e]N_P, \circledast, ^{-1}, \alpha)$ *is a Hom-polygroup.*

Proof. For Hom-associativity, let $[p]N_P, [q]N_P, [r]N_P \in [P : N]$ be arbitrary where $p, q, r \in P$ and let $\alpha([p]N_P) = [\alpha(p)]N_P\ \forall p \in P$. Then

$$\begin{aligned}
\alpha([p]N_P) \circledast ([q]N_P \circledast [r]N_P) &= [\alpha(p)]N_P \circledast ([q]N_P \circledast [r]N_P) \\
&= [\alpha(p)]N_P \circledast \{[x]N_P : x \in ([q]N_P) * ([r]N_P)\} \\
&= \{[y]N_P : y \in (\alpha([p]N_P) * [x]N_P, x \in ([q]N_P) * ([r]N_P)\} \\
&= \{[y]N_P : y \in [\alpha(p)]N_P) * ((([q]N_P) * ([r]N_P))N_P\} \text{[by Lemma 3.28,]} \\
&= \{[y]N_P : y \in [\alpha(p)]N_P) * ((([q]N_P) * ([r]N_P))\} \text{[by Lemma 3.26,]} \\
&= \{[y]N_P : y \in (\alpha(p)N) * ((qN) * (rN))\} \\
&= \{[y]N_P : y \in (\alpha(p)N) * ((q * r)N))\} \\
&= \{[y]N_P : y \in ((\alpha(p) * (q * r))N))\} \\
&= \{[y]N_P : y \in ((p * q) * \alpha(r))N))\} \\
&= \{[y]N_P : y \in ((pN) * (qN)) * (\alpha(r)N))\} \\
&= \{[y]N_P : y \in (([p]N_P) * ([q]N_P)) * ([\alpha(r)]N_P))\} \\
&= \{[z]N_P : z \in (([p]N_P) * ([q]N_P))\} \circledast ([\alpha(r)]N_P)) \\
&= \{[w]N_P : w \in ([z]N_P) * [\alpha(r)]N_P\}, z \in ([p]N_P) * ([q]N_P) \\
&= \{[w]N_P : w \in ((([p]N_P) * ([q]N_P))N_P\} \circledast [\alpha(r)]N_P \\
&= \{[w]N_P : w \in ([p]N_P) * ([q]N_P))\} \circledast [\alpha(r)]N_P \\
&= ([p]N_P) \circledast ([q]N_P)) \circledast \alpha([r]N_P)
\end{aligned}$$

For multiplicativity of α, let $[x]N_P, [y]N_P \in [P : N]$ where $x, y \in P$. Then

$$\begin{aligned}\alpha([x]N_P \circledast [y]N_P) &= \{\alpha([z]N_P) : z \in [x]N_P * [y]N_P\} \\ &= \{[\alpha(z)]N_P : z \in [x]N_P * [y]N_P\} \\ &= \{[\alpha(z)]N_P : z \in (xN) * (yN)\} \\ &= \{[\alpha(z)]N_P : z \in (x * y)N\} \\ &= \alpha((x * y)N) \\ &= (\alpha(x) * \alpha(y))N \\ &= (\alpha(x)N) * (\alpha(y)N) \\ &= ([\alpha(x)]N_P) \circledast ([\alpha(y)]N_P) \\ &= \alpha([x]N_P) \circledast \alpha([y]N_P).\end{aligned}$$

Next, it is clear that $[e]N_P$ is a unit element in $[P : N]$ and $\forall [x]N_P \in [N : P]$, $([x]N_P)^{-1} = [x^{-1}]N_P$ is the inverse of $[x]N_P$. Now,

$$\begin{aligned}([e]N_P) \circledast ([x]N_P) &= \{[y]N_P : y \in ([e]N_P) * ([x]N_P)\} \\ &= \{[y]N_P : y \in (eN) * (xN)\} \\ &= \{[y]N_P : y \in (e * x)N\} \\ &= \{[y]N_P : y \in \alpha(x)N\} \\ &= \{[y]N_P : y \in [\alpha(x)]N_P\} \\ &= \{[\alpha(x)]N_P\} := \alpha([x]N_P) = ([x]N_P) \circledast ([e]N_P).\end{aligned}$$

Also, $\alpha([e]N_P) = [e]N_P$ and $\alpha^{-1}([e]N_P) = [e]N_P$.

For Hom-reversibility, suppose that $[x]N_P \subseteq ([y]N_P) \circledast ([z]N_P)$. Then $\exists u \in [x]N_P, v \in [y]N_P, w \in [z]N_P$ with $u, v, w \in P$ such that $u \in v * w$ which implies that $v \in u * w^{-1}$ and $w \in v^{-1} * u$. Therefore, $[v]N_P \subseteq ([u]N_P) * ([w^{-1}]N_P)$ and $[w]N_P \subseteq ([v^{-1}]N_P) * ([u]N_P)$. Hence, $([y]N_P) \subseteq ([x]N_P) \circledast ([z^{-1}]N_P)$ and $[z]N_P \subseteq ([y^{-1}]N_P) \circledast ([x]N_P)$.

Lastly, consider

$$\begin{aligned}[x]N_P \circledast [x^{-1}]N_P &= \{[y]N_P : y \in [x]N_P * [x^{-1}]N_P\} \\ &= \{[y]N_P : y \in (xN) * (x^{-1}N)\} \\ &= \{[y]N_P : y \in (x * x^{-1})N\} \\ &= \{[y]N_P : y \in eN\} \\ &= \{[y]N_P : y \in [e]N_P\} \\ &= \{[e]N_P\}.\end{aligned}$$

It follows that $[e]N_P \subseteq [x]N_P \circledast [x^{-1}]N_P = [x^{-1}]N_P \circledast [x]N_P$.
According to Definition 3.1, $([P : N], [e]N_P, \circledast, ^{-1}, \alpha)$ is a Hom-polygroup and the proof is complete. □

Corollary 3.30. *Let (P, α) be a Hom-polygroup and let $N \lhd P$. Then $(P/N, eN, \circledast, ^{-1}, \alpha)$ is a Hom-polygroup called Hom-quotient polygroup where $xN \circledast yN = \{zN : z \in xy\}$ and $(xN)^{-1} = x^{-1}N$.*

Definition 3.31. Let $(P, e_P, *, ^{-1}, \alpha)$ and $(Q, e_Q, \star, ^{-1}, \beta)$ be two Hom-polygroups. The mapping $\phi : P \to Q$ is called a strong Hom-polygroup homomorphism or a good Hom-polygroup homomorphism if the following conditions hold:

(i) $\phi(e_P) = e_Q$;

(ii) $\phi(x * y) = \phi(x) \star \phi(y)\ \forall x, y \in P$;

(iii) $\beta(\phi(x)) = \phi(\alpha(x)) \; \forall x \in P$.

If in addition ϕ is a bijection, then ϕ is called a Hom-polygroup isomorphism and we write $P \cong Q$.

Definition 3.32. Let (P, α) and (Q, β) be two Hom-polygroups. The mapping $\phi : P \to Q$ is called a weak Hom-good polygroup homomorphism if

$$\phi(e_P) = e_Q \;\text{ and }\; \beta \circ \phi(x * y) = (\phi \circ \alpha(x)) \star (\phi \circ \alpha(y)) \;\; \forall x, y \in P.$$

Definition 3.33. Let $\phi : (P, \alpha) \to (Q, \beta)$ be a good Hom-polygroup homomorphism.

(i) The image of ϕ denoted by Imϕ is defined by

$$\text{Im}\phi = \{y \in Q : y = \phi(x) \;\text{ for some }\; x \in P\}.$$

(ii) The kernel of ϕ denoted by Kerϕ is defined by

$$\text{Ker}\phi = \{x \in P : \phi(x) = e_Q\}.$$

Proposition 3.34. *Let $\phi : (P, \alpha) \to (Q, \beta)$ be a good Hom-polygroup homomorphism. Then $\phi(x^{-1}) = (\phi(x))^{-1} \; \forall x \in P$.*

Proof. Since P is a Hom-polygroup, we have $e_P \in \alpha(xx^{-1})$ and since ϕ is a strong Hom-polygroup homomorphism, we have

$$\begin{aligned}
\phi(e_P) &\in \phi(\alpha(xx^{-1})) \\
&= \beta(\phi(xx^{-1})) \\
&= \beta(\phi(x)\phi(x^{-1})) \\
&= \beta(\phi(x))\beta(\phi(x^{-1})) \quad \text{[by Hom-reversibility, we have]} \\
\beta(\phi(x^{-1})) &\in (\beta(\phi(x)))^{-1}\phi(e_P) \\
&= \beta((\phi(x))^{-1})\phi(e_P) \\
&= \beta((\phi(x))^{-1})e_Q \\
&= \beta((\phi(x))^{-1}) \quad \text{[since } \beta \text{ is invertible, we have]} \\
\phi(x^{-1}) &= (\phi(x))^{-1}.
\end{aligned}$$

□

Proposition 3.35. *Let $\phi : (P, \alpha) \to (Q, \beta)$ be a strong Hom-polygroup homomorphism. Then*

(i) Im$\phi \leq Q$;

(ii) Ker$\phi \lhd P$

Proof. (i) Since $\phi(e_P) = e_Q$, it follows that Im$\phi \neq \emptyset$. Let $u, v \in$ Imϕ be arbitrary. Then $\exists x, y \in P$ such that $u = \phi(x)$ and $v = \phi(y)$. Now,

$$\begin{aligned}
uv^{-1} &\subseteq \phi(x)(\phi(y))^{-1} \\
&= \phi(x)(\phi(y^{-1}) \\
&= \phi(xy^{-1}) \in \text{Im}\phi \\
\therefore \text{Im}\phi &\leq Q.
\end{aligned}$$

(ii) Since $\phi(e_P) = e_Q$, it follows that Ker$\phi \neq \emptyset$. Let $u, v \in$ Kerϕ be arbitrary. Then $\phi(u) = \phi(v) = e_Q$. Now,

$$\begin{aligned}
\phi(uv^{-1}) &= \phi(u)(\phi(v^{-1}) \\
&= \phi(u)(\phi(v))^{-1} \\
&= e_Q e_Q = \beta(e_Q) = e_Q \\
\therefore uv^{-1} &\subseteq \text{Ker}\phi \\
\therefore \text{Ker}\phi &\leq P.
\end{aligned}$$

For Hom-normality, let $p \in P$ and consider the following:

$$\begin{aligned}
\phi((pu)\alpha(p^{-1})) &= \phi(pu)\phi(\alpha(p^{-1})) \\
&= \phi(pu)\beta(\phi(p^{-1})) \\
&= [\phi(p)\phi(u)]\beta(\phi(p^{-1})) \\
&= [\phi(p)e_Q]\beta(\phi(p^{-1})) \\
&= \beta(\phi(p))\beta(\phi(p^{-1})) \\
&= \beta(\phi(p))\beta((\phi(p))^{-1}) \\
&= \beta(\phi(p)(\phi(p))^{-1}) \quad \text{[since } e_Q \in \beta(\phi(p)(\phi(p))^{-1})\text{, we have]} \\
\phi((pu)\alpha(p^{-1})) &= e_Q \\
\therefore (pu)\alpha(p^{-1}) &\subseteq \text{Ker}\phi \\
\therefore \text{Ker}\phi &\lhd P.
\end{aligned}$$

□

Remark 3.36. The result of Proposition 3.35 (ii) is different from what is obtainable in the classical polygroup. In the classical polygroup, Kerϕ is generally not normal in P.

Proposition 3.37. *Let $\phi : (P, \alpha) \rightarrow (Q, \beta)$ be a strong Hom-polygroup homomorphism. Then ϕ is injective if and only if Ker$\phi = \{e_P\}$.*

Proof. Suppose that ϕ is injective. Let $x \in$ Kerϕ be arbitrary. Then $\phi(x) = e_Q = \phi(e_P)$ so that $x = e_P$. Consequently, Ker$\phi = \{e_P\}$.

Conversely, suppose that Ker$\phi = \{e_P\}$ and suppose that $\phi(x) = \phi(y)$ $\forall x, y \in P$. Since β is invertible, we have $\beta(\phi(x)) = \beta(\phi(y))$ and by the definition of Hom-polygroup homomorphism, we have $\phi(\alpha(x)) = \phi(\alpha(y))$. By postmultiplying both sides by $\phi((\alpha(y))^{-1}))$, we have $\phi(\alpha(x))\phi((\alpha(y))^{-1})) = \phi(\alpha(y))\phi((\alpha(y))^{-1}))$ which implies that $\phi(\alpha(xy^{-1})) = \phi(\alpha(yy^{-1}))$. Since $e_P \in \alpha(yy^{-1})$, we have that $\phi(\alpha(xy^{-1})) = \phi(e_p) = e_Q$. Now, let $z \in xy^{-1}$. Then we have $\phi(\alpha(z)) = e_Q$ and so, $\alpha(z) \in$ Ker$\phi = \{e_P\}$. This means that $e_P \in xy^{-1}$ and by Hom-reversibility, we have $x = y$. Hence, ϕ is injective. The proof is complete. □

Example 3.38. Let $(P, *, \alpha)$ be a Hom-polygroup and let $N \lhd P$. Then the mapping $\psi : (P, *, \alpha) \rightarrow (P/N, \circledast, \beta)$ defined by $\psi(x) = xN$ is a strong Hom-polygroup epimorphism called strong Hom-canonical homomorphism with Ker$\phi = N$.

Proof. Suppose that $x, y \in P$ are arbitrary. Then

$$\begin{aligned}
\psi(x * y) &= \psi(\{z : z \in x * y\}) \\
&= \{\psi(z) : z \in x * y\} \\
&= \{zN : z \in xy\} \\
&= xN \circledast yN \\
&= \psi(x) \circledast \psi(y).
\end{aligned}$$

Also, $\forall x \in P$,

$$\beta(\psi(x)) = \beta(xN) = \alpha(x)N = \psi(\alpha(x)).$$

Thus, ψ is a strong Hom-polygroup homomorphism. Since ψ is clearly surjective, it follows that ψ is a strong Hom-polygroup epimorphism.
Finally,

$$\begin{aligned}
\text{Ker}\phi &= \{x \in P : \psi(x) = e_{P/N}\} \\
&= \{x \in P : \psi(x) = eN\} \\
&= \{x \in P : xN = N\} \\
&= N.
\end{aligned}$$

The proof is complete. □

Theorem 3.39. *[First Isomorphism Theorem] Let $\phi : (P, *, \alpha) \to (Q, \circ, \beta)$ be a strong Hom-polygroup homomorphism with $K = Ker\phi$. Then $P/K \cong Im\phi$.*

Proof. By Proposition 3.35 (i), Im$\phi \leq Q$ which is a Hom-polygroup, by Proposition 3.35 (ii), $K \lhd P$ and by Corollary 3.30, P/K is a Hom-polygroup. Now, let $\psi : (P/K, \circledast, \mu) \to (\text{Im}\phi, \circ, \beta)$ be a mapping defined by $\psi(xK) = \phi(x)\ \forall x \in P$. We first show that ψ is well-defined. Suppose that $xK = yK$. Then $\exists k \in K$ such that $xk = yk$. Now consider

$$\begin{aligned} \phi(x * k) &= \phi(y * k) \\ \Rightarrow \phi(x) \circ \phi(k) &= \phi(y) \circ \phi(k) \\ \Rightarrow \phi(x) \circ e_Q &= \phi(y) \circ e_Q \\ \Rightarrow \beta(\phi(x)) &= \beta(\phi(y)) \quad \text{[since } \beta \text{ is invertible, we have]} \\ \phi(x) &= \phi(y). \end{aligned}$$

Hence, ψ is well-defined. Next, we show that ψ is a strong Hom-polygroup homomorphism. To this end, let $xK, yK \in P/K$ be arbitrary where $x, y \in P$. Then

$$\begin{aligned} \psi(xK \circledast yK) &= \psi(\{z : z \in x * y\}) \\ &= \{\psi(z) : z \in x * y\} \\ &= \{zK : z \in xy\} \\ &= xK \circledast yK \\ &= \psi(x) \circledast \psi(y). \end{aligned}$$

Also, $\forall xK \in P/K$, we have

$$\beta(\psi(xK)) = \beta(\phi(x)) = \phi(\alpha(x)) = \psi(\alpha(x)K) = \psi(\mu(xK)).$$

Hence, ψ is a strong Hom-polygroup homomorphism. Lastly, we show that ψ is a bijection. By the definition of ψ, it is clear that ψ is surjective. For injective, consider the following:

$$\begin{aligned} \text{Ker}\psi &= \{xK \in P/K : \psi(xK) = e_{\text{Im}\phi}\} \\ &= \{xK \in P/K : \phi(x) = \phi(e_{P/K})\} \\ &= \{xK \in P/K : x \in K\} \\ &= \{K\}. \end{aligned}$$

Accordingly, ψ is an injection and therefore a bijection. Since ψ is a bijective strong Hom-polygroup homomorphism, it follows that $P/K \cong$ Imϕ. The proof is complete. □

Corollary 3.40. *If $\phi : (P, *, \alpha) \to (Q, \circ, \beta)$ is a strong Hom-polygroup epimomorphism with $K = Ker\phi$, then $P/K \cong Q$.*

Theorem 3.41. *[Second Isomorphism Theorem] Let H and N be Hom-subpolygroups of a Hom-polygroup $(P, *, \alpha)$ with $N \lhd P$. Then $H/(H \cap N) \cong (HN)/N$.*

Proof. By Proposition 3.22 (iv), $H \cap N \lhd H$, by Proposition 3.22 (v), $HN = NH \leq P$ and by Proposition 3.22 (vi), $N \lhd HN$. Let $\phi : (H, \alpha) \to ((HN)/N, \circledast, \beta)$ be a mapping defined by $\phi(x) = xN\ \forall x \in H$. We then show that ϕ is a strong Hom-polygroup epimorphism. To this end, let $x, y \in H$ be arbitrary. Then by Proposition 3.22 (ii), we have

$$\begin{aligned} \phi(x * y) &= (x * y)N \\ &= (xN) \circledast (yN) \\ &= \phi(x) \circledast \phi(y). \end{aligned}$$

Also,

$$\beta(\phi(x)) = \beta(xN) = \alpha(x)N = \phi(\alpha(x)).$$

Accordingly, ϕ is a strong Hom-polygroup homomorphism. We claim that ϕ is surjective. For the proof of the claim, let $xN \in (HN)/N$ be arbitrary with $x \in HN$. Then $\exists h \in H$ and $\exists n \in N$ such that $x \in hn$ and by Proposition 3.22, we have

$$xN = hnN = (hN)(nN) = hN = \phi(h).$$

This shows that every element of $(HN)/N$ has a preimage in H. Hence, ϕ is surjective and thus, ϕ is a strong Hom-polygroup epimorphism. Lastly, consider the following:

$$\begin{aligned}
\text{Ker}\phi &= \{x \in H : \phi(x) = e_{(HN)/N}\} \\
&= \{x \in H : \phi(x) = N\} \\
&= \{x \in H : xN = N\} \\
&= \{x \in H : x \in N\} \\
&= \{x \in H : x \in H \cap N\} \\
&= H \cap N.
\end{aligned}$$

By invoking Corollary 3.40, we conclude that $H/(H \cap N) \cong (HN)/N$. This completes the proof. □

Theorem 3.42. *[Third Isomorphism Theorem] Let H and N be Hom-normal subpolygroups of a Hom-polygroup $(P, *, \alpha)$ with $N \leq H$. Then*

(i) $H/N \lhd P/N$;

(ii) $(P/N)/(H/N) \cong P/H$.

Proof. (i) Obviously, $N \lhd H$. It is clear that H/N and P/N are Hom-polygroups and $H/N \subseteq P/N$. For Hom-normality, $\forall hN \in H/N, pN \in P/N$, with $h \in H, p \in P$, we have

$$\begin{aligned}
(pN)(hN)\beta((pN)^{-1}) &= ((ph)N)\beta(p^{-1}N) \\
&= ((ph)N)(\alpha(p^{-1})N) \\
&= [(ph)\alpha(p^{-1})]N \\
&\subseteq H/N \\
\therefore H/N &\lhd P/N.
\end{aligned}$$

(ii) Since by (i) $H/N \lhd P/N$, it follows that $(P/N)/(H/N)$ is a Hom-polygroup. Now, let $\phi : (P/N, \circledast, \beta) \to (P/H, \circledast, \mu)$ be a mapping defined by $\phi(xN) = xH \ \forall xN \in P/N, yN \in P/H$. We first show that ϕ is well-defined. To do this, we note that since $N \leq H$, then $NH = H$. Suppose that $xN = yN$. Then $(xN)H = (yN)H$ and $\exists h \in H, n \in N$ such that

$$\begin{aligned}
(xn)h &= (yn)h \\
\Rightarrow (xn)\alpha\alpha^{-1}(h) &= (yn)\alpha\alpha^{-1}(h) \\
\Rightarrow \alpha(x)(n\alpha^{-1}(h)) &= \alpha(y)(n\alpha^{-1}(h)) \quad [\text{since } n\alpha^{-1}(h) \subseteq NH = H, \text{ we have}] \\
\alpha(x)(NH) &= \alpha(y)(NH) \\
\Rightarrow \alpha(x)H &= \alpha(y)H \\
\Rightarrow \mu(xH) &= \mu(yH) \quad [\text{since } \mu \text{ is invertible, we have}] \\
xH &= yH \\
\therefore \phi(xN) &= \phi(yN) \quad [\text{and } \phi \text{ is well-defined}].
\end{aligned}$$

Next, we show that ϕ is a strong Hom-polygroup epimorphism. By the definition of ϕ, it is clear that ϕ is surjective. Let $pN, qN \in P/N$ be arbitrary with $p, q \in P$. Then by Proposition 3.22 (ii), we have

$$\begin{aligned}
\phi((pN) \circledast (qN)) &= \phi((pq)N) \\
&= (pq)H \\
&= (pH)(qH) \\
&= \phi(pN)\phi(qN).
\end{aligned}$$

Also,

$$\mu(\phi(pN)) = \mu(pH) = \alpha(p)H = \phi(\alpha(p)N) = \phi(\beta(pN)).$$

Accordingly, ϕ is a strong Hom-polygroup epimorphism. Lastly, consider the following:

$$\begin{aligned} \mathrm{Ker}\phi &= \{pN \in P/N : \phi(pN) = e_{(P/H}\} \\ &= \{pN \in P/N : pH = H\} \\ &= \{pN \in P/N : p \in H\} \\ &= H/N. \end{aligned}$$

Hence by Corollary 3.40, we have that $(P/N)/(H/N) \cong P/H$. The proof is complete. □

4 Conclusion, Possible areas of Application and Future Research

In this paper, we have introduced the concept of a Hom-polygroup as a generalization of a Hom-group. We presented several foundational properties of Hom-polygroups with examples. We studied homomorphisms of Hom-polygroups, and showed that the kernel of a Hom-polygroup homomorphism is normal. We also studied the quotient structure of Hom-polygroups, and presented the first, second and third isomorphism theorems of Hom-polygroups. Hom-polygroups provide a framework for studying generalizations of group theory, which can be useful in exploring new algebraic structures and relationships. They serve as a bridge between classical group theory and broader algebraic systems like hypergroups and quasigroups. We hope in future works the non-commutative and generalized operations in Hom-polygroups can be utilized to design cryptographic systems that are resistant to traditional cryptanalysis technique. Also, Hom-polygroups can model symmetry and connectivity in graphs and hypergraphs, particulary in cases involving multi-valued relationship.

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Author information

E.O. Adeleke[1], Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
E-mail: `yemi376@yahoo.com`

A.A.A. Agboola[2*], Department of Mathematics, Federal University of Agriculture, Abeokuta, Nigeria.
E-mail: `agboolaaaa@funaab.edu.ng`

B. Davvaz[3], Department of Mathematical Sciences, Yazd University, Yazd, Iran.
E-mail: `davvaz@yazd.ac.ir`

M.A. Ibrahim[4], Department of Mathematics, Auburn University, Auburn, AL 36849, USA.
E-mail: `mai0015@auburn.edu`

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