

# A FRACTIONAL DIFFERENTIAL SYSTEM OF BEAM TYPE WITH TWO GENERALIZED NONLINEARITIES

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**Abstract** his study utilizes the Caputo derivative to investigate a nonlinear coupled system related to beam deflection. The system includes new derivative parameters  $0 < \alpha_1, \alpha_2 \leq 1$ ,  $2 < \beta_1, \beta_2 \leq 3$  that are used for sequential derivatives. These parameters do not adhere to the standard semi-group and commutativity rules. Additionally, the system features two generalized nonlinearities on its right-hand sides. Our system can be simplified to a fourth-order ordinary coupled system reminiscent of beam dynamics in certain cases. We employ the Banach contraction principle and the Schauder fixed point theorem to establish two primary results: the uniqueness of solutions and the existence of at least one solution. Furthermore, stability in the context of Ulam-Hyers stability is studied, and 2 examples are provided to support our results.

## 1 Introduction

Recently, there has been a lot of interest in the development of fractional calculus (FC) to explore integration and differentiation to non-integer order. This branch of FC has important uses in many scientific and technological domains where mathematical equations can be used to describe real-world issues, see [1, 8, 32, 33, 34]. fractional differential equations have gained popularity due to their important applications in various fields of science and engineering, including viscoelasticity, electrochemistry, control systems, biology, and aerodynamics. For further information and detailed insights, we recommend referring the reader to [12, 15, 19, 20, 21, 22, 25]. the elastic beam, often referred to as the Euler-Bernoulli equation, is fundamental in material science and engineering. It describes how a beam behaves under bending loads and can be used to calculate the stress distribution, deflection, and strain in a beam made of elastic material. The equation is given by :

$$\frac{d^2}{dx^2}(EI \frac{d^2}{dx^2}u(x)) = -dl(x),$$

where  $u$  represents the deflection of the beam,  $E$  denotes Young's modulus of the material,  $I$  symbolizes the moment of inertia of the beam's cross-sectional area, and  $dl$  indicates the distributed load on the beam.

Beam equations have extensive applications in engineering and applied sciences. In structural engineering, they are used to study the bending, shear, and vibrational characteristics of beams in buildings [11, 30], as well as the behavior of bridges under moving vehicles [35]. Beam equations also find applications in biomechanics for modeling bone mechanics and prosthetics [24, 36]. These examples highlight only a few of the applications of beam equations.

In pure mathematics, the elastic beam equations have inspired the study of the following

family of differential equations, as referenced in [2, 3, 31]:

$$\begin{cases} u^4(t) = g(t, u(t), u''(t)), & 0 < t < 1 \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

Fractional beam-type differential equations have also been utilized to model the behavior of beams. For instance, A. Almbaidin and I. Abu-Alshaikh, in [4], proposed an equation to study vibrations in a graded simply supported beam that is subjected to a moving oscillator. Their proposed equation is given by:

$$C_{\alpha} I \frac{\partial^4}{\partial x^4} \left( D_t^{(\alpha)} u(x, t) \right) + (EI)_{eq} \frac{\partial^4 u(x, t)}{\partial x^4} + \rho_{eq} \frac{\partial^2 u(x, t)}{\partial t^2} = -m \left( \frac{d^2 q(t)}{dt^2} + g \right) \delta(x - vt),$$

where,  $u$  indicates the displacement of the beam,  $q$  gives us the displacement of the oscillator,  $g$  symbolizes for us the acceleration, and the expression  $D_t^{\alpha}$  refers to the Caputo-Fabrizio derivative. In the work of J. Villa-Morales et al. [27], the authors proposed fractional differential equations using the Caputo derivative to model the deflection of beams, for which they derived a closed-form solution. Their problem is given by:

$$EID_{0^+}^{\alpha+1} u(x) = \frac{x^{1-\alpha}}{\Gamma(2-\alpha)} M(x).$$

In [5], R. AlSaleh et al. investigated the dynamics of a beam with fractional order damping. They proposed a method to obtain a closed-form solution by integrating Green’s functions with Laplace transforms and the composition method.

In addition to finding solutions, it is essential to study the existence and uniqueness of solutions to nonlinear fractional differential equations. This exploration is critical for obtain a deeper understanding of complex nonlinear physical phenomena and their practical applications in fractional order systems. For more details and various, we refer to [23, 26]. After exploring the results related to the uniqueness and existence of solutions, many mathematicians became interested in examining Ulam-Hyer stability within mathematical analysis. For further clarification on this topic, see for instance [10, 13, 14, 16, 17, 28]. In this context, I. Bachar et al. [7] investigated the existence and uniqueness of a solution for the following problem:

$$\begin{cases} \mathbb{D}^{\alpha} (\mathbb{D}^{\beta} \theta) (\varsigma) + \Upsilon(\varsigma, \vartheta(\varsigma)) = 0, & 0 < \varsigma < 1, \\ \vartheta(0) = \mathbb{D}^{\beta} \theta(0) = (\mathbb{D}^{\beta} \theta)' (0) = \mathbb{D}^{\beta} \vartheta(1) = 0, \end{cases}$$

where  $2 < \alpha \leq 3, 0 < \beta \leq 1$  and  $\Upsilon \in C([0, 1] \times \mathbb{R}^2, \mathbb{R})$  satisfying some sufficient conditions. In [6], the authors focused on the existence of solutions and their Ulam-stability for a non-sequential problem:

$$\begin{cases} D^{\alpha} u(x) = f_1(x, u(x), v(x)) + a_1 g_1(x, u(x)) + b_1 h_1(x, u''(x)), \\ D^{\beta} u(x) = f_2(x, u(x), v(x)) + a_2 g_2(x, u(x)) + b_2 h_2(x, u''(x)), \end{cases}$$

with the conditions:

$$\begin{cases} u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

where  $x \in [0, 1], 3 < \alpha, \beta \leq 4, D^{\alpha}, D^{\beta}$  denote the fractional derivatives in the sense of Caputo, for  $i=1,2 f_i \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), g_i, h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$  and  $a_i, b_i \in \mathbb{R}$ .

In [9], K. Bensassa et al. examined the existence and uniqueness of solutions as well as stability in the context of Ulam-Hyers for the system:

$$\begin{cases} D^{\alpha_1} D^{\alpha_2} u(x) = f_1(x, u(x), v(x)) + a_1 g_1(x, u(x)) + b_1 h_1(x, D^{\delta} u(x)), \\ D^{\beta_1} D^{\beta_2} v(x) = f_2(x, u(x), v(x)) + a_2 g_2(x, u(x)) + b_2 h_2(x, D^{\delta} u(x)), \end{cases}$$

under the following conditions of type flexible and / or fixed end:

$$\begin{cases} u(0) = u(1) = a, \\ u'(0) = u'(1) = 0, \\ v(0) = v(1) = b, \\ v'(0) = v'(1) = 0, \end{cases}$$

where, for  $i = 1, 2$ ,  $a_i, b_i \in \mathbb{R}$ ,  $D^{\alpha_i}, D^{\beta_i}, D^\delta$  are Caputo fractional derivatives,  $0 < \delta \leq 1$   $f_i \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ ,  $g_i, h_i \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ . The absence of semi group and commutativity properties are guaranteed by the conditions  $0 < \alpha_1, \beta_1 \leq 1$  and  $2 < \alpha_2, \beta_2 \leq 3$ .

Inspired by the research paper conducted by K. Bensassa et al. [9], we aim to study a generalized problem related to the elastic beam system. In this context, we introduce an additional fractional derivative in the sense of Caputo, of order  $\delta - 1$ , where  $1 < \delta < 2$ , along with two general nonlinearities,  $F_1$  and  $F_2$ .

$$\begin{aligned} D^{\alpha_1} D^{\beta_1} u_1(x) &= F_1(x, u_1(x), u_2(x), D^{\delta-1} u_1(x), D^\delta u_1(x)), \\ D^{\alpha_2} D^{\beta_2} u_2(x) &= F_2(x, u_1(x), u_2(x), D^{\delta-1} u_2(x), D^\delta u_2(x)), \end{aligned} \tag{1.1}$$

under the conditions

$$\begin{cases} u_1(0) = u_1(1) = \theta_1, \\ u'_1(0) = u'_1(1) = 0, \\ u_2(0) = u_2(1) = \theta_2, \\ u'_2(0) = u'_2(1) = 0, \end{cases} \tag{1.2}$$

where, for  $i = 1, 2$ ,  $\theta_i \in \mathbb{R}^+$ ,  $F_i \in C([0, 1] \times \mathbb{R}^4, \mathbb{R})$  and  $D^{\alpha_i}, D^{\beta_i}, D^\delta$  are Caputo fractional derivatives, with  $0 < \alpha_1, \alpha_2 \leq 1$ ,  $2 < \beta_1, \beta_2 \leq 3$ . We analyze this system using the Banach contraction principle and the Schauder fixed point theorem to establish two key results: the uniqueness of solutions and the existence of at least one solution. Additionally, we provide examples to illustrate the uniqueness of solutions. Furthermore, we discuss Ulam-Hyers stability.

## 2 Preliminaries

In this section, we review some essential notions from fractional calculus, along with the Lemmas and theorems used in our proofs, see [21, 25].

**Definition 2.1.** The Riemann–Liouville fractional integral of order  $\alpha > 0$  for a function  $f \in C([0, 1])$  is defined as

$$I^\alpha f(x) = \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x) dx, \tag{2.1}$$

and for  $\alpha = 0$ , as

$$I^0 f(x) = f(x).$$

**Definition 2.2.** The Caputo derivative of positive order  $\alpha$  of  $f \in C^n([0, 1])$  is defined as

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^n(x) dx, \tag{2.2}$$

where,  $n = [\alpha] + 1$  denotes the integer part of  $\alpha$ .

The following lemmas in [21, 25] will be used later in this paper.

**Lemma 2.3.** Let  $\alpha > 0$ . The the differential equation

$$D^\alpha f(x) = 0, \tag{2.3}$$

has a general solution on  $[0, 1]$  given by

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_{n-1} x^{n-1},$$

where,  $c_i \in \mathbb{R}; i = 0, 1, 2, \dots, n - 1$ .

**Lemma 2.4.** *Let  $\alpha > 0$ . then we have*

$$I^\alpha D^\alpha f(x) = f(x) + c_0 + c_1x + c_2x^2 + \dots + c_{n-1}x^{n-1}, \tag{2.4}$$

where  $c_i \in \mathbb{R}, i = 0, 1, 2, \dots, n - 1$ .

**Lemma 2.5.** *For any  $x \in [0, 1]$  the following properties are true:*

- $I^\alpha I^\beta f(x) = I^{\alpha+\beta} f(x)$ , for all  $\alpha > 0$  and  $\beta > 0$ .
- $D^\alpha I^\alpha f(x) = f(x)$ , for all  $\alpha > 0$ .
- $D^\alpha I^\beta f(x) = I^{\beta-\alpha} f(x)$ , for all  $\beta > \alpha > 0$ .

**Lemma 2.6** (Schauder fixed-point Theorem). *Let  $E$  be a Banach space,  $B_R = \{x \in E, \|x\| < R\}$ . If  $f$  is a compact map from  $B_R$  to  $B_R$ , then  $f$  admits a fixed point.*

**Lemma 2.7.** *Let  $K_1, K_2 \in C([0, 1], \mathbb{R})$ . The linear problem*

$$\begin{cases} D^{\alpha_1} D^{\beta_1} u_1(x) = K_1(x), \\ D^{\alpha_2} D^{\beta_2} u_2(x) = K_2(x), \end{cases} \tag{2.5}$$

with the conditions

$$\begin{cases} u_1(0) = u_1(1) = \theta_1, \\ u'_1(0) = u'_1(1) = 0, \\ u_2(0) = u_2(1) = \theta_2, \\ u'_2(0) = u'_2(1) = 0, \end{cases} \tag{2.6}$$

admits a unique solution given by

$$\begin{aligned} u_1(x) &= I^{\alpha_1+\beta_1} K_1(x) + \frac{2I^{\alpha_1+\beta_1} K_1(1) - I^{\alpha_1+\beta_1-1} K_1(1)}{\beta_1 - 2} x^{\beta_1} + \theta_1 \\ &+ \frac{\beta_1 I^{\alpha_1+\beta_1} K_1(1) - I^{\alpha_1+\beta_1-1} K_1(1)}{2 - \beta_1} x^2, \\ u_2(x) &= I^{\alpha_2+\beta_2} K_2(x) + \frac{2I^{\alpha_2+\beta_2} K_2(1) - I^{\alpha_2+\beta_2-1} K_2(1)}{\beta_2 - 2} x^{\beta_2} + \theta_2 \\ &+ \frac{\beta_2 I^{\alpha_2+\beta_2} K_2(1) - I^{\alpha_2+\beta_2-1} K_2(1)}{2 - \beta_2} x^2. \end{aligned}$$

*Proof.* We have

$$\begin{cases} u_1(x) = I^{\alpha_1+\beta_1} K_1(x) + I^{\beta_1} a_0 + a_1 + a_2x + a_3x^2, \\ u_2(x) = I^{\alpha_2+\beta_2} K_2(x) + I^{\beta_2} e_0 + e_1 + e_2x + e_3x^2. \end{cases}$$

Also we know that

$$\begin{cases} u'_1(x) = I^{\alpha_1+\beta_1-1} K_1(x) - \frac{a_0\beta_1}{\Gamma(\beta_1 + 1)} x^{\beta_1-1} + a_2 + 2a_3x, \\ u'_2(x) = I^{\alpha_2+\beta_2-1} K_2(x) - \frac{e_0\beta_2}{\Gamma(\beta_2 + 1)} x^{\beta_2-1} + e_2 + 2e_3x. \end{cases}$$

The solution of (2.5)-(2.6) is given by

$$\begin{cases} u_1(x) = I^{\alpha_1+\beta_1} K_1(x) + \frac{2x^{\beta_1} - \beta_1x^2}{\beta_1 - 2} I^{\alpha_1+\beta_1} K_1(1) + \theta_1 + \frac{\beta_1x^2 - x^{\beta_1}}{2 - \beta_1} I^{\alpha_1+\beta_1-1} K_1(1), \\ u_2(x) = I^{\alpha_2+\beta_2} K_2(x) + \frac{2x^{\beta_2} - \beta_2x^2}{\beta_2 - 2} I^{\alpha_2+\beta_2} K_2(1) + \theta_2 + \frac{x^2 - x^{\beta_2}}{2 - \beta_2} I^{\alpha_2+\beta_2-1} K_2(1), \end{cases}$$

and we have

$$\begin{aligned}
 a_0 &= \left( \frac{2I^{\alpha_1+\beta_1} K_1(1) - I^{\alpha_1+\beta_1-1} K_1(1)}{2 - \beta_1} \right) \times \Gamma(\beta_1 + 1), a_1 = \theta_2, a_2 = 0, \\
 a_3 &= \frac{\beta_1 I^{\alpha_1+\beta_1} K_1(1) - I^{\alpha_1+\beta_1-1} K_1(1)}{2 - \beta_1}, \\
 e_0 &= \left( \frac{2I^{\alpha_2+\beta_2} K_2(1) - I^{\alpha_2+\beta_2-1} K_2(1)}{2 - \beta_2} \right) \times \Gamma(\beta_2 + 1), e_1 = \theta_2, e_2 = 0, \\
 e_3 &= \frac{\beta_2 I^{\alpha_2+\beta_2} K_2(1) - I^{\alpha_2+\beta_2-1} K_2(1)}{2 - \beta_2}.
 \end{aligned}$$

□

### 3 Analysis of the system outlined in equations (1.1) and (1.2)

Consider the Banach space given by

$$E = (x : x \in C([0, 1]), D^{\delta-1}x \in C([0, 1]), D^\delta u \in C([0, 1])),$$

and the norm

$$\|x\|_E = \|x\|_\infty + \|{}^c D^\delta x\|_\infty + \|{}^c D^{\delta-1}x\|_\infty,$$

such that

$$\|x\|_\infty = \max_{t \in [0,1]} |x(t)|, \quad \|{}^c D^\delta x\|_\infty = \max_{t \in [0,1]} |D^\delta x|, \quad \|{}^c D^{\delta-1}x\|_\infty = \max_{t \in [0,1]} |D^{\delta-1}x|,$$

In addition, we introduce the product space  $E \times E$ , such that its norm is given by:

$$\|(x, y)\|_{E \times E} = \|x\|_E + \|y\|_E.$$

Let us consider the sufficient conditions.

- (G<sub>1</sub>) The functions  $F_1, F_2 : [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$  are continuous.
- (G<sub>2</sub>) There exist positive constants  $h_{ij}, i = 1, 2, j = 1, 2, 3, 4$ , such that, for all  $x \in [0, 1]$  and  $(y, z, t, w), (y', z', t', w') \in \mathbb{R}^4$ , we have

$$\begin{aligned}
 |F_i(x, y, z, t, w) - F_i(x', y', z', t', w')| &\leq h_{i1}|y - y'| + h_{i2}|z - z'| + h_{i3}|t - t'| \\
 &+ h_{i4}|w - w'|,
 \end{aligned}$$

where

$$H_i = \max(h_{i1}, h_{i2}, h_{i3}, h_{i4}), i = 1, 2.$$

- (G<sub>3</sub>) There exist positive constants  $z_1, z_2$ :

$$\begin{aligned}
 |F_1(x, y, z, t, w)| &\leq z_1, \\
 |F_2(x, y, z, t, w)| &\leq z_2.
 \end{aligned}$$

For  $i = 1, 2$ , we shall take:

$$\begin{aligned}
 \Delta_i &= \left( 1 + \frac{2 + \beta_i}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1)} + \left( \frac{2}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i)}, \\
 \Delta'_i &= \left( 1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 1)} + \frac{2\beta_i}{(\beta_i - 2)(\Gamma(3 - \delta))} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} \\
 &+ \left( \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 1 - \delta)} + \frac{2\beta_i}{(\beta_i - 2)\Gamma(3 - \delta)} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta)},
 \end{aligned}$$

$$\Delta_i'' = \left( 1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 2)} + \frac{3\beta_i}{(\beta_i - 2)(\Gamma(4 - \delta))} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} + \left( \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 2 - \delta)} + \frac{3\beta_i}{(\beta_i - 2)\Gamma(4 - \delta)} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta + 1)},$$

$$\Lambda = (\Delta_1 + \Delta_1' + \Delta_1'' + \Delta_2 + \Delta_2' + \Delta_2''),$$

$$A_i = \left( \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1)} + \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} + \frac{z_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} \right),$$

$$C = \left( 1 + \frac{2}{\Gamma(3 - \delta)} + \frac{2}{(4 - \delta)} \right),$$

$$B_i = \left( 1 + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 1 - \delta)} + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 2 - \delta)} \right).$$

**3.1 Uniqueness of solution to system (1)-(2)**

We begin this section by defining the operator  $Y : E \times E \rightarrow E \times E$  as follows

$$Y(u_1(x), u_2(x)) = (Y_1(u_1(x), u_2(x)), Y_2(u(x), v(x))),$$

where for  $i = 1, 2$ , we have:

$$\begin{aligned} Y_i(u_1(x), u_2(x)) = & I^{\alpha_i + \beta_i} F_i(x, u_1(x), u_2(x), D^{\delta-1}u_i(x), D^\delta u_i(x)) \\ & + \frac{2 I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i} \\ & - \frac{I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i} \\ & + \theta_i + \frac{\beta_i I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^2 \\ & - \frac{I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^2. \end{aligned} \tag{3.1}$$

**Theorem 3.1.** Assume that  $(G_1)$ ,  $(G_2)$  are satisfied and  $\Lambda = \Delta_1 + \Delta_1' + \Delta_1'' + \Delta_2 + \Delta_2' + \Delta_2'' < 1$ . Then (1.1)-(1.2) admits a unique solution  $(u_1, u_2)$  over  $[0, 1]$ .

*Proof.* As  $E$  is a Banach space and  $Y : E \times E \rightarrow E \times E$ , stability is trivial. Let  $x \in [0, 1]$ , we have

$$\begin{aligned} |Y_i(u_1(x), u_2(x)) - Y_i(v_1(x), v_2(x))| \leq & \left( 1 + \frac{2 + \beta_i}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E) \\ & + \left( \frac{2}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E). \end{aligned}$$

Hence, we obtain

$$\|Y_i(u_1(x), u_2(x)) - Y_i(v_1(x), v_2(x))\|_\infty \leq \Delta \| (u_1(x), u_2(x)) - (v_1(x), v_2(x)) \|_{E \times E}. \tag{3.2}$$

On the other hand,

$$\begin{aligned}
 & D^\delta Y_i(u_1(x), u_2(x)) I^{\alpha_i + \beta_i - \delta} F_i(x, u_1(x), u_2(x), D^{\delta-1} u_i(x), D^\delta u_i(x)) \\
 & + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 1 - \delta)} \times \frac{2 I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i - \delta} \\
 & - \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 1 - \delta)} \times \frac{I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i - \delta} \\
 & + \frac{\Gamma(3)}{\Gamma(3 - \delta)} \times \frac{\beta_i I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^{2 - \delta} \\
 & - \frac{\Gamma(3)}{\Gamma(3 - \delta)} \times \frac{I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^{2 - \delta},
 \end{aligned}$$

$$\begin{aligned}
 & |D^\delta Y_i(u_1(x), u_2(x)) - D^\delta Y_i(v_1(x), v_2(x))| \\
 & \leq \left( 1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 1)} + \frac{2\beta_i}{(\beta_i - 2)(\Gamma(3 - \delta))} \right) \\
 & \times \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E) \\
 & + \left( \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 1 - \delta)} + \frac{2\beta_i}{(\beta_i - 2)\Gamma(3 - \delta)} \right) \\
 & \times \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E).
 \end{aligned}$$

Thus, we obtain

$$\|D^\delta Y_i(u_1(x), u_2(x)) - D^\delta Y_i(v_1(x), v_2(x))\|_\infty \leq \Delta' \| (u_1(x), u_2(x)) - (v_1(x), v_2(x)) \|_{E \times E}. \tag{3.3}$$

Also,

$$\begin{aligned}
 & D^{\delta-1} Y_i(u_1(x), u_2(x)) = I^{\alpha_i + \beta_i - \delta + 1} F_i(x, u_1(x), u_2(x), D^{\delta-1} u_i(x), D^\delta u_i(x)) \\
 & + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 2 - \delta)} \times \frac{2 I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i - \delta + 1} \\
 & - \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 2 - \delta)} \times \frac{I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i - \delta + 1} \\
 & + \frac{\Gamma(4)}{\Gamma(4 - \delta)} \times \frac{\beta_i I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^{3 - \delta} \\
 & - \frac{\Gamma(4)}{\Gamma(4 - \delta)} \times \frac{I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1} u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^{3 - \delta},
 \end{aligned}$$

$$\begin{aligned}
 & |D^{\delta-1} Y_i(u_1(x), u_2(x)) - D^{\delta-1} Y_i(v_1(x), v_2(x))| \\
 & \leq \left( 1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 2)} + \frac{3\beta_i}{(\beta_i - 2)\Gamma(4 - \delta)} \right) \\
 & \times \frac{H_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E) \\
 & + \left( \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 2 - \delta)} + \frac{3\beta_i}{(\beta_i - 2)\Gamma(4 - \delta)} \right) \\
 & \times \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta + 1)} (\|u_1(x) - v_1(x)\|_E + \|u_2(x) - v_2(x)\|_E).
 \end{aligned}$$

We get

$$\begin{aligned} & \|D^{\delta-1}Y_i(u_1(x), u_2(x)) - D^{\delta-1}Y_i(v_1(x), v_2(x))\|_{\infty} \\ & \leq \Delta'' \|(u_1(x), u_2(x)) - (v_1(x), v_2(x))\|_{E \times E}. \end{aligned} \tag{3.4}$$

Thanks to (3.2), (3.3) and (3.4), we get

$$\begin{aligned} & \|Y_i(u_1(x), u_2(x)) - Y_i(v_1(x), v_2(x))\|_E \\ & \leq (\Delta_i + \Delta'_i + \Delta''_i) \|(u_1(x), u_2(x)) - (v_1(x), v_2(x))\|_{E \times E}. \end{aligned}$$

Therefore,

$$\begin{aligned} & \|Y_1(u_1(x), u_2(x)) - Y_1(v_1(x), v_2(x))\|_E \\ & \leq (\Delta_1 + \Delta'_1 + \Delta''_1) \|(u_1(x), u_2(x)) - (v_1(x), v_2(x))\|_{E \times E} \end{aligned}$$

and

$$\begin{aligned} & \|Y_2(u_1(x), u_2(x)) - Y_2(v_1(x), v_2(x))\|_E \\ & \leq (\Delta_2 + \Delta'_2 + \Delta''_2) \|(u_1(x), u_2(x)) - (v_1(x), v_2(x))\|_{E \times E}. \end{aligned}$$

Finally,

$$\|Y(u_1(x), u_2(x)) - Y(v_1(x), v_2(x))\|_{E \times E} \leq \Lambda \|(u_1(x), u_2(x)) - (v_1(x), v_2(x))\|_{E \times E}.$$

Thus,  $Y$  is a contraction. By the Banach contraction principle, we get that  $Y$  has a unique fixed point. □

### 3.2 Existence of solutions to system (1)-(2)

let us now establish the following second main result.

**Theorem 3.2.** *Suppose that  $(G_1)$  and  $(G_3)$  are valid. Then, (1.1)–(1.2) has at least one solution.*

*Proof.* let  $\epsilon > 0$  be a positive constant and  $B_\epsilon = \{(u, v) \in E \times E, \|(u, v)\|_{E \times E} \leq \epsilon\}$ . The proof will be developed as follows

**Claim 1:** Let us take the subset  $B_\epsilon$ . Since it is trivially is bounded, convex and closed of  $E$ ; so we need in this first claim to prove that  $Y(B_\epsilon)$  is bounded. To do this, we take  $(u_1, u_2) \in B_\epsilon$  and  $x \in [0, 1]$ . So, we have for  $i = 1, 2$ ,

$$\begin{aligned} |Y_i(u_1(x), u_2(x))| &= \left| I^{\alpha_i + \beta_i} F_i(x, u_1(x), u_2(x), D^{\delta-1}u_i(x), D^\delta u_i(x)) \right. \\ & \quad + \frac{2 I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i} \\ & \quad - \frac{I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{\beta_i - 2} x^{\beta_i} \\ & \quad + \theta_1 + \frac{\beta_i I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^2 \\ & \quad \left. - \frac{I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{2 - \beta_i} x^2 \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \|Y_i(u_1(x), u_2(x))\|_\infty &\leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1)} \\ &\quad + \frac{2 |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_1(1), D^\delta u_1(1))|}{\beta_i - 2} \\ &\quad + \frac{I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))}{\beta_i - 2} \\ &\quad + |\theta_i| + \frac{|\beta_i I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2} \\ &\quad + \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \|Y_i(u_1(x), u_2(x))\|_\infty &\leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1)} + \frac{2z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} + \frac{z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i)} \\ &\quad + |\theta_i| + \frac{\beta_i z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} + \frac{z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i)}. \end{aligned} \tag{3.5}$$

Also, we state that

$$\begin{aligned} |D^\delta Y_i(u_1(x), u_2(x))| &\leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} \\ &\quad + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 1 - \delta)} \times \frac{2 |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_1(1), D^\delta u_1(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 1 - \delta)} \times \frac{|I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(3)}{\Gamma(3 - \delta)} \times \frac{\beta_i |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(3)}{\Gamma(3 - \delta)} \times \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2}. \end{aligned}$$

Thus,

$$\begin{aligned} \|D^\delta Y_i(u_i(x), v_i(x))\|_\infty &\leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} \\ &\quad + \frac{\Gamma(\beta_i + 1)2z_i}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} + \frac{\Gamma(\beta_i + 1)z_i}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i)} \\ &\quad + \frac{2\beta_i z_i}{\Gamma(3 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} + \frac{2z_i}{\Gamma(3 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i)}. \end{aligned} \tag{3.6}$$

We have also

$$\begin{aligned} |D^{\delta-1} Y_i(u_1(x), u_2(x))| &\leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} \\ &\quad + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 2 - \delta)} \times \frac{2 |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_1(1), D^\delta u_1(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(\beta_i + 1)}{\Gamma(\beta_i + 2 - \delta)} \times \frac{|I^{\alpha_i + \beta_i - 1} F_1(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(4)}{\Gamma(4 - \delta)} \times \frac{\beta_i |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2} \\ &\quad + \frac{\Gamma(4)}{\Gamma(4 - \delta)} \times \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\beta_i - 2}. \end{aligned}$$

So, it yields that

$$\begin{aligned} & \|D^{\delta-1}Y_i(u_1(x), u_2(x))\|_{\infty} \leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} \\ & + \frac{\Gamma(\beta_i + 1)2z_i}{\Gamma(\beta_i + 2 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} + \frac{\Gamma(\beta_i + 1)z_i}{\Gamma(\beta_i + 2 - \delta)(\beta_i - 2)\Gamma(\alpha_i + \beta_i)} \\ & + \frac{\Gamma(3)\beta_i z_i}{\Gamma(4 - \delta)(2 - \beta_i)\Gamma(\alpha_i + \beta_i + 1)} + \frac{\Gamma(3)z_i}{\Gamma(4 - \delta)(2 - \beta_i)\Gamma(\alpha_i + \beta_i)}. \end{aligned} \tag{3.7}$$

We have

$$\begin{aligned} \|Y_i(u_1(x), u_2(x))\|_E & \leq A_i \\ & + B_i \frac{2z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} \\ & + B_i \frac{z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i)} \\ & + |\theta_i| + C \frac{\beta_i z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i + 1)} \\ & + C \frac{z_i}{(\beta_i - 2)\Gamma(\alpha_i + \beta_i)} < \infty. \end{aligned} \tag{3.8}$$

This ends the proof of Claim 1.

**Claim 2:**  $Y$  is continuous.

Let  $(u_n, v_n) \rightarrow (u, v)$ ; where  $(u_n, v_n)$  converge in  $B_{\epsilon}$  in the sense of the above norm. We have

$$\begin{aligned} & \left| Y_i(u_n(x), v_n(x)) - Y_i(u(x), v(x)) \right| \leq \\ & \left( 1 + \frac{2 + \beta_i}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \\ & + \left( \frac{2}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} & \|Y_i(u_n(x), v_n(x)) - Y_i(u(x), v(x))\|_{\infty} \leq \\ & \left( 1 + \frac{2 + \beta_i}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \\ & + \left( \frac{2}{\beta_i - 2} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E). \end{aligned}$$

Similarly, we have

$$\begin{aligned} & \|D^{\delta}Y_i(u_n(x), v_n(x)) - D^{\delta}Y_i(u(x), v(x))\|_{\infty} \\ & \leq \left( 1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 1)} + \frac{2\beta_i}{(\beta_i - 2)\Gamma(3 - \delta)} \right) \\ & \quad \times \frac{H_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \\ & + \left( \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 1 - \delta)} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \\ & + \left( \frac{2\beta_i}{(\beta_i - 2)\Gamma(3 - \delta)} \right) \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \end{aligned}$$

and

$$\begin{aligned} & \|D^{\delta-1}Y_i(u_n(x), v_n(x) - D^{\delta-1}Y_i(u(x), v(x)))\|_{\infty} \\ & \leq \left(1 + \frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i - \delta + 2)} + \frac{3\beta_i}{(\beta_i - 2)(\Gamma(4 - \delta))}\right) \\ & \quad \times \frac{H_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E) \\ + & \left(\frac{2\Gamma(\beta_i + 1)}{(\beta_i - 2)\Gamma(\beta_i + 2 - \delta)}\right) \frac{H_i}{\Gamma(\alpha_i + \beta_i - \delta + 1)} (\|u_1(x) - v_1(x)\|_E + \|v_n(x) - v(x)\|_E) \\ & + \left(\frac{3\beta_i}{(\beta_i - 2)\Gamma(4 - \delta)}\right) \frac{H_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} (\|u_n(x) - u(x)\|_E + \|v_n(x) - v(x)\|_E). \end{aligned}$$

Consequently,

$$\|Y(u_n(x), v_n(x)) - Y(u(x), v(x))\|_{E \times E} \leq \Lambda (\|(u_n(x), v_n(x)) - (u(x), v(x))\|_{E \times E}).$$

Therefore,

$$\|Y(u_n(x), v_n(x)) - Y(u(x), v(x))\|_{E \times E} \rightarrow 0 \quad \text{for } n \rightarrow \infty$$

So,  $Y$  is a continuous operator over  $B_{\epsilon}$ .

**Claim 3 :** We prove the equicontinuity.

let  $x_1, x_2 \in [0, 1]$ ; where  $x_1 < x_2$  for  $(y, z, t, w) \in B_{\epsilon}$ , we have

$$\begin{aligned} & \left|Y_i(u_1(x_2), u_2(x_2)) - Y_i(u_1(x_1), u_2(x_1))\right| \leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1)} \times (x_2^{\alpha_i + \beta_i} - x_1^{\alpha_i + \beta_i}) \\ & + \frac{2 |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\beta_i - 2} \times (x_2^{\beta_i} - x_1^{\beta_i}) \\ & + \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\beta_i - 2} \times (x_2^{\beta_i} - x_1^{\beta_i}) \\ & + \frac{\beta_i |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\beta_i - 2} \times (x_2^2 - x_1^2) \\ & + \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\beta_i - 2} \times (x_2^2 - x_1^2). \end{aligned}$$

Also, we can observe that

$$\begin{aligned} & \left|D^{\delta}Y_i(u_1(x_2), u_2(x_2)) - D^{\delta}Y_i(u_1(x_1), u_2(x_1))\right| \\ & \leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} \times (x_2^{\alpha_i + \beta_i - \delta} - x_1^{\alpha_i + \beta_i - \delta}) \\ & + \frac{\Gamma(\beta_i + 1) 2 |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)} (x_2^{\beta_i - \delta} - x_1^{\beta_i - \delta}) \\ & + \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)} (x_2^{\beta_i - \delta} - x_1^{\beta_i - \delta}) \\ & + \frac{2\beta_i |I^{\alpha_i + \beta_i} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\Gamma(3 - \delta)(\beta_i - 2)} (x_2^{2 - \delta} - x_1^{2 - \delta}) \\ & + \frac{|I^{\alpha_i + \beta_i - 1} F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^{\delta}u_i(1))|}{\Gamma(3 - \delta)(\beta_i - 2)} (x_2^{2 - \delta} - x_1^{2 - \delta}). \end{aligned}$$

Hence,

$$\begin{aligned}
 & \left| D^{\delta-1}Y_i(u_1(x_2), u_2(x_2)) - D^{\delta-1}Y_i(u_1(x_1), u_2(x_1)) \right| \\
 & \leq \frac{z_i}{\Gamma(\alpha_i + \beta_i + 2 - \delta)} \times \left( x_2^{\alpha_i + \beta_i - \delta + 1} - x_1^{\alpha_i + \beta_i + 1 - \delta} \right) \\
 & + \frac{(\Gamma(\beta_i + 1)2 I^{\alpha_i + \beta_i} |F_i(1, u_1(1), u_2(1), D^{\delta-1}u_1(1), D^\delta u_1(1))|}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)} \times \left( x_2^{\beta_i - \delta + 1} - x_1^{\beta_i - \delta + 1} \right) \\
 & + \frac{I^{\alpha_i + \beta_i - 1} |F_1(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\Gamma(\beta_i + 1 - \delta)(\beta_i - 2)} \times \left( x_2^{\beta_i - \delta + 1} - x_1^{\beta_i - \delta + 1} \right) \\
 & + \frac{2\beta_i I^{\alpha_i + \beta_i} |F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\Gamma(4 - \delta)(\beta_i - 2)} \times \left( x_2^{3 - \delta} - x_1^{3 - \delta} \right) \\
 & + \frac{I^{\alpha_i + \beta_i - 1} |F_i(1, u_1(1), u_2(1), D^{\delta-1}u_i(1), D^\delta u_i(1))|}{\Gamma(4 - \delta)(\beta_i - 2)} \times \left( x_2^{3 - \delta} - x_1^{3 - \delta} \right).
 \end{aligned}$$

When  $x_1 \rightarrow x_2$ , the right-hand side of the last inequality tends to 0. Consequently,  $Y$  is equicontinuous. Thus, Arzella-ascoli and the above three Claims allow us to confirm that  $Y$  is completely continuous.

Thanks to Shauder theorem, we conclude that (1)-(2) has at least one solution over  $[0, 1]$ . □

### 3.3 Example

Let consider the problem:

$$\begin{cases}
 D^{\frac{1}{5}} D^{\frac{5}{3}} u_1(x) = \frac{|u_1(x) + u_2(x) + D^{\frac{1}{2}} u_1(x) + D^{\frac{3}{2}} u_1(x)|}{100e^{x^2+1} \left( 1 + |u_1(x) + u_2(x) + D^{\frac{1}{2}} u_1(x) + D^{\frac{3}{2}} u_1(x)| \right)} \\
 D^{\frac{1}{4}} D^{\frac{7}{3}} u_2(x) = \frac{\cos(2u_1(x) - u_2(x))}{40 \times \Pi^3(x^2 + 1)} + \frac{1}{150} \times \frac{|D^{\frac{1}{2}} u_2(x) + D^{\frac{3}{2}} u_2(x)|}{1 + (D^{0.7} u_2(x) + D^{1.7} u_2(x))} \\
 u_1(0) = u_1(1) = 2, \\
 u_1'(0) = u_1'(1) = 0, \\
 u_2(0) = u_2(1) = 1, \\
 u_2'(0) = u_2'(1) = 0,
 \end{cases}$$

where,

$$\alpha_1 = \frac{1}{5}, \quad \beta_1 = \frac{5}{2}, \quad \alpha_2 = \frac{1}{4}, \quad \beta_2 = \frac{7}{3}, \quad \delta = \frac{3}{2},$$

and

$$\begin{aligned}
 & F_1(x, u_1(x), u_2(x), D^{\delta-1}u_1(x), D^\delta u_1(x)) \\
 & = \frac{|u_1(x) + u_2(x) + D^{\frac{1}{2}} u_1(x) + D^{\frac{3}{2}} u_1(x)|}{100e^{x^2+1} \left( 1 + |u_1(x) + u_2(x) + D^{\frac{1}{2}} u_1(x) + D^{\frac{3}{2}} u_1(x)| \right)}, \\
 & F_2(x, u_1(x), u_2(x), D^{\delta-1}u_2(x), D^\delta u_2(x)) \\
 & = \frac{\cos(2u_1(x) - u_2(x))}{40 \times \Pi^3(x^2 + 1)} + \frac{1}{150} \times \frac{|D^{\frac{1}{2}} u_2(x) + D^{\frac{3}{2}} u_2(x)|}{1 + (D^{0.7} u_2(x) + D^{1.7} u_2(x))}.
 \end{aligned}$$

After calculation, we have

$$H = \max\left(\frac{1}{150 \times \pi}, \frac{1}{100 \times \exp(1)}\right),$$

$$\Delta_1 = 0.0148, \quad \Delta'_1 = 0.1217, \quad \Delta''_1 = 0.1724,$$

$$\Delta_2 = 0.0166, \quad \Delta'_1 = 0.1661, \quad \Delta''_1 = 0.2275,$$

$$\Lambda = 0.0148 + 0.1217 + 0.1724 + 0.0166 + 0.1661 + 0.2275 < 1.$$

Thanks to Theorem 8, we can understand that the system presented in this example has a unique solution.

### 4 Stability of solutions to system (1)-(2)

To provide a foundation for our study, let's begin by defining Ulam-Hyers stability. This concept will play a crucial role in our analysis.

**Definition 4.1.** We say that the system (1)-(2) is stable in the sense of Ulam Hyers if there exists  $M > 0$ , such that for each positive  $\xi_1, \xi_2$  and for an arbitrary solution  $(u_1, u_2) \in E^2$ , solution of the inequality.

$$\begin{aligned} & \left| D^{\alpha_1} D^{\beta_1} u_1(x) - F_1(x, u_1(x), u_2(x), D^{\delta-1} u_1(x), D^\delta u_1(x)) \right| \leq \xi_1, \\ & \left| D^{\alpha_2} D^{\beta_2} u_2(x) - F_2(x, u_1(x), u_2(x), D^{\delta-1} u_2(x), D^\delta u_2(x)) \right| \leq \xi_2, \end{aligned} \tag{4.1}$$

(with the same conditions as in (2)), we can obtain a solution  $(u'_1, u'_2) \in E^2$  of system (1)-(2) that satisfies

$$\|(u_1 - u'_1, u_2 - u'_2)\|_{E \times E} \leq M(\xi_1 + \xi_2).$$

Let's embark on a proof of the following theorem concerning the stability of Ulam-Hyers in relation to our problem.

**Theorem 4.2.** *Suppose that the conditions of Theorem 8 are satisfied. Then, (1)-(2) is stable in Ulam Hyers sense.*

*Proof.* We take  $(u_1, u_2) \in E \times E$  as a solution of (4.1) and let  $(u'_1, u'_2) \in E \times E$  be the unique solution of (1.1).

Using (4.1), we can write

$$\begin{aligned} & \left| u_1(x) - I^{\alpha_1+\beta_1} F_1(x, u_1(x), u_2(x), D^{\delta-1} u_1(x), D^\delta u_1(x)) \right. \\ & + \frac{2 I^{\alpha_1+\beta_1} F_1(1, u_1(1), u_2(1), D^{\delta-1} u_1(1), D^\delta u_1(1))}{\beta_1 - 2} x^{\beta_1} \\ & - \frac{I^{\alpha_1+\beta_1-1} F_1(x, u_1(1), u_2(1), D^{\delta-1} u_1(1), D^\delta u_1(1))}{\beta_1 - 2} x^{\beta_1} \\ & + \theta_1 + \frac{\beta_1 I^{\alpha_1+\beta_1} F_1(1, u_1(1), u_2(1), D^{\delta-1} u_1(1), D^\delta u_1(1))}{2 - \beta_1} x^2 \\ & \left. - \frac{I^{\alpha_1+\beta_1-1} F_1(1, u_1(1), u_2(1), D^{\delta-1} u_1(1), D^\delta u_1(1))}{2 - \beta_1} x^2 \right| \leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1)}, \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} & \left| u_2(x) - I^{\alpha_2+\beta_2} F_2(x, u_1(x), u_2(x), D^{\delta-1} u_2(x), D^\delta u_2(x)) \right. \\ & + \frac{2 I^{\alpha_2+\beta_2} F_2(1, u_1(1), u_2(1), D^{\delta-1} u_2(1), D^\delta u_2(1))}{\beta_1 - 2} x^{\beta_2} \\ & - \frac{I^{\alpha_2+\beta_2-1} F_2(x, u_1(1), u_2(1), D^{\delta-1} u_2(1), D^\delta u_2(1))}{\beta_1 - 2} x^{\beta_2} \\ & + \theta_2 + \frac{\beta_1 I^{\alpha_2+\beta_2} F_2(1, u_1(1), u_2(1), D^{\delta-1} u_2(1), D^\delta u_2(1))}{2 - \beta_1} x^2 \\ & \left. - \frac{I^{\alpha_2+\beta_2-1} F_2(1, u_1(1), u_2(1), D^{\delta-1} u_2(1), D^\delta u_2(1))}{2 - \beta_2} x^2 \right| \leq \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1)}. \end{aligned} \tag{4.3}$$

By (4.1) and (4.3), we have the following three inequalities:

$$\begin{aligned} \| (u_1 - u'_1) \|_\infty &= \| -Y_1 u_1 + u_1 + Y_1 u_1 - Y_1 u'_1 \|_\infty \\ &\leq \| u_1 - Y_1 u_1 \|_\infty + \| Y_1 u_1 - Y_1 u'_1 \|_\infty \\ &\leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \Delta_1 (\| (u_1) - (u'_1) \|_E), \end{aligned}$$

$$\begin{aligned} \| D_1^\delta(u_1) - D_1^\delta(u'_1) \|_\infty \\ \leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1 - \delta)} + \Delta'_1 (\| (u_1) - (u'_1) \|_E) \end{aligned}$$

and

$$\begin{aligned} \| D_1^{\delta-1}(u_1) - D_1^{\delta-1}(u'_1) \|_\infty \\ \leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 2 - \delta)} + \Delta''_1 (\| (u_1) - (u'_1) \|_E). \end{aligned}$$

So

$$\begin{aligned} \| (u_1 - u'_1) \|_E \leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1 - \delta)} + \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 2 - \delta)} \tag{4.4} \\ + (\Delta_1 + \Delta'_1 + \Delta''_1) (\| (u_1(x)) - (u'_1(x)) \|_E). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \| u_2 - u'_2 \|_\infty &= \| -Y_2 u_2 + u_2 + Y_2 u_2 - Y_2 u'_2 \|_\infty \\ &\leq \| u_2 - Y_2 u_2 \|_\infty + \| Y_2 u_2 - Y_2 u'_2 \|_\infty \\ &\leq \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1)} + \Delta_2 (\| (u_2) - (u'_2) \|_E), \end{aligned}$$

$$\begin{aligned} \| D_2^\delta(u_2) - D_2^\delta(u'_2) \|_\infty \\ \leq \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1 - \delta)} + \Delta'_2 (\| (u_2) - (u'_2) \|_E) \end{aligned}$$

and

$$\begin{aligned} \| D_2^{\delta-1}(u_2) - D_2^{\delta-1}(u'_2) \|_\infty \\ \leq \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 2 - \delta)} + \Delta''_2 (\| (u_2) - (u'_2) \|_E). \end{aligned}$$

Hence,

$$\begin{aligned} \| (u_2 - u'_2) \|_E \leq \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1 - \delta)} + \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 2 - \delta)} \tag{4.5} \\ + (\Delta_2 + \Delta'_2 + \Delta''_2) (\| (u_2) - (u'_2) \|_E). \end{aligned}$$

Finally, by (4.4) and (4.5), we get

$$\begin{aligned} \| (u_1 - u'_1, u_2 - u'_2) \|_{E \times E} \leq \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1)} + \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 1 - \delta)} + \frac{\xi_1}{\Gamma(\alpha_1 + \beta_1 + 2 - \delta)} \\ + \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1)} + \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 1 - \delta)} + \frac{\xi_2}{\Gamma(\alpha_2 + \beta_2 + 2 - \delta)} \\ + (\Delta_1 + \Delta'_1 + \Delta''_1 + \Delta_2 + \Delta'_2 + \Delta''_2) \times (\| (u_1, u_2) - (u'_1, u'_2) \|_{E \times E}). \end{aligned}$$

Hence,

$$\|(u_1 - u'_1, u_2 - u'_2)\|_{E \times E} \leq (\xi_1 + \xi_2) \times A^* + \Lambda (\|(u_1, u_2) - (u'_1, u'_2)\|_{E \times E}),$$

where,

$$A^* = \sum_{i=1}^2 \frac{1}{\Gamma(\alpha_i + \beta_i + 1)} + \frac{1}{\Gamma(\alpha_i + \beta_i + 1 - \delta)} + \frac{1}{\Gamma(\alpha_i + \beta_i + 2 - \delta)}.$$

Consequently,

$$(1 - \Lambda) \times \|(u_1 - u'_1, u_2 - u'_2)\|_{E \times E} \leq (\xi_1 + \xi_2) \times A^*.$$

So, we obtain

$$\|(u_1 - u'_1, u_2 - u'_2)\|_{E \times E} \leq \frac{A^*}{(1 - \Lambda)} \times (\xi_1 + \xi_2).$$

Finally,

$$\|(u_1 - u'_1, u_2 - u'_2)\|_{E \times E} \leq M \times (\xi_1 + \xi_2).$$

In consequence, (1.1) and (1.2) is stable in the sense of Ulam Hyers .

### 4.1 Example

We can see that the earlier example illustrating existence and uniqueness is Ulam-Hyers stable, as it fulfills the criteria established in Theorem 8. Let’s build on this understanding by presenting another example.

The following system satisfies all the conditions in Theorem 8. Thus, it is Ulam-Hyers stable.

$$\begin{cases} D^{\frac{1}{5}} D^{\frac{24}{10}} u_1(x) = \frac{1}{200} \times \left( \frac{\sin(u_1(x))}{e^{x+2}} + \frac{\cos(u_2(x))}{e^{x+3}} + \frac{\cos(D^{\delta-1}u_1(x) + D^{\delta}u_1(x))}{e^{x+4}} \right) \\ D^{\frac{5}{3}} D^{\frac{7}{3}} u_2(x) = \frac{1}{100} \times \left( \frac{(u_1(x) + u_2(x))^2 + |u_1(x) + u_2(x)|}{e^{x+2}(1 + |u_1(x) + u_2(x)|)} + \frac{|D^{\delta-1}u_2(x) + D^{\delta}u_2(x)|}{e^{x+4}(1 + |x + D^{\delta}u_2(x)|)} \right) \\ u_1(0) = u_1(1) = 10, \\ u'_1(0) = u'_1(1) = 0, \\ u_2(0) = u_2(1) = 2, \\ u'_2(0) = u'_2(1) = 0, \end{cases}$$

where,

$$\alpha_1 = \frac{1}{9}, \quad \beta_1 = \frac{24}{10}, \quad \alpha_2 = \frac{3}{5}, \quad \beta_2 = \frac{7}{3}, \quad \delta = \frac{4}{3},$$

and

$$\begin{aligned} & F_1(x, u_1(x), u_2(x), D^{\delta-1}u_1(x), D^{\delta}u_2(x)) \\ &= \frac{1}{200} \times \left( \frac{\sin(u_1(x))}{e^{x+2}} + \frac{\cos(u_2(x))}{e^{x+3}} + \frac{\cos(D^{\delta-1}u_1(x) + D^{\delta}u_1(x))}{e^{x+4}} \right), \\ & F_2(x, u_1(x), u_2(x), D^{\delta-1}u_2(x), D^{\delta}u_2(x)) \\ &= \frac{1}{100} \times \left( \frac{(u_1(x) + u_2(x))^2 + |u_1(x) + u_2(x)|}{e^{x+2}(1 + |u_1(x) + u_2(x)|)} + \frac{|D^{\delta-1}u_2(x) + D^{\delta}u_2(x)|}{e^{x+4}(1 + |x + D^{\delta}u_2(x)|)} \right). \end{aligned}$$

In fact,

$$\begin{aligned}
 H &= \max\left(\frac{1}{100 \times e^2}, \frac{1}{200 \times \exp(2)}\right), \\
 \Delta_1 &= 0.0043 \quad \Delta'_1 = 0.0361 \quad \Delta''_1 = 0.0536, \\
 \Delta_2 &= 0.0068 \quad \Delta'_1 = 0.0731 \quad \Delta''_1 = 0.0855, \\
 \Lambda &= 0.0043 + 0.0361 + 0.0855 + 0.0068 + 0.0731 + 0.0855 < 1, \\
 A^* &= 2.7871, \\
 M &= \frac{A^*}{(1 - \Lambda)} = 3.7633.
 \end{aligned}$$

□

## 5 Conclusion remarks

In this study, we investigated a nonlinear coupled system related to beam deflection, incorporating the Caputo derivative with new parameters and various general nonlinearities on both sides of the equation. These parameters, which are used for sequential derivatives, do not adhere to the standard semi-group and commutativity rules. We demonstrated that, under certain conditions, this system can be simplified to a fourth-order ordinary coupled system similar to beam dynamics. By applying the Banach contraction principle and the Schauder fixed point theorem, we established two main results: the uniqueness of solutions and the existence of at least one solution. To validate our findings, we provided an illustrative example for the first main result and discussed Ulam-Hyers stability using an additional example.

For future research, several avenues can be explored to enhance the understanding and application of this nonlinear coupled system:

1. Extending the analysis to systems with variable-order derivatives could offer deeper insights into more complex beam deflection scenarios.
2. Investigating the impact of different boundary conditions on stability and solution behavior could lead to broader applications in engineering and physics.
3. Developing numerical simulations and methods could help approximate solutions and validate theoretical results more robustly.

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