Intrinsic square functions and their commutators on weighted variable Herz spaces

S. Dehimi and Z. Zeghad

Communicated by: Sarika Verma

Corresponding Author: S. Dehimi

Abstract *In this work, we use the properties of two-weighted Herz spaces with variable exponents to establish the boundedness properties of intrinsic square functions and their associated commutators on these function spaces. The results obtained extend and generalize previously known operator-theoretic results within a broader framework.*

1 Introduction

The intrinsic square function was initially introduced by Wilson [23]. For $0 < \beta \leq 1$, let $\mathcal{C}_\beta$ denote the set of all functions $\varphi : \mathbb{R}^n \to \mathbb{R}$ that satisfy the following conditions:

(i) $\operatorname{supp} \varphi \subset \overline{B(0,1)}$.

(ii) $\int_{\mathbb{R}^n} \varphi(x)\,dx = 0$.

(iii) $|\varphi(x) - \varphi(y)| \leq |x - y|^\beta$ *for all $x, y \in \mathbb{R}^n$.*

For $(y,t) \in \mathbb{R}^{n+1}_+$ and $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, set

$$A_\beta(f)(y,t) = \sup_{\varphi \in \mathcal{C}_\beta} |f * \varphi_t(y)| = \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} \varphi_t(y - z) f(z)\,dz \right|,$$

where $\varphi_t(\cdot) = t^{-n} \varphi\left(\frac{\cdot}{t}\right)$. The intrinsic square function of f, denoted by $S_{\beta,\gamma}(f)$, is then defined as:

$$S_{\beta,\gamma}(f)(x) = \left(\iint_{\Gamma_\gamma(x)} (A_\beta(f)(y,t))^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}, \gamma > 0,$$

where $\Gamma_\gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x - y| \leq \gamma t\}$. For the special case $\gamma = 1$, we denote $S_{\beta,1}(f)$ simply by $S_\beta(f)$.

The intrinsic Littlewood-Paley $\mathcal{G}$-function and $\mathcal{G}^*_\lambda$-function are defined as follows, respectively:

$$\mathcal{G}_\beta(f)(x) = \left(\int_0^\infty (A_\beta(f)(y,t))^2 \frac{dt}{t} \right)^{\frac{1}{2}},$$

and

$$\mathcal{G}^*_{\lambda,\beta}(f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} (A_\beta(f)(y,t))^2 \frac{dy\,dt}{t^{n+1}} \right)^{\frac{1}{2}}, \quad \lambda > 1.$$

Intrinsic square functions play a significant role in harmonic analysis and partial differential equations, as highlighted in [2, 12, 13, 14, 25]. The boundedness of these functions has been

extensively studied. For example, Wilson [24] established the boundedness of S_β on weighted Lebesgue spaces, while Wang [17] demonstrated that $S_\beta, \mathcal{G}_\beta$ and $\mathcal{G}^*_{\lambda,\beta}$ are bounded on weighted Morrey spaces. For additional details and further remarks, we refer readers to recent works such as [1, 5, 6, 7, 15, 16, 18, 20, 21].

For any locally integrable function b on $\mathbb{R}^n$, the commutators $[b, S_\beta], [b, \mathcal{G}_\beta]$ and $[b, \mathcal{G}^*_{\lambda,\beta}]$ are defined, respectively, as follows:

$$[b, S_\beta](f)(x) = \left(\iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}},$$

$$[b, \mathcal{G}_\lambda](f)(x) = \left(\int_0^\infty \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f(z) dz \right|^2 \frac{dydt}{t} \right)^{\frac{1}{2}},$$

and

$$[b, \mathcal{G}^*_{\lambda,\beta}](f)(x) = \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t}{t + |x - y|} \right)^{\lambda n} \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{\mathbb{R}^n} [b(x) - b(z)] \varphi_t(y - z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

where $\lambda > 1$.

Wang [17] investigated the boundedness of the commutators $[b, S_\beta], [b, \mathcal{G}_\lambda]$ and $[b, \mathcal{G}^*_{\lambda,\beta}]$ on weighted Morrey spaces. Furthermore, strong type and weak type estimates on weighted Herz spaces were established in [19].

Hu and Wang [9] established the boundedness of commutators generated by BMO *functions and intrinsic square functions on weighted Herz spaces. Izuki and Noi [10] proved the boundedness of the intrinsic square function on weighted Herz spaces with variable exponent. Wang [22] further showed that the commutator $[b, S_\beta]$ is bounded on Morrey spaces with variable exponents.*

More recently, Izuki and Noi [11] introduced the two weighted Herz spaces with variable exponents and provided a characterization of these spaces in terms of two critical constant exponents. This characterization provides valuable tools for studying the boundedness of various operators on such spaces.

Motivated by the aforementioned studies, the primary objective of this paper is to investigate the boundedness of the intrinsic square functions $S_\beta, \mathcal{G}_\beta, \mathcal{G}^*_{\lambda,\beta}$, as well as their associated commutators, on two-weighted Herz spaces with variable exponents.

2 Preliminaries

For any measurable set $\Omega \subset \mathbb{R}^n$, the notation $|\Omega|$ represents the Lebesgue measure of Ω, and χ_Ω denotes its characteristic function. The expression $f \lesssim g$ indicates the existence of an independent positive constant c, such that $f \le c\,g$. Additionally, $f \approx g$ signifies that both $f \lesssim g$ and $g \lesssim f$ hold.

A nonnegative function w defined on $\mathbb{R}^n$ is called a weight if it is locally integrable. For a given weight w and a measurable set Ω, we write $w(\Omega) := \int_\Omega w(x) dx$.

The set $\mathcal{P}_0(\mathbb{R}^n)$ consists of all measurable functions $p(\cdot)$ defined on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. For $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$, we set

$$p^+ = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x) \quad and \quad p^- = \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x).$$

The subset of $\mathcal{P}_0(\mathbb{R}^n)$ consisting of all measurable functions $p(\cdot)$ satisfying $1 < p^- < p^+ < \infty$, is denoted by $\mathcal{P}(\mathbb{R}^n)$. The modular $\varrho_{p(\cdot)}$ associated with $p(\cdot)$ is defined by

$$\varrho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx.$$

Definition 2.1. Let $p(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ with $p^- \ge 1$ and $p^+ < \infty$. The weighted variable Lebesgue space $L^{p(\cdot)}(w)(\mathbb{R}^n)$ is defined as the space of all measurable functions f on $\mathbb{R}^n$ for which the norm

$$\|f\|_{L^{p(\cdot)}(w)} := \inf \left\{ \mu > 0 : \varrho_{p(\cdot)} \left(\frac{1}{\mu} w^{1/p(\cdot)} f \right) \le 1 \right\}$$

is finite.

Similarly, the space $L_{loc}^{p(\cdot)}(w)(\mathbb{R}^n)$ consists of all measurable functions f such that $f \in L^{p(\cdot)}(w)(K)$ for every compact set $K \subset \mathbb{R}^n$.

Definition 2.2. Let h be a measurable function. We say that h is locally log-Hölder continuous, if there exists a positive constant $c_{\log}$ such that

$$|h(x) - h(y)| \le \frac{c_{\log}}{\ln(e + 1/|x - y|)},$$

for all $x, y \in \mathbb{R}^n$. If

$$|h(x) - h(0)| \le \frac{c_{\log}}{\ln(e + 1/|x|)},$$

for all $x \in \mathbb{R}^n$, then h is said to be log-Hölder continuous at the origin (or has a log decay at the origin). Furthermore, if there exist $h_\infty \in \mathbb{R}$ and $c_{\log} > 0$, such that

$$|h(x) - h_\infty| \le \frac{c_{\log}}{\ln(e + |x|)},$$

for all $x \in \mathbb{R}^n$, then h is said to be log-Hölder continuous at infinity (or has a log *decay at infinity*).

The set $\mathcal{P}^{\ln}(\mathbb{R}^n)$ consists of all exponent functions $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ that are locally log-*Hölder continuous and have a* log *decay at infinity, where* $p_\infty := \lim_{|x| \to \infty} p(x)$*. Notably,* $p(\cdot) \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ *if and only if* $p'(\cdot) \in \mathcal{P}^{\ln}(\mathbb{R}^n)$*, where* $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$*.*

Definition 2.3. Let $f \in L_{\text{loc}}^1(\mathbb{R}^n)$. The Hardy-Littlewood maximal operator $\mathcal{M}$ is defined as follows:

$$\mathcal{M}f(x) = \sup_{B:\text{Ball}, x \in B} \frac{1}{|B|} \int_B |f(y)| dy.$$

Definition 2.4. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. We say that $w \in A_{p(\cdot)}$ if w satisfies the following condition:

$$\sup_{B:\text{Ball}} \frac{1}{|B|} \left\| w^{1/p(\cdot)} \chi_B \right\|_{L^{p(\cdot)}} \left\| w^{-1/p(\cdot)} \chi_B \right\|_{L^{p'(\cdot)}} < \infty.$$

We say that w is an A_1 weight if the inequality $\mathcal{M}w(x) \lesssim w(x)$ holds for almost every $x \in \mathbb{R}^n$.

It is evident that if $w \in A_{p(\cdot)}$ then $w^{-p'(\cdot)/p(\cdot)} \in A_{p'(\cdot)}$. In [11], the authors showed that $A_{p_1(\cdot)} \subseteq A_{p(\cdot)}$ whenever $p_1(\cdot), p(\cdot) \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $p_1(\cdot) \le p(\cdot)$.

We conclude this section by stating a series of theorems and technical lemmas that will be used extensively throughout this work. The following lemma is known as a Hardy-type inequality.

Lemma 2.5. *Let* $0 < a < 1, 0 < q \le \infty$ *and* $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ *be a positive sequence, such that*

$$\left(\sum_{k=-\infty}^{\infty} \varepsilon_k^q \right)^{1/q} < \infty.$$

Then

$$\left(\sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{\infty} a^{|k-j|} \varepsilon_j \right)^q \right)^{1/q} \le \left(\sum_{k=-\infty}^{\infty} \varepsilon_k^q \right)^{1/q}.$$

Wang [17, Theorem 3.1] established the weighted L^p-boundedness of the commutators of intrinsic square functions, as stated in the following result.

Theorem 2.6. *Let* $0 < \beta \le 1, 1 < p \le \infty$ *and* $w \in A_p$*. Then the operators* $[b, S_\beta], [b, \mathcal{G}_\beta]$ *and* $[b, \mathcal{G}_{\lambda,\beta}^*]$ *are bounded on* $L^p(w)(\mathbb{R}^n)$*.*

Definition 2.7. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and w be a weight. We say that $(p(\cdot), w)$ is an $\mathcal{M}$-pair if the maximal operator is bounded on both $L^{p(\cdot)}(w)(\mathbb{R}^n)$ and $L^{p'(\cdot)}\big(w^{-p'(\cdot)/p(\cdot)}\big)(\mathbb{R}^n)$.

Remark 2.8. If $p(\cdot) \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$, then $(p(\cdot), w)$ is an $\mathcal{M}$-pair.

Let F be a family of pairs (f, g) of measurable functions, and let $1 \leq p, q < \infty$. We say that the inequality

$$\|f\|_{L^p(w)} \lesssim \|g\|_{L^p(w)} \tag{2.1}$$

holds for any $(f, g) \in F$ and $w \in A_q$ if (2.1) is valid for every pair in F such that $f \in L^{p(\cdot)}(w)(\mathbb{R}^n)$.

The following extrapolation theorem was established by Cruz-Uribe and Wang [3].

Theorem 2.9. *Let $1 < p_0 < \infty$ and $w \in A_{p_0}$. If $\|f\|_{L^{p_0}(w)} \lesssim \|g\|_{L^{p_0}(w)}$, then $\|f\|_{L^{p(\cdot)}(w)} \lesssim \|g\|_{L^{p(\cdot)}(w)}$ for every $\mathcal{M}$-pair $(p(\cdot), w)$ and all $(f, g) \in F$.*

By combining Theorem 2.6 and Theorem 2.9, we derive the following result.

Theorem 2.10. *Let $0 < \beta \leq 1, p(\cdot) \in \mathcal{P}^{\ln}(\mathbb{R}^n)$ and $w \in A_{p(\cdot)}$. Then the operators $[b, S_\beta], [b, \mathcal{G}_\beta]$ and $[b, \mathcal{G}^*_{\lambda, \beta}]$ are bounded on $L^{p(\cdot)}(w)(\mathbb{R}^n)$.*

Let $k \in \mathbb{Z}$, we set

$$B_k := B(0, 2^k), \quad R_k := B_k \setminus B_{k-1} \quad and \quad \chi_k = \chi_{R_k},$$

where χ_{R_k} denotes the characteristic function of R_k. The following theorems provide various estimates associated with Muckenhoupt weights.

Theorem 2.11 ([4]). *Let $w \in A_p$ with $p \in [1, \infty[$. Let $j, k \in \mathbb{Z}$ such that $j < k$, then*

$$\frac{w(B_k)}{w(B_j)} \lesssim 2^{np(k-j)},$$

and

$$\frac{w(B_j)}{w(B_k)} \lesssim 2^{n\delta_w(j-k)}$$

where $\delta_w \in]0, 1[$ is a constant independent of B_k and B_j.

Theorem 2.12 ([11]). *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), w \in A_{p_1(\cdot)}$ with $p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $j, k \in \mathbb{Z}$ such that $j \leq k$. Then*

$$\frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_j\|_{L^{p(\cdot)}(w)}} \lesssim 2^{\frac{(k-j)np_1^+}{\min(p(0), p_\infty)}}.$$

Theorem 2.13 ([11]). *Let $p(\cdot), p_1(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), w \in A_{p_1(\cdot)}$, and $j, k \in \mathbb{Z}$ such that $k \leq j$. Then*

$$\frac{\|\chi_k\|_{L^{p(\cdot)}(w)}}{\|\chi_j\|_{L^{p(\cdot)}(w)}} \lesssim 2^{\frac{(k-j)n\delta'_w}{\max(p(0), p_\infty)}},$$

where $\delta'_w \in]0, 1[$ is a constant independent of B_k and B_j.

Recall that the space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions b satisfying

$$\|b\|_{\mathrm{BMO}} := \sup_B \frac{1}{|B|} \int_B |b(x) - b_B|\, dx < \infty,$$

where $B = B(x, r)$ denotes a ball in $\mathbb{R}^n$, and

$$b_B = \frac{1}{|B|} \int_B b(y)\, dy.$$

The following result was established by Ho [8].

Theorem 2.14. *Let X be a Banach function spaces. If the Hardy-Littlewood maximal operator is bounded on the associate space X', then*

$$\|b\|_{\mathrm{BMO}} \approx \sup_{B:Ball} \frac{1}{\|\chi_B\|_X} \|(b - b_B)\chi_B\|_X,$$

for all $b \in \mathrm{BMO}(\mathbb{R}^n)$.

3 Boundedness of the intrinsic square functions and their commutators

We begin by stating the definition of the mixed Lebesgue-sequence spaces.

Definition 3.1. Let $p\left(\cdot\right), q\left(\cdot\right) \in \mathcal{P}_0(\mathbb{R}^n)$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined via the modular

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\left\{\lambda_v > 0 : \varrho_{p(\cdot)}\left(\frac{f_v}{\lambda_v^{1/q(\cdot)}}\right) \le 1\right\}.$$

$\ell^{q(\cdot)}(L^{p(\cdot)})$ *is a quasi-Banach space, and its quasi-norm is given by*

$$\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\left\{\mu > 0 : \varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\left(\frac{1}{\mu}(f_v)_v\right) \le 1\right\}.$$

Remark 3.2. If $q^+ < \infty$, then the modular can be expressed as

$$\varrho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) = \sum_v \left\||f_v|^{q(\cdot)}\right\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Moreover, if $p\left(\cdot\right)$ and $q\left(\cdot\right)$ are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)})$ coincides with $\ell^q(L^p)$.

For a given weight function w, the space $\ell^{q(\cdot)}(L^{p(\cdot)}(w))$ consists of all sequences $(f_v)_v$ such that

$$\left\|(f_v)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w))} = \left\|\left(w^{1/p(\cdot)} f_v\right)_v\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

Definition 3.3 ([11]). Let w_1, w_2 be weights on $\mathbb{R}^n$, and let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n)$ and $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$. The inhomogeneous weighted Herz space, denoted by $K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, is defined as the set of all $f \in L_{\mathrm{loc}}^{p(\cdot)}(w_2)(\mathbb{R}^n)$ such that

$$\|f\|_{K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)} := \left\|f\,\chi_{B_0}\right\|_{L^{p(\cdot)}(w_2)} + \left\|\left((w_1(B_k))^{\alpha(\cdot)/n} f\,\chi_k\right)_{k \ge 1}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))} < \infty.$$

Similarly, the homogeneous weighted Herz space, denoted by $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, consists of all $f \in L_{\mathrm{loc}}^{p(\cdot)}(w_2)(\mathbb{R}^n \setminus \{0\})$ such that

$$\left\|\left((w_1(B_k))^{\alpha(\cdot)/n} f\,\chi_k\right)_{k \in \mathbb{Z}}\right\|_{\ell^{q(\cdot)}(L^{p(\cdot)}(w_2))} < \infty.$$

For any $f \in K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, we set

$$\|\{f_k\}\|_{\ell_>^q(L^{p(\cdot)}(w_2))} := \left(\sum_{k=0}^\infty \|f_k\|_{L^{p(\cdot)}(w_2)}^q\right)^{1/q},$$

and

$$\|\{f_k\}\|_{\ell_<^q(L^{p(\cdot)}(w_2))} := \left(\sum_{k=-\infty}^{-1} \|f_k\|_{L^{p(\cdot)}(w_2)}^q\right)^{1/q}.$$

The following theorem is a fundamental result and plays a pivotal role in this paper.

Theorem 3.4 ([11]). *Let $p(\cdot), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n), w_1 \in A_{p_1}$ with $p_1 \in [0, \infty[$ and w_2 be a weight. If $q(\cdot)$ and $\alpha(\cdot)$ are* log-*Hölder continuous at infinity, then*

$$K_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n) = K_{p(\cdot),q_\infty}^{\alpha_\infty}(w_1, w_2)(\mathbb{R}^n).$$

Moreover, if $q(\cdot)$ and $\alpha(\cdot)$ are log-*Hölder continuous at the origin, then*

$$\|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)} \approx \left\|\{(w_1(B_k))^{\frac{\alpha(0)}{n}} f\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)}(w_2))}$$

$$+ \left\|\{(w_1(B_k))^{\frac{\alpha_\infty}{n}} f\}\right\|_{\ell_>^{q_\infty}(L^{p(\cdot)}(w_2))}.$$

Let $\alpha(\cdot) \in L^\infty, p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), w_2 \in A_{p_2(\cdot)}$ and $w_1 \in A_{p_1}$, where $p_1 \in [1, \infty[$. By Theorem 2.11 and Theorem 2.13, there exist two positive constants δ_{w_1} and δ_{w_2} such that, for all $j < k$, the following inequalities hold:

$$\frac{w_1(B_j)}{w_1(B_k)} \lesssim 2^{(j-k)n\delta_{w_1}},$$

and

$$\frac{\|\chi_j\|_{L^{p(\cdot)}(w_2)}}{\|\chi_k\|_{L^{p(\cdot)}(w_2)}} \lesssim 2^{\frac{(j-k)n\delta_{w_2}}{\max(p(0),p_\infty)}}.$$

Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, we set

$$w_0^- = \begin{cases} \delta_{w_1} & if \quad \alpha(0) \geq 0 \\ p_1 & if \quad \alpha(0) < 0 \end{cases}, \quad w_0^+ = \begin{cases} p_1 & if \quad \alpha(0) \geq 0 \\ \delta_{w_1} & if \quad \alpha(0) < 0 \end{cases},$$

$$w_\infty^- = \begin{cases} \delta_{w_1} & if \quad \alpha_\infty \geq 0 \\ p_1 & if \quad \alpha_\infty < 0 \end{cases}, \quad w_\infty^+ = \begin{cases} p_1 & if \quad \alpha_\infty \geq 0 \\ \delta_{w_1} & if \quad \alpha_\infty < 0 \end{cases},$$

$$\delta_1 = \max\{\frac{\delta_{w_2}}{\max(p(0),p_\infty)}, 1 - \frac{p_2^+}{\min(p'(0),p'_\infty)}\} \text{ and } \delta_2 = \max\{\frac{\delta_{w_2}}{\max(p'(0),p'_\infty)}, 1 - \frac{p_2^+}{\min(p(0),p_\infty)}\}.$$

We now present the main results of this section.

Theorem 3.5. Let $0 < \beta \leq 1, p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Assume $w_1 \in A_{p_1}$ with $p_1 \in [1, \infty[$ and $w_2 \in A_{p_2(\cdot)}$, where $p_2(\cdot) \leq p(\cdot)$. Suppose further that $q(\cdot)$ and $\alpha(\cdot)$ are log-Hölder continuous at the origin and infinity, such that

$$-\frac{n\delta_1}{p_1} < \alpha(0), \alpha_\infty < \frac{n\delta_2}{p_1}.$$

Then, the operators S_β and $[b, S_\beta]$ are bounded on $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Proof. We focus solely on proving the boundedness of $[b, S_\beta]$ on $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, as the boundedness of S_β on $\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$ can be established using similar arguments. Let $f \in \dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1, w_2)(\mathbb{R}^n)$ such that

$$f(x) = \sum_{j=-\infty}^\infty f(x)\chi_j(x) = \sum_{j=-\infty}^\infty f_j(x).$$

By Theorem 3.4, we have

$$\begin{aligned}
\left\|[b, S_\beta]f\right\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)} &\approx \left\|\{(w_1(B_k))^{\alpha(0)/n}([b, S_\beta]f)\}\right\|_{\ell_<^{q(0)}(L^{p(\cdot)}(w_2))} \\
&\quad + \left\|\{(w_1(B_k))^{\alpha_\infty/n}([b, S_\beta]f)\}\right\|_{\ell_>^{q_\infty}(L^{p(\cdot)}(w_2))} \\
&\lesssim E_1 + E_2 + E_3 + E_4 + E_5 + E_6,
\end{aligned}$$

where

$$E_1 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=-\infty}^{k-2} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$E_2 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k-1}^{k+1} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$E_3 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k+2}^{\infty} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$E_4 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=-\infty}^{k-2} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty},$$

$$E_5 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=k-1}^{k+1} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty},$$

and

$$E_6 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=k+2}^{\infty} \big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty}.$$

Since $[b,S_\beta]$ is bounded on $L^{p(\cdot)}(w_2)(\mathbb{R}^n)$, it follows that

$$
\begin{aligned}
E_2 + E_5 \lesssim \|b\|_{\mathrm{BMO}} & \left(\Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k-1}^{k+1} \big\| f_j \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)} \right. \\
& \left. + \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=k-1}^{k+1} \big\| f_j \big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty} \right) \\
\lesssim \|b\|_{\mathrm{BMO}} & \left(\Big(\sum_{k=-\infty}^{-1} (w_1(B_k))^{\alpha(0)q(0)/n} \big\| f_k \big\|_{L^{p(\cdot)}(w_2)}^{q(0)} \Big)^{1/q(0)} \right. \\
& \left. + \Big(\sum_{k=0}^{\infty} (w_1(B_k))^{\alpha_\infty q_\infty/n} \big\| f_k \big\|_{L^{p(\cdot)}(w_2)}^{q_\infty} \Big)^{1/q_\infty} \right) \\
\lesssim \|b\|_{\mathrm{BMO}} & \|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

We now proceed to estimate E_1. We have

$$\big\| ([b,S_\beta]f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} \lesssim \big\| ((b-b_{B_j})S_\beta f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)} + \big\| (S_\beta(b-b_{B_j})f_j)\chi_k \big\|_{L^{p(\cdot)}(w_2)}$$
$$\lesssim E_{1.1} + E_{1.2}.$$

First we estimate $E_{1.1}$. Since $\varphi \in \mathcal{C}_\beta$ and $(y,t) \in \Gamma(x)$, we deduce that

$$\left| \int_{R_j} \varphi_t(y-z)f(z)dz \right| \lesssim t^{-n} \int_{R_j \cap \{z: |y-z| \le t\}} |f(z)| \, dz. \tag{3.1}$$

For any $x \in R_k$, $(y,t) \in \Gamma(x)$ and $z \in R_j \cap \{z: |y-z| \le t\}$ such that $j \le k-2$, we have

$$2t \ge |y-z| + |x-y| \ge |x-z| \ge |x| - |z| \ge \frac{|x|}{2}. \tag{3.2}$$

Consequently, combining (3.1) and (3.2), we obtain

$$
\begin{aligned}
S_\beta\left(f\chi_j\right)(x) &\lesssim \left(\iint_{\Gamma(x)} \sup_{\varphi\in\mathcal{C}_\beta} \left|\int_{R_j}\varphi_t(y-z)f(z)dz\right|^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{\frac{|x|}{4}}\int_{|x-y|<t}\left(t^{-n}\int_{R_j}|f(z)|\,dz\right)^2 \frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_{R_j}|f(z)|\,dz\right)\left(\int_{\frac{|x|}{4}}\frac{dt}{t^{2n+1}}\right)^{\frac{1}{2}} \\
&\lesssim |x|^{-n}\|f\chi_j\|_{L^1} \\
&\lesssim 2^{-kn}\|f\chi_j\|_{L^1}.
\end{aligned}
\tag{3.3}
$$

Therefore,

$$
\begin{aligned}
E_{1.1} &\lesssim 2^{-kn}\|f\chi_j\|_{L^1}\left\|(b-b_{B_j})\chi_k\right\|_{L^{p(\cdot)}(w_2)} \\
&\lesssim 2^{-kn}\|f\chi_j\|_{L^1}\left(\left\|(b-b_{B_k})\chi_k\right\|_{L^{p(\cdot)}(w_2)}+\left\|(b_{B_k}-b_{B_j})\chi_k\right\|_{L^{p(\cdot)}(w_2)}\right).
\end{aligned}
$$

By applying Theorem 2.14 with $X=L^{p(\cdot)}(w_2)(\mathbb{R}^n)$ and using the inequality

$$
\left|b_{B_k}-b_{B_j}\right|\leq|j-k|\,\|b\|_{\mathrm{BMO}},
$$

we obtain

$$
\begin{aligned}
E_{1.1} &\lesssim 2^{-kn}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^1}\left(\|\chi_k\|_{L^{p(\cdot)}(w_2)}+(k-j)\|\chi_k\|_{L^{p(\cdot)}(w_2)}\right) \\
&\lesssim (k-j)2^{-kn}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^1}\|\chi_k\|_{L^{p(\cdot)}(w_2)}.
\end{aligned}
$$

Using the generalized Hölder's inequality, we conclude that

$$
E_{1.1}\lesssim (k-j)2^{-kn}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w_2)}.
$$

From Theorem 2.13, it follows that

$$
\begin{aligned}
2^{-kn}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w_2)} &\lesssim \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\|\chi_k\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}^{-1} \\
&\lesssim 2^{\frac{(j-k)n\delta_{w_2}}{\max(p'(0),p'_\infty)}}.
\end{aligned}
$$

On the other hand, Theorem 2.12 yields

$$
\begin{aligned}
2^{-kn}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w_2)} &\lesssim 2^{(j-k)n}\|\chi_j\|_{L^{p(\cdot)}(w_2)}^{-1}\|\chi_k\|_{L^{p(\cdot)}(w_2)} \\
&\lesssim 2^{(j-k)n\left(1-\frac{p_2^+}{\min(p(0),p_\infty)}\right)}.
\end{aligned}
$$

This implies

$$
2^{-kn}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\|\chi_k\|_{L^{p(\cdot)}(w_2)}\lesssim 2^{(j-k)n\delta_2}.
\tag{3.4}
$$

Consequently,

$$
E_{1.1}\lesssim (k-j)2^{(j-k)n\delta_2}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}.
\tag{3.5}
$$

For $E_{1.2}$, observe that for any $\varphi\in\mathcal{C}_\beta$ and $(y,t)\in\Gamma(x)$, we have

$$
\left|\int_{R_j}(b(z)-b_{B_j})\varphi_t(y-z)f(z)dz\right|\lesssim t^{-n}\int_{R_j\cap\{z:|y-z|\leq t\}}\left|(b(z)-b_{B_j})\right||f(z)|\,dz.
\tag{3.6}
$$

Using similar arguments as in the estimate of (3.3), we conclude

$$S_\beta\left(f\chi_j\right) \lesssim 2^{-kn}\int_{R_j}\left|(b(z)-b_{B_j})\right|\left|f(z)\right|dz.$$

By applying the generalized Hölder's inequality, Theorem 2.14 and (3.4), we obtain

$$
\begin{aligned}
E_{1.2} \lesssim & 2^{-kn}\big\|\chi_k\big\|_{L^{p(\cdot)}(w_2)}\int_{R_j}\left|(b(z)-b_{B_j})\right|\left|f(z)\right|dz\\
\lesssim & 2^{-kn}\big\|\chi_k\big\|_{L^{p(\cdot)}(w_2)}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\big\|(b-b_{B_j})\chi_j\big\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\\
\lesssim & 2^{-kn}\left\|b\right\|_{\mathrm{BMO}}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\big\|\chi_k\big\|_{L^{p(\cdot)}(w_2)}\big\|\chi_j\big\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\\
\lesssim & 2^{(j-k)n\delta_2}\left\|b\right\|_{\mathrm{BMO}}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}.
\end{aligned}
\tag{3.7}
$$

Combining (3.5) and (3.7), we deduce

$$\big\|([b,S_\beta]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \lesssim (k-j)2^{(j-k)n\delta_2}\left\|b\right\|_{\mathrm{BMO}}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}, \tag{3.8}$$

where $j \leq k-2$. From (3.8) and Theorem 2.11, it follows that

$$
\begin{aligned}
E_1 \lesssim & \Big(\sum_{k=-\infty}^{-1}\Big((w_1(B_k))^{\alpha(0)/n}\sum_{j=-\infty}^{k-2}(k-j)2^{(j-k)n\delta_2}\left\|b\right\|_{\mathrm{BMO}}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{1/q(0)}\\
\lesssim & \left\|b\right\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-2}(k-j)2^{(j-k)n\delta_2}\Big(\frac{w_1(B_k)}{w_1(B_j)}\Big)^{\alpha(0)/n}\\
& \times(w_1(B_j))^{\alpha(0)/n}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{1/q(0)}\\
\lesssim & \left\|b\right\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)\alpha(0)w_0^+}2^{(j-k)n\delta_2}\\
& \times(w_1(B_j))^{\alpha(0)/n}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{1/q(0)}\\
\lesssim & \left\|b\right\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}\Big(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha(0)w_0^+-n\delta_2)}\\
& \times(w_1(B_j))^{\alpha(0)/n}\big\|f\chi_j\big\|_{L^{p(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{1/q(0)}.
\end{aligned}
$$

Since $\alpha(0)w_0^+ - n\delta_2 < 0$, it follows from Lemma 2.5 that

$$
\begin{aligned}
E_1 \lesssim & \left\|b\right\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}(w_1(B_k))^{\alpha(0)q(0)/n}\big\|f\chi_k\big\|_{L^{p(\cdot)}(w_2)}^{q(0)}\Big)^{1/q(0)}\\
\lesssim & \left\|b\right\|_{\mathrm{BMO}}\big\|f\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

We now proceed to estimate E_3. We have

$$
\begin{aligned}
\big\|([b,S_\beta]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \lesssim & \big\|((b-b_{B_j})S_\beta f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)}+\big\|(S_\beta(b-b_{B_j})f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)}\\
\lesssim & E_{3.1}+E_{3.2},
\end{aligned}
$$

where $k \leq j-2$. For any $x \in R_k, (y,t) \in \Gamma(x)$ and $z \in R_j \cap \{z : |y-z| \leq t\}$, it can be shown that

$$2t \geq |y-z|+|x-y| \geq |x-z| \geq |z|-|x| \geq \frac{|z|}{2}, \tag{3.9}$$

and

$$S_\beta \left(f\chi_j \right)(x) \lesssim \left(\iint_{\Gamma(x)} \sup_{\varphi \in \mathcal{C}_\beta} \left| \int_{R_j} \varphi_t(y-z) f(z) dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\frac{|z|}{4}} \int_{|x-y|<t} \left(t^{-n} \int_{R_j} |f(z)| \, dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}}$$

$$\lesssim \left(\int_{R_j} |f(z)| \, dz \right) \left(\int_{\frac{|z|}{4}} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}}$$

$$\lesssim |z|^{-n} \|f\chi_j\|_{L^1}$$

$$\lesssim 2^{-jn} \|f\chi_j\|_{L^1}.$$

Following a similar approach to the estimates of $E_{1.1}$ and $E_{1.2}$, we obtain

$$E_{3.1} \lesssim (j-k) 2^{-jn} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)} \|\chi_k\|_{L^{p(\cdot)}(w_2)}, \qquad (3.10)$$

and

$$E_{3.2} \lesssim 2^{-jn} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \|\chi_k\|_{L^{p(\cdot)}(w_2)} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)}. \qquad (3.11)$$

From Theorem 2.13, it follows that

$$2^{-jn} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)} \|\chi_k\|_{L^{p(\cdot)}(w_2)} \lesssim \|\chi_j\|_{L^{p(\cdot)}(w_2)}^{-1} \|\chi_k\|_{L^{p(\cdot)}(w_2)}$$

$$\lesssim 2^{\frac{-(j-k)n\delta_{w_2}}{\max(p(0),p_\infty)}}.$$

On the other hand, Theorem 2.12 yields

$$2^{-jn} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)} \|\chi_k\|_{L^{p(\cdot)}(w_2)} \lesssim 2^{(k-j)n} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)} \|\chi_k\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)}^{-1}$$

$$\lesssim 2^{-(j-k)n(1-\frac{p_2^+}{\min(p'(0),p_\infty')})}.$$

This implies

$$2^{-jn} \|\chi_j\|_{L^{p'(\cdot)} \left(w_2^{-p'(\cdot)/p(\cdot)} \right)} \|\chi_k\|_{L^{p(\cdot)}(w_2)} \lesssim 2^{-(j-k)n\delta_1}. \qquad (3.12)$$

Therefore, combining (3.10), (3.11) and (3.12), we deduce

$$E_{3.1} \lesssim (j-k) 2^{-(j-k)n\delta_1} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)},$$

and

$$E_{3.2} \lesssim 2^{-(j-k)n\delta_1} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)}.$$

Hence

$$\left\| ([b, S_\beta] f_j) \chi_k \right\|_{L^{p(\cdot)}(w_2)} \lesssim (j-k) 2^{-(j-k)n\delta_1} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)}.$$

Consequently,

$$
\begin{aligned}
E_3 &\lesssim \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k+2}^{\infty} (j-k)2^{-(j-k)n\delta_1} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)} \\
&\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k+2}^{\infty} (j-k)2^{-(j-k)n\delta_1} \Big(\frac{w_1(B_k)}{w_1(B_j)} \Big)^{\alpha(0)/n} \\
&\quad \times (w_1(B_j))^{\alpha(0)/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)} \\
&\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k+2}^{\infty} (j-k)2^{-(j-k)\alpha(0)w_0^-} 2^{-(j-k)n\delta_1} \\
&\quad \times (w_1(B_j))^{\alpha(0)/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)} \\
&\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=k+2}^{\infty} (j-k)2^{(j-k)(-\alpha(0)w_0^- - n\delta_1)} \\
&\quad \times (w_1(B_j))^{\alpha(0)/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)}.
\end{aligned}
$$

Since $-\alpha(0)w_0^- - n\delta_1 < 0$, it follows from Lemma 2.5 that

$$
\begin{aligned}
E_3 &\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} (w_1(B_k))^{\alpha(0)q(0)/n} \|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{q(0)} \Big)^{1/q(0)} \\
&\lesssim \|b\|_{\mathrm{BMO}} \|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

Using the same reasoning as in the estimates of E_1 and E_3, we can establish that

$$
\begin{aligned}
E_4 &\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=0}^{\infty} \Big(\sum_{j=-\infty}^{k-2} (k-j)2^{(k-j)(\alpha_\infty w_\infty^+ - n\delta_2)} \\
&\quad \times (w_1(B_j))^{\alpha_\infty/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty},
\end{aligned}
$$

and

$$
\begin{aligned}
E_6 &\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=0}^{\infty} \Big(\sum_{j=k+2}^{\infty} (j-k)2^{(j-k)(-\alpha_\infty w_\infty^- - n\delta_1)} \\
&\quad \times (w_1(B_j))^{\alpha_\infty/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty}.
\end{aligned}
$$

Since $\alpha_\infty w_\infty^+ - n\delta_2 < 0$ and $-\alpha_\infty w_\infty^- - n\delta_1 < 0$, it follows from Lemma 2.5 that

$$
\begin{aligned}
E_4 + E_6 &\lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=0}^{\infty} (w_1(B_k))^{\alpha_\infty q_\infty/n} \|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{\infty} \Big)^{1/q_\infty} \\
&\lesssim \|b\|_{\mathrm{BMO}} \|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

$\square$

By applying the same reasoning as in Theorem 3.5, we derive the following theorem.

Theorem 3.6. *Let* $0 < \beta \le 1, p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ *and* $b \in \mathrm{BMO}(\mathbb{R}^n)$. *Assume* $w_1 \in A_{p_1}$ *with* $p_1 \in [1, \infty[$ *and* $w_2 \in A_{p_2(\cdot)}$, *where* $p_2(\cdot) \le p(\cdot)$. *Suppose further that* $q(\cdot)$ *and* $\alpha(\cdot)$ *are* log-*Hölder continuous at infinity, such that*

$$
-\frac{n\delta_1}{p_1} < \alpha_\infty < \frac{n\delta_2}{p_1}.
$$

Then, the operators S_β and $[b, S_\beta]$ are bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Remark 3.7. Theorem 3.5 can be regarded as a generalization of Theorem 3 in [19], Theorem 5 in [10], Theorem 1.4 in [9] and Corollary 21 in [22].

Theorem 3.8. *Let $0 < \beta \leq 1, p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n)$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Assume $w_1 \in A_{p_1}$ with $p_1 \in [1, \infty[$ and $w_2 \in A_{p_2(\cdot)}$, where $p_2(\cdot) \leq p(\cdot)$. Suppose further that $q(\cdot)$ and $\alpha(\cdot)$ are* log-*Hölder continuous at the origin and infinity, such that*

$$-\frac{n\delta_1}{p_1} < \alpha(0), \alpha_\infty < \frac{n\delta_2}{p_1}.$$

Then $\mathcal{G}_\beta$ and $[b, \mathcal{G}_\beta]$ are bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Proof. We focus solely on proving the boundedness of $[b, \mathcal{G}_\beta]$ on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, as the boundedness of $\mathcal{G}_\beta$ on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$ can be established using similar arguments. Wilson [23] proved that for any $0 < \beta \leq 1$, the functions S_β and $\mathcal{G}_\beta$ are pointwise comparable, with comparability constants depending only on β and n. We have

$$\left\|([b, \mathcal{G}_\beta]f_j)\chi_k\right\|_{L^{p(\cdot)}(w_2)} \lesssim \left\|((b - b_{B_j})\mathcal{G}_\beta f_j)\chi_k\right\|_{L^{p(\cdot)}(w_2)} + \left\|(\mathcal{G}_\beta(b - b_{B_j})f_j)\chi_k\right\|_{L^{p(\cdot)}(w_2)}$$

$$\lesssim \Big(\left\|((b - b_{B_j})S_\beta f_j)\chi_k\right\|_{L^{p(\cdot)}(w_2)}$$

$$+ \left\|(S_\beta(b - b_{B_j})f_j)\chi_k\right\|_{L^{p(\cdot)}(w_2)} \Big). \tag{3.13}$$

From (3.13) and the boundedness of $[b, \mathcal{G}_\beta]$ on $L^{p(\cdot)}(w_2)(\mathbb{R}^n)$, and using arguments analogous to the proof of Theorem 3.5, we can establish that $[b, \mathcal{G}_\beta]$ is bounded on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$. $\square$

By applying the same reasoning as in Theorem 3.8, we derive the following theorem.

Theorem 3.9. *Let $0 < \beta \leq 1, p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Assume $w_1 \in A_{p_1}$ with $p_1 \in [1, \infty[$ and $w_2 \in A_{p_2(\cdot)}$, where $p_2(\cdot) \leq p(\cdot)$. Suppose further that $q(\cdot)$ and $\alpha(\cdot)$ are* log-*Hölder continuous at infinity, such that*

$$-\frac{n\delta_1}{p_1} < \alpha_\infty < \frac{n\delta_2}{p_1}.$$

Then, the operators $\mathcal{G}_\beta$ and $[b, \mathcal{G}_\beta]$ are bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Next, we establish the boundedness of $\mathcal{G}^*_{\lambda,\beta}$ and $[b, \mathcal{G}^*_{\lambda,\beta}]$ on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Theorem 3.10. *Let λ, β be real numbers such that $\lambda > 2$ and $0 < \beta \leq 1$. Let $p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n), b \in \mathrm{BMO}(\mathbb{R}^n), w_1 \in A_{p_1}$ where $p_1 \in [1, \infty[$ and $w_2 \in A_{p_2(\cdot)}$ with $p_2(\cdot) \leq p(\cdot)$. Assume further that $q(\cdot)$ and $\alpha(\cdot)$ are* log-*Hölder continuous at the origin and at infinity, such that*

$$-\frac{n\delta_1}{p_1} < \alpha(0), \alpha_\infty < \frac{n\delta_2}{p_1}.$$

Then $\mathcal{G}^*_{\lambda,\beta}$ and $[b, \mathcal{G}^*_{\lambda,\beta}]$ are bounded on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Proof. We focus solely on proving the boundedness of $[b, \mathcal{G}^*_{\lambda,\beta}]$ on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$, as the boundedness of $\mathcal{G}^*_{\lambda,\beta}$ on $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$ can be established using similar arguments. Let $f \in \dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)$ such that

$$f(x) = \sum_{k=-\infty}^{\infty} f(x)\chi_j(x) = \sum_{j=-\infty}^{\infty} f_j(x).$$

From Theorem 3.5, it follows that

$$\begin{aligned}
\big\|[b,\mathcal{G}^*_{\lambda,\beta}]f\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)} \approx & \big\|\{(w_1(B_k))^{\alpha(0)/n}([b,\mathcal{G}^*_{\lambda,\beta}]f)\}\big\|_{\ell^{q(0)}_<(L^{p(\cdot)}(w_2))} \\
& + \big\|\{(w_1(B_k))^{\alpha_\infty/n}([b,\mathcal{G}^*_{\lambda,\beta}]f)\}\big\|_{\ell^{q_\infty}_>(L^{p(\cdot)}(w_2))} \\
\lesssim & F_1 + F_2 + F_3 + F_4 + F_5 + F_6,
\end{aligned}$$

where

$$F_1 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=-\infty}^{k-2} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$F_2 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k-1}^{k+1} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$F_3 = \Big(\sum_{k=-\infty}^{-1} \Big((w_1(B_k))^{\alpha(0)/n} \sum_{j=k+2}^{\infty} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q(0)} \Big)^{1/q(0)},$$

$$F_4 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=-\infty}^{k-2} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty},$$

$$F_5 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=k-1}^{k+1} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty},$$

and

$$F_6 = \Big(\sum_{k=0}^{\infty} \Big((w_1(B_k))^{\alpha_\infty/n} \sum_{j=k+2}^{\infty} \big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} \Big)^{q_\infty} \Big)^{1/q_\infty}.$$

Since $[b,\mathcal{G}^*_{\lambda,\beta}]$ is bounded on $L^{p(\cdot)}(w_2)(\mathbb{R}^n)$, it follows that

$$F_2 + F_5 \lesssim \|b\|_{\text{BMO}} \big\|f\big\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)}.$$

We now proceed to estimate F_1. From (3.1). We have

$$\begin{aligned}
[b,\mathcal{G}^*_{\lambda,\beta}] = & \left(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|} \Big)^{\lambda n} \sup_{\varphi\in\mathcal{C}_\beta} \left| \int_{R_j} [b(x)-b(z)]\, \varphi_t(y-z)f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
\lesssim & \left(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|} \Big)^{\lambda n} \left| t^{-n}\int_{R_j}[b(x)-b(z)]f(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{\frac{1}{2}} \\
\lesssim & \int_{R_j} |b(x)-b(z)||f(z)| \left(\int_0^{|x|} \int_{|y-z|<t} \Big(\frac{t}{t+|x-y|} \Big)^{\lambda n} \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} dz \\
& + \int_{R_j} |b(x)-b(z)||f(z)| \left(\int_{|x|}^{\infty} \int_{|y-z|<t} \Big(\frac{t}{t+|x-y|} \Big)^{\lambda n} \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} dz \\
= & F_{1.1} + F_{1.2}.
\end{aligned}$$

For any $x\in R_k$ and $z\in R_j\cap\{z:|y-z|\le t\}$ such that $j\le k-2$, we obtain

$$t+|x-y| \ge |y-z|+|x-y| \ge |x-z| \ge |x|-|z| \ge \frac{|x|}{2}. \tag{3.14}$$

From (3.14) and the condition $\lambda > 2$, it follows that

$$
\begin{aligned}
F_{1.1} &\lesssim \int_{R_j} |b(x) - b(z)| |f(z)| \left(\int_0^{|x|} \frac{1}{|x|^{\lambda n}} \int_{|y-z|<t} \frac{dydt}{t^{3n+1-\lambda n}} \right)^{\frac{1}{2}} dz \\
&\lesssim \int_{R_j} |b(x) - b(z)| |f(z)| \left(\int_0^{|x|} \frac{1}{|x|^{\lambda n}} \frac{dt}{t^{2n+1-\lambda n}} \right)^{\frac{1}{2}} dz \\
&\lesssim \frac{1}{|x|^n} \int_{R_j} |b(x) - b(z)| |f(z)| \, dz \\
&\lesssim 2^{-kn} \|(b(x) - b)f\chi_j\|_{L^1}.
\end{aligned}
\tag{3.15}
$$

Analogous to the estimate of $F_{1.1}$, we derive

$$
\begin{aligned}
F_{1.2} &\lesssim \int_{R_j} |b(x) - b(z)| |f(z)| \left(\int_{|x|}^{\infty} \int_{|y-z|<t} \frac{dydt}{t^{3n+1}} \right)^{\frac{1}{2}} dz \\
&\lesssim \int_{R_j} |b(x) - b(z)| |f(z)| \left(\int_{|x|}^{\infty} \frac{dt}{t^{2n+1}} \right)^{\frac{1}{2}} dz \\
&\lesssim \frac{1}{|x|^n} \int_{R_j} |b(x) - b(z)| |f(z)| \, dz \\
&\lesssim 2^{-kn} \|(b(x) - b)f\chi_j\|_{L^1}.
\end{aligned}
\tag{3.16}
$$

By applying the generalized Hölder's inequality, along with (3.15) and (3.16), we obtain

$$
\begin{aligned}
F_{1.1} + F_{1.2} &\lesssim 2^{-kn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \|(b(x) - b)\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\lesssim 2^{-kn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \left(|b(x) - b_{B_j}| \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right. \\
&\quad \left. + \|(b_{B_j} - b)\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|([b, \mathcal{G}_{\lambda,\beta}^*]f_j)\chi_k\|_{L^{p(\cdot)}(w_2)} &\lesssim 2^{-kn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \left(\|(b - b_{B_j})\chi_k\|_{L^{p(\cdot)}(w_2)} \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right. \\
&\quad \left. + \|(b_{B_j} - b)\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right) \\
&\lesssim 2^{-kn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \left(\|(b - b_{B_k})\chi_k\|_{L^{p(\cdot)}(w_2)} \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right. \\
&\quad + \|(b_{B_k} - b_{B_j})\chi_k\|_{L^{p(\cdot)}(w_2)} \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\quad \left. + \|(b_{B_j} - b)\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \right).
\end{aligned}
$$

Using similar arguments as in the estimate of $E_{1.1}$, we derive

$$
\begin{aligned}
\|([b, \mathcal{G}_{\lambda,\beta}^*]f_j)\chi_k\|_{L^{p(\cdot)}(w_2)} &\lesssim (k-j)2^{-kn} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \|\chi_k\|_{L^{p(\cdot)}(w_2)} \\
&\quad \times \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\lesssim (k-j)2^{(j-k)n\delta_2} \|b\|_{\mathrm{BMO}} \|f\chi_j\|_{L^{p(\cdot)}(w_2)}.
\end{aligned}
$$

Furthermore, by applying the same reasoning as in the estimate of E_1, we establish that

$$F_1 \lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} \Big(\sum_{j=-\infty}^{k-2} 2^{(k-j)(\alpha(0)w_0^+ - n\delta_2)}$$
$$\times (w_1(B_j))^{\alpha(0)/n} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} (k-j) \Big)^{q(0)} \Big)^{1/q(0)}.$$

Since $\alpha(0)w_0^+ - n\delta_2 < 0$, it follows from Lemma 2.5 that

$$F_1 \lesssim \|b\|_{\mathrm{BMO}} \Big(\sum_{k=-\infty}^{-1} (w_1(B_k))^{\alpha(0)q(0)/n} \|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{q(0)} \Big)^{1/q(0)}$$
$$\lesssim \|b\|_{\mathrm{BMO}} \|f\|_{\dot{K}_{p(\cdot),q(\cdot)}^{\alpha(\cdot)}(w_1,w_2)}.$$

We now proceed to estimate F_3. From (3.1), it follows that

$$[b, \mathcal{G}_{\lambda,\beta}^*] = \Bigg(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t + |x-y|} \Big)^{\lambda n}$$
$$\times \sup_{\varphi \in \mathcal{C}_\beta} \Big| \int_{R_j} [b(x) - b(z)] \, \varphi_t(y-z) f(z) dz \Big|^2 \frac{dydt}{t^{n+1}} \Bigg)^{\frac{1}{2}}$$
$$\lesssim \Bigg(\iint_{\mathbb{R}_+^{n+1}} \Big(\frac{t}{t + |x-y|} \Big)^{\lambda n} \Big| t^{-n} \int_{R_j} [b(x) - b(z)] \, f(z) dz \Big|^2 \frac{dydt}{t^{n+1}} \Bigg)^{\frac{1}{2}}$$
$$\lesssim \int_{R_j} |b(x) - b(z)| \, |f(z)| \Bigg(\int_0^{|z|} \int_{|y-z|<t} \Big(\frac{t}{t + |x-y|} \Big)^{\lambda n} \frac{dydt}{t^{3n+1}} \Bigg)^{\frac{1}{2}} dz$$
$$+ \int_{R_j} |b(x) - b(z)| \, |f(z)| \Bigg(\int_{|z|}^\infty \int_{|y-z|<t} \Big(\frac{t}{t + |x-y|} \Big)^{\lambda n} \frac{dydt}{t^{3n+1}} \Bigg)^{\frac{1}{2}} dz$$
$$= F_{3.1} + F_{3.2}.$$

For any $x \in R_k$ and $z \in R_j \cap \{z : |y-z| \le t\}$ such that $k \le j-2$, we have

$$t + |x-y| \ge |y-z| + |x-y| \ge |x-z| \ge |z| - |x| \ge \frac{|z|}{2}.$$

By employing analogous reasoning to that used in the estimates of $F_{1.1}$, $F_{1.2}$ and E_3, we deduce

$$F_{3.1} + F_{3.2} \lesssim 2^{-jn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \big\|(b(x) - b)\chi_j\big\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}$$
$$\lesssim 2^{-jn} \|f\chi_j\|_{L^{p(\cdot)}(w_2)} \Big(|b(x) - b_{B_j}| \, \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}$$
$$+ \big\|(b_{B_j} - b)\chi_j\big\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \Big).$$

Hence

$$
\begin{aligned}
\big\|([b,\mathcal{G}^*_{\lambda,\beta}]f_j)\chi_k\big\|_{L^{p(\cdot)}(w_2)} &\lesssim 2^{-jn}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\Big(\big\|(b-b_{B_k})\chi_k\big\|_{L^{p(\cdot)}(w_2)}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\quad + \big\|(b_{B_k}-b_{B_j})\chi_k\big\|_{L^{p(\cdot)}(w_2)}\|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\quad + \big\|(b_{B_j}-b)\chi_j\big\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)}\Big) \\
&\lesssim (j-k)2^{-jn}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\|\chi_k\|_{L^{p(\cdot)}(w_2)} \\
&\quad \times \|\chi_j\|_{L^{p'(\cdot)}\left(w_2^{-p'(\cdot)/p(\cdot)}\right)} \\
&\lesssim (j-k)2^{-(j-k)n\delta_1}\|b\|_{\mathrm{BMO}}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}.
\end{aligned}
$$

Analogous to the estimate of E_3, we derive

$$
\begin{aligned}
F_3 \lesssim \|b\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}\Big(\sum_{j=k+2}^{\infty}(j-k)2^{(j-k)(-\alpha(0)w_0^- - n\delta_1)} \\
\times (w_1(B_j))^{\alpha(0)/n}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\Big)^{q(0)}\Big)^{1/q(0)}.
\end{aligned}
$$

Since $-\alpha(0)w_0^- - n\delta_1 < 0$, it follows from Lemma 2.5 that

$$
\begin{aligned}
F_3 &\lesssim \|b\|_{\mathrm{BMO}}\Big(\sum_{k=-\infty}^{-1}(w_1(B_k))^{\alpha(0)q(0)/n}\|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{q(0)}\Big)^{1/q(0)} \\
&\lesssim \|b\|_{\mathrm{BMO}}\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

Using the same reasoning as in the estimates of F_1 and F_3, we can establish that

$$
\begin{aligned}
F_4 \lesssim \|b\|_{\mathrm{BMO}}\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=-\infty}^{k-2}(k-j)2^{(k-j)(\alpha_\infty w_\infty^+ - n\delta_2)} \\
\times (w_1(B_j))^{\alpha_\infty/n}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\Big)^{q_\infty}\Big)^{1/q_\infty}
\end{aligned}
$$

and

$$
\begin{aligned}
F_6 \lesssim \|b\|_{\mathrm{BMO}}\Big(\sum_{k=0}^{\infty}\Big(\sum_{j=k+2}^{\infty}(j-k)2^{(j-k)(-\alpha_\infty w_\infty^- - n\delta_1)} \\
\times (w_1(B_j))^{\alpha_\infty/n}\|f\chi_j\|_{L^{p(\cdot)}(w_2)}\Big)^{q_\infty}\Big)^{1/q_\infty}.
\end{aligned}
$$

Since $\alpha_\infty w_\infty^+ - n\delta_2 < 0$ and $-\alpha_\infty w_\infty^- - n\delta_1 < 0$, it follows from Lemma 2.5 that

$$
\begin{aligned}
F_4 + F_6 &\lesssim \|b\|_{\mathrm{BMO}}\Big(\sum_{k=0}^{\infty}(w_1(B_k))^{\alpha_\infty q_\infty/n}\|f\chi_k\|_{L^{p(\cdot)}(w_2)}^{\infty}\Big)^{1/q_\infty} \\
&\lesssim \|b\|_{\mathrm{BMO}}\|f\|_{\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1,w_2)}.
\end{aligned}
$$

$\square$

By applying the same reasoning as in Theorem 3.10, we derive the following theorem.

Theorem 3.11. *Let λ, β be real numbers such that $\lambda > 2$ and $0 < \beta \leq 1$. Let $p(\cdot), p_2(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n), q(\cdot) \in \mathcal{P}_0(\mathbb{R}^n), \alpha(\cdot) \in L^\infty(\mathbb{R}^n), b \in \mathrm{BMO}(\mathbb{R}^n), w_1 \in A_{p_1}$ where $p_1 \in [1,\infty[$ and*

$w_2 \in A_{p_2(\cdot)}$ with $p_2(\cdot) \leq p(\cdot)$. *Suppose further that $q(\cdot)$ and $\alpha(\cdot)$ are* log-*Hölder continuous at infinity, such that*

$$-\frac{n\delta_1}{p_1} < \alpha_\infty < \frac{n\delta_2}{p_1}.$$

Then $\mathcal{G}^*_{\lambda,\beta}$ and $\left[b, \mathcal{G}^*_{\lambda,\beta}\right]$ are bounded on $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(w_1, w_2)(\mathbb{R}^n)$.

Remark 3.12. Theorem 3.10 can be regarded as a generalization of Theorem 5 in [19] and Theorem 1.6 in [9].

4 Conclusion

Function spaces with variable exponents play a fundamental role in various areas of mathematical analysis, including partial differential equations, fluid dynamics, variational calculus and image processing. The boundedness of singular integral operators on Herz-type spaces is an essential tool in investigating the regularity properties of solutions of elliptic equations with discontinuous coefficients. In this work, we establish the boundedness of the intrinsic square functions $S_\beta, \mathcal{G}_\beta$ and $\mathcal{G}^*_{\lambda,\beta}$, as well as their commutators $[b, S_\beta], [b, \mathcal{G}_\lambda]$ and $[b, \mathcal{G}^*_{\lambda,\beta}]$, generated by BMO *functions, on two-weighted Herz spaces. These results are obtained in the context where all three parameters of the Herz spaces are variable, while the parameters λ and β associated with the operators remain fixed. Our findings are novel and provide a generalization of several established results concerning intrinsic square functions.*

References

[1] *M. Asim, F. Gürbüz,* Some variable exponent boundedness and commutators estimates for fractional rough Hardy operators on central Morrey space, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat., 73 (2), 802-819, (2024).*

[2] *S. Chang, J. Wilson, T. Wolff,* Some weighted norm inequalities concerning the Schödinger operators, *Comment Math Helv., 60, 217-246, (1985).*

[3] *D. Cruz-Uribe, L.A Wang,* Extrapolation and weighted norm inequalities in the variable Lebesgue spaces, *Trans. Am. Math. Soc., 369, 1205-1235, (2017).*

[4] *J. García-Cuerva, J.L. de Rubio de Francia,* Weighted Norm Inequalities and Related Topics, *NorthHolland Mathematics Studies, vol. 116. North-Holland, Amsterdam (1985)*

[5] *F. Gürbüz,* Some estimates for intrinsic square functions and commutators on homogeneous Herz spaces with variable exponents, *Eğitim Publishing, Editor: Gürbüz, Ferit, 1st Edition, April 2022, Volume:1, pp. 40, ISBN: 978-625-8108-88-0 (In Turkish).*

[6] *F. Gürbüz, S. Ding, H. Han, P. Long,* Norm inequalities on variable exponent vanishing Morrey type spaces for the rough singular type integral operators, *Int. J. Nonlinear Sci. Numer. Simul., 22 (6), 721-739, (2021).*

[7] *F. Gürbüz, S. Ding, H. Han, P. Long,* Characterizations of rough fractional-type integral operators on variable exponent vanishing Morrey-type spaces, *Topics in contemporary mathematical analysis and applications, 95-123, Math. Appl. Model. Engin. Soc. Sci., CRC Press, Boca Raton, FL, 2021.*

[8] *K.-P. Ho,* Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, *Anal. Math., 38, 173-85, (2012).*

[9] *Y. Hu, Y. Wang,* The commutators of intrinsic square function on weighted Herz spaces, *Bull. Malays. Math. Sci. Soc., 39, 1421-1437, (2016).*

[10] *M. Izuki, T. Noi,* An intrinsic square function on weighted Herz spaces with variable exponent, *J.Math. Inequal., 11, 799-816, (2017).*

[11] *M. Izuki, T. Noi,* Two weighted Herz spaces with variable exponents, *Bull. Malays. Math. Sci. Soc., 43, 169-200, (2020).*

[12] *C. Kenig,* Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems., *Providence: Am. Math. Soc, 1994.*

[13] *E. Pişkin, Y. Dinç, C. Tunç,* Lower and Upper Bounds for the Blow up Time for Generalized Heat Equations with Variable Exponents, *Palest. J. Math., 10 (2), 601-608, (2021).*

[14] *E. M. Stein,* The development of square functions in the work of A.Zygmund., *Bull Amer Math Soc 1982; 7 (2), 359-75, (1982).*

[15] *S. Sultan, A. Hussain, M. Sultan, F. Gürbüz,* Boundedness of higher order commutators of sublinear operators on grand variable Herz-Morrey spaces, *Rom. J. Math. Comput. Sci.,* **14 (2)**, *1-11, (2024).*

[16] *S. Sultan, M. Sultan, F. Gürbüz,* BMO estimate for the higher order commutators of Marcinkiewicz integral operator on grand variable Herz-Morrey spaces, *Commun. Fac. Sci. Univ. Ank. Ser. A1. Math. Stat.,* **72 (4)**, *1000-1018, (2023).*

[17] *H. Wang,* Intrinsic square functions on the weighted Morrey spaces, *J.Math. Anal. Appl.,* **396** *, 302-314, (2012).*

[18] *H. Wang,* Boundedness of intrinsic square functions on the weighted weak Hardy spaces, *Integral Equations Operator Theory.,* **75**, *135-149, (2013).*

[19] *H. Wang,* The boundedness of intrinsic square functions on the weighted Herz spaces, *J. Funct Spaces 2014, Art. ID 274521, 14 pp.*

[20] *H. Wang, H. P. Liu,* Weak type estimates of intrinsic square functions on the weighted Hardy spaces, *Arch. Math.,* **97**, *49-59, (2011).*

[21] *H. Wang, H. P. Liu,* The intrinsic square function characterizations of weighted Hardy spaces, *Illinois J. Math.,* **56**, *367-381, (2012).*

[22] *L. Wang,* Boundedness of the commutator of the intrinsic square function in variable exponent spaces, *J Korean Math Soc.,* **55 (4)**, *939-962, (2018).*

[23] *M. Wilson,* The intrinsic square function, *Rev. Mat. Iberoam.,* **23**, *771-791, (2007).*

[24] *M. Wilson,* Weighted Littlewood-Paley Theory and Exponential-Square Integrability, *Lecture Notes in Math., Vol 1924. Springer, Berlin, (2008).*

[25] *N. Yılmaz, E. Pişkin* Blow up of solutions for a logarithmic Petrovsky equation with variable exponent, *Palest. J. Math.,* **23 (4)**, *948-954, (2024).*

Author information

S. Dehimi, Department of Mathematics, Faculty of Mathematics and Informatics, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arreridj, Bordj Bou Arreridj, Algeria.
E-mail: `souheyb.dehimi@univ-bba.dz, sohayb20091@gmail.com`

Z. Zeghad, Ecole Normale Supérieure de Bousaada, Laboratory of Functional Analysis and Geometry of Spaces, Bousaada 28001, Algeria.
E-mail: `zeghad.zouheyr@ens-bousaada.dz`