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Taylor and Laurent Series Expansions for α-conformable Fractional Analytic Functions

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Abstract *In this paper we utilize the concepts of α-conformable fractional axes and α-conformable fractional analytic function to present the α-conformable fractional versions of the Taylor series and Laurent series expansions.*

1 Introduction

Complex variable functions are often much better behaved than real variable functions, especially if they are analytic. It is known that, analyticity of a complex valued function over a domain within the complex plane is a central idea of complex theory. Analytic functions have a wide range of nice and strong properties, for instance, the limit of a uniformly convergent sequence of analytic functions is itself analytic. Moreover, unlike the real variable functions, all analytic functions have Taylor series expansion. Structurally, classical theorems for series, that can be applied to the case of real variable functions, are also applicable to the case of complex variable functions. The only difference is that intervals of convergence will be replaced by disks or circles of convergence. Therefore, a much wider range of results and consequences can be drawn about complex analytic functions than about real differentiable functions [*17*], [*3*] [*12*], [*13*], [*2*].

Let us now state both Taylor series and Laurent series in classical complex analysis.

Theorem 1.1. [*12*] *(Taylor series of complex functions) If f is an analytic function throughout the open disk $|z-z_\circ|<R$, that is centered at $z_\circ$ and has a radius of R, then f has the following power series expansion*

$$f(z)=\sum_{n=0}^{\infty}c_n\left(z-z_\circ\right)^n, \qquad |z-z_\circ|<R, \tag{1.1}$$

where

$$c_n=\frac{f^{(n)}(z_\circ)}{n!}=\frac{1}{2\pi i}\oint_C\frac{f(z)}{\left(z-z_\circ\right)^{n+1}}dz \quad \left(n=1,2,\cdots\right), \tag{1.2}$$

and C denote any positively oriented simple closed contour around $z_\circ$ and lying in open disk $|z-z_\circ|<R$. Moreover, the series converges uniformly to $f(z)$ for all z lies in that open disk.

For complex valued functions that are not analytic everywhere (have few singularities), we are still able to find power series expansions for such functions. In this particular case, we need to include negative powers of the variable within the series expansions. Such series expansions are called Laurent's series. It should be noted that, depending on the terms with negative powers *in the Laurent's series expansion, one can track the nature of isolated singularities. Moreover, Laurent's series leads to the most important series in contemporary mathematics that is Fourier series.*

Theorem 1.2. [12] *(Laurent series of complex functions) Consider an annular region $r < |z - z_\circ| < R$ that is bounded by two concentric circles centered at $z_\circ$. Let f be analytic on these two circles as well as analytic throughout this annular region such that few singularities occur inside the small circle. Then at each point z within the annular region, f can be represented as positive and negative powers of $(z - z_\circ)$ as follows*

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_\circ)^n, \qquad r < |z - z_\circ| < R, \tag{1.3}$$

where

$$c_n = \frac{f^{(n)}(z_\circ)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_\circ)^{n+1}} dz \quad (n = 0, \pm 1, \pm 2, \cdots)$$

and C denote any positively oriented simple closed contour around $z_\circ$ and lying in the annular domain.

In 2018, Khalil, R. et al., presented the concept of α-conformable fractional analytic functions by using the α-conformable fractional derivative where $\alpha \in (0, 1)$ [18]. Additionally, for α-conformable fractional analytic functions, they introduced the notions of both α-conformable fractional Cauchy theorem and α-conformable fractional Cauchy formula.

Lately, in 2023, Adm, M. and Khalil, R. [14], reinterduced an updated version of α-conformable fractional analytic functions. Moreover, they established α-conformable fractional forms for both Cauchy integral formula as well as Cauchy Riemann equations.

In this paper, we prove Taylor and Laurent series expansions for α-conformable fractional analytic functions.

2 Preliminaries

In 2014, Khalil, R. et al., published an article about a novel definition of fractional derivatives. They introduced the following definition of α-conformable fractional derivative.

Definition 2.1. [19] Suppose $u : (0, \infty) \to \mathbb{R}$. Then for $x \in (0, \infty)$ the α-conformable fractional derivative of u at x denoted by $D^\alpha u(x)$ is given as

$$D^\alpha u(x) = \lim_{\epsilon \to 0} \frac{u(x + \epsilon x^{1-\alpha}) - u(x)}{\epsilon}, \qquad \text{where } \alpha \in (0, 1). \tag{2.1}$$

By the above definition, we have the following differentiation rules:

$$\begin{aligned} &(i)\ D^\alpha(c_1 u + c_2 v) = c_1 D^\alpha(u) + c_2 D^\alpha(v), \text{ for all } c_1, c_2 \in \mathbb{R}, \\ &(ii)\ D^\alpha(k) = 0, \text{ for all constant functions } f(x) = k, \\ &(iii)\ D^\alpha(uv) = uD^\alpha(v) + vD^\alpha(u), \\ &(iv)\ D^\alpha(\tfrac{u}{v}) = \tfrac{vD^\alpha(u) - uD^\alpha(v)}{v^2},\ v(x) \neq 0. \end{aligned} \tag{2.2}$$

Also, it can be shown that

$$\begin{aligned} &(i)\ D^\alpha(x^p) = px^{p-\alpha}, \\ &(ii)\ D^\alpha(\sin(\tfrac{1}{\alpha}x^\alpha)) = \cos(\tfrac{1}{\alpha}x^\alpha), \\ &(iii)\ D^\alpha(\cos(\tfrac{1}{\alpha}x^\alpha)) = -\sin(\tfrac{1}{\alpha}x^\alpha), \\ &(iv)\ D^\alpha(e^{\frac{1}{\alpha}x^\alpha}) = e^{\frac{1}{\alpha}x^\alpha}. \end{aligned} \tag{2.3}$$

It should be noted that, the case $\alpha = 1$ reduces the above differentiation rules into the corresponding classical rules for ordinary derivatives of order one.

Various differential equations, that include α-conformable fractional derivatives, have been, precisely, modeled many complex real world problems [4], [5], [6], [16], [7], [1], [15], [8], [9], [10], [11].

Now, the standard notation for any complex valued function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ in complex variable z, where $z = x + iy$ for some $x; y \geq 0$, is

$$f(z) = f(x, y) = u(x, y) + iv(x, y), \tag{2.4}$$

where $u(x, y)$ is the real part and $v(x, y)$ is the imaginary part of f.

Definition 2.2. [14] Let $f : D \subseteq \{(x, y) : x, y > 0\} \to \mathbb{C}$. Then f is called α-conformable fractional differentiable at $z_\circ = x_\circ + iy_\circ \in D$ and denoted by $f^\alpha(z_\circ)$, if

$$\lim_{(\epsilon_1,\epsilon_2)\to(0,0)} \frac{f(x_\circ + \epsilon_1 x_\circ^{1-\alpha}, y_\circ + \epsilon_2 y_\circ^{1-\alpha}) - f(x, y)}{\epsilon_1 + i\epsilon_2}, \quad \text{where } \alpha \in (0, 1)\,, \tag{2.5}$$

exists. Moreover, if there exists $\delta > 0$ such that f is α-differentiable for all $z \in B(\delta, z_\circ)$ where $B(\delta, z_\circ)$ is an open disc centered at $z_\circ$, then f is said to be α-fractional analytic at $z_\circ$.

Theorem 2.3. [14] *(α-Conformable Fractional Cauchy Integral Formula) Let f be α-conformable fractional analytic everywhere inside and on a simple closed α-contour C taken in the positive sense such that*

$$C := \{z^\alpha = x^\alpha + iy^\alpha : |(x^\alpha + iy^\alpha) - (\xi_\circ^\alpha + i\xi_\circ^\alpha)| = r\}\,.$$

Then for all $z_\circ \in C^\circ$, the interior of C, we have

$$f(z_\circ^\alpha) = \frac{\alpha}{2\pi i} \oint_C \frac{f(z^\alpha)}{z^\alpha - z_\circ^\alpha} dz^\alpha. \tag{2.6}$$

3 Main Results

3.1 $\alpha-$Fractional Plane in the Euclidean Space

For $\mathbb{R}^2 = \{(x, y) : x,\, y \in \mathbb{R}\}$, we have

$$E_1 = \{(x, 0) : x \in \mathbb{R}\}\,, \text{ is the } x\text{-axis,}$$

and

$$E_2 = \{(0, y) : y \in \mathbb{R}\}\,, \text{ is the } y\text{-axis.}$$

Consequently, we present the following definition.

Definition 3.1. ($x^\alpha y^\alpha$-fractional plane) Let

$$\begin{aligned} A_1 &= \{x^\alpha : x \geq 0,\ \alpha \in (0, 1)\ \} \equiv [0, \infty) \\ &\text{and} \\ A_2 &= \{-x^\alpha : x \geq 0,\ \alpha \in (0, 1)\ \} \equiv (-\infty, 0)\,. \end{aligned} \tag{3.1}$$

Then $A = A_1 \cup A_2$ is called the x^α-fractional axis.

Similarly, let

$$\begin{aligned} B_1 &= \{y^\alpha : y \geq 0,\ \alpha \in (0, 1)\ \} \equiv [0, \infty) \\ &\text{and} \\ B_2 &= \{-y^\alpha : y \geq 0,\ \alpha \in (0, 1)\} \equiv (-\infty, 0)\,. \end{aligned} \tag{3.2}$$

Then $B = B_1 \cup B_2$ is called the y^α-fractional axis.

Moreover, the plane that consists of both x^α-fractional axis and y^α- fractional axis is said to be $x^\alpha y^\alpha$-fractional palne. This new plane contains the point $(0^\alpha, 0^\alpha) \equiv (0, 0)$ as its origin.

It should be noted that, from (3.1) *and* (3.2), *for any point* z^α *in the first quadrant,* z^α *has the form* (x^α, y^α), *where* $x, y \geq 0$. *It should be noted that, we do not define* $z^\alpha = x^\alpha + iy^\alpha$, *but rather, we took all the pairs* (x^α, y^α) *for* $x, y \geq 0$ *and here come both the* x^α*-fractional axis and the* y^α*-fractional axis. Any element in such axes is a pair* (x^α, y^α) *and is written as* z^α.

Remark 3.2. There is a simple mapping

$$T_\alpha : (x, y) \;\mapsto\; (x^\alpha, y^\alpha),$$

which sends each point of the classical plane to a corresponding point in the $x^\alpha y^\alpha$–fractional plane. This mapping keeps the overall structure but changes the geometry. Ordinary circles, for example, become α-fractional circles whose shapes depend on α. As $\alpha \to 1$, the usual complex plane is recovered, while for $\alpha < 1$ the distortion reflects the new geometry where our fractional expansions are developed (See Figure 1).

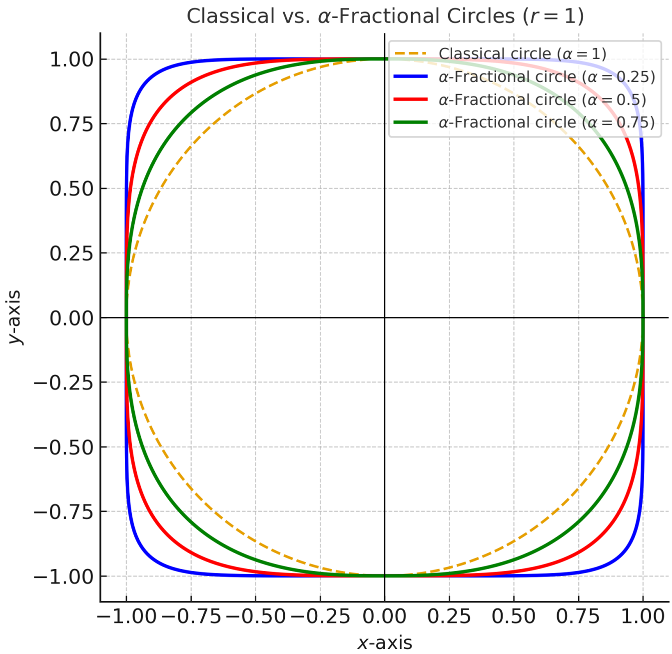

Figure 1. Comparison between a classical circle ($\alpha = 1$) and several α-fractional circles with $\alpha = 0.25, 0.5, 0.75$. As α decreases from 1 toward 0, the shape of the circle distorts, highlighting the geometric differences between the classical plane and the $x^\alpha y^\alpha$-fractional plane.

3.2 Taylor Series for Fractional Analytic Functions

In this section, elements of $\mathbb{C}$ *are written as* $z^\alpha = x^\alpha + iy^\alpha$ $(0 < \alpha < 1)$. *Also, the idea of* $x^\alpha y^\alpha$*-fractional palne* (3.1, 3.2) *is adopted for all analytical descriptions of any* α*-fractional region or contour over* $\mathbb{C}$.

Theorem 3.3. *(Taylor Series for Fractional Analytic Functions) Let* f *be an* α*-fractional analytic function throughout an open* α*-fractional disk*

$$|z^\alpha - z_\circ^\alpha| < r,$$

that is centered at $z_\circ^\alpha$ *and has a radius of* r, *then* f *has the following* α*-fractional power series expansion*

$$f(z^\alpha) = \sum_{n=0}^{\infty} c_n \left(z^\alpha - z_\circ^\alpha\right)^n, \quad |z^\alpha - z_\circ^\alpha| < r, \tag{3.3}$$

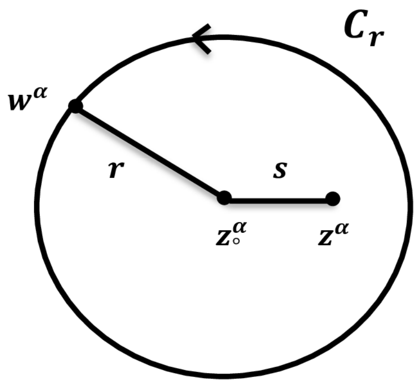

Figure 2. Taylor's series

where $c_n = \frac{f^{(n)\alpha}(z_\circ^\alpha)}{n!}$ $(n = 1, 2, \cdots)$ *such that*

$$f^{(n)\alpha}(z_\circ^\alpha) = \frac{n!\alpha}{2\pi i} \oint_{C_r} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha, \tag{3.4}$$

for any positively oriented simple closed α*-fractional circle* C_r *around* $z_\circ^\alpha$ *that lies in the open* α*-fractional disk. Such* α*-power series converges to* $f(z^\alpha)$ *for all* z^α *lies in the open* α*-fractional disk* $|z^\alpha - z_\circ^\alpha| < r$ *and is called Taylor series for* α*-fractional analytic function.*

Proof. *Let* $f(z^\alpha)$ *be an* α*-fractional analytic function throughout the open* α*-fractional disk* $|z^\alpha - z_\circ^\alpha| < r$ *centered at* $z_\circ^\alpha$ *(See Figure 2).*

From the α*-fractional Cauchy integral formula* (2.6)*, we have*

$$f(z^\alpha) = \frac{\alpha}{2\pi i} \oint_{C_r} \frac{f(w^\alpha)}{w^\alpha - z^\alpha} dw^\alpha, \quad |z^\alpha - z_\circ^\alpha| < r \tag{3.5}$$

where C_r *any positively oriented simple closed* α*-fractional contour around* z^α *and lying in the open* α*-disk. Without loss of generality, let us consider* C_r *to be* α*-fractional circle as follows*

$$C_r := \left\{(x^\alpha, y^\alpha) : (x^\alpha - x_\circ^\alpha)^2 + (y^\alpha - y_\circ^\alpha)^2 = r^2\right\}. \tag{3.6}$$

Now, from (3.5)*, we have*

$$\begin{aligned} f(z^\alpha) &= \frac{\alpha}{2\pi i} \oint_{C_r} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha) - (z^\alpha - z_\circ^\alpha)} dw^\alpha \\ &= \frac{\alpha}{2\pi i} \oint_{C_r} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)\left[1 - \frac{(z^\alpha - z_\circ^\alpha)}{(w^\alpha - z_\circ^\alpha)}\right]} dw^\alpha. \end{aligned} \tag{3.7}$$

But, noting that $|z^\alpha - z_\circ^\alpha| = s < r = |w^\alpha - z_\circ^\alpha|$ *(See figure 1). Thus,* $\left|\frac{z^\alpha - z_\circ^\alpha}{w^\alpha - z_\circ^\alpha}\right| < 1$ *and also,*

$$\frac{1}{1 - \frac{(z^\alpha - z_\circ^\alpha)}{(w^\alpha - z_\circ^\alpha)}} = \sum_{n=0}^{\infty} \frac{(z^\alpha - z_\circ^\alpha)^n}{(w^\alpha - z_\circ^\alpha)^n}. \tag{3.8}$$

Therefore, by invoking the geometric series argument that is, integral of uniformly convergent

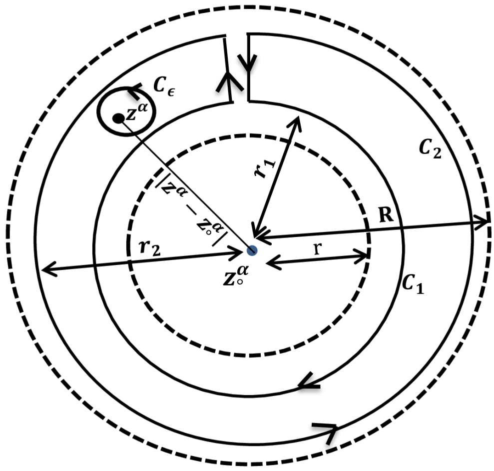

Figure 3. Laurent's series

series converges to the sum of the integrals, (3.7) *can be written as*

$$f(z^\alpha) = \frac{\alpha}{2\pi i}\oint_{C_r}\left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)}\sum_{n=0}^{\infty}\frac{(z^\alpha - z_\circ^\alpha)^n}{(w^\alpha - z_\circ^\alpha)^n}\right)dw^\alpha$$
$$= \frac{\alpha}{2\pi i}\sum_{n=0}^{\infty}\oint_{C_r}\left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}}dw^\alpha\right)(z^\alpha - z_\circ^\alpha)^n$$
$$= \sum_{n=0}^{\infty}\frac{f^{(n)\alpha}(z_\circ^\alpha)}{n!}(z^\alpha - z_\circ^\alpha)^n. \tag{3.9}$$

Since the geometric series converges uniformly on the α–fractional disk, we may differentiate and integrate term by term, as justified in [20, Theorem 4.1]. This establishes the validity of the expansion and completes the proof.

Theorem 3.4. *(Laurent Series for Fractional Analytic Functions) Let f be an α–fractional analytic function throughout an α-fractional annular region $r < |z^\alpha - z_\circ^\alpha| < R$ centered at $z_\circ^\alpha$. Then for every z^α belongs to the α-fractional annular region, f has the following α-fractional power series expansion*

$$f(z^\alpha) = \sum_{n=-\infty}^{\infty} c_n (z^\alpha - z_\circ^\alpha)^n, \tag{3.10}$$

where $c_n = \frac{f^{(n)\alpha}(z_\circ^\alpha)}{n!}$ $(n = 0, \pm1, \pm2, \cdots)$ such that

$$f^{(n)\alpha}(z_\circ^\alpha) = \frac{n!\alpha}{2\pi i}\oint_C \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}}dw^\alpha, \tag{3.11}$$

for any positively oriented simple closed α-contour C around $z_\circ^\alpha$and lying in the α-fractional annular region. Such α-power series converges to $f(z^\alpha)$ for all z^α lies in the α-fractional annular region and is called Laurent series for the α-fractional analytic function f.

Proof. *Let $f(z^\alpha)$ be an α-fractional analytic function throughout the α-fractional annular region $r < |z^\alpha - z_\circ^\alpha| < R$ centered at $z_\circ^\alpha$ (See Figure 3).*

For any z^α belongs to this region, one can apply the α-fractional Cauchy integral formula (2.6) *to get*

$$f(z^\alpha) = \frac{\alpha}{2\pi i}\oint_{C_\in}\frac{f(w^\alpha)}{w^\alpha - z^\alpha}dw^\alpha, \tag{3.12}$$

where $C_\in$ is an α-fractional circle centered at z^α and small enough to be, completely, contained in the α-fractional annular region (See figure 1).

Now, by the principle of deformation of paths, $C_\in$ can be continuously deformed into a closed α-fractional annular region $r_1 < |z^\alpha - z_\circ^\alpha| < r_2$ that is contained in the original α-fractional annular region and whose interior contains the point z^α. Hence, the right hand side of (3.12) can be reduced into integral around the two α-fractional circles C_1 and C_2 as follows

$$\begin{aligned} f(z^\alpha) &= \frac{\alpha}{2\pi i} \oint\limits_{C_2 \cup \{-C_1\}} \frac{f(w^\alpha)}{w^\alpha - z^\alpha} dw^\alpha \\ &= \frac{\alpha}{2\pi i} \oint\limits_{C_2} \frac{f(w^\alpha)}{w^\alpha - z^\alpha} dw^\alpha - \frac{\alpha}{2\pi i} \oint\limits_{C_1} \frac{f(w^\alpha)}{w^\alpha - z^\alpha} dw^\alpha \\ &= \frac{\alpha}{2\pi i} \oint\limits_{C_2} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha) - (z^\alpha - z_\circ^\alpha)} dw^\alpha \\ &+ \frac{\alpha}{2\pi i} \oint\limits_{C_1} \frac{f(w^\alpha)}{(z^\alpha - z_\circ^\alpha) - (w^\alpha - z_\circ^\alpha)} dw. \end{aligned} \tag{3.13}$$

Clearly, the last two integrals of (3.13) act differently depending on the location of the integral variable w^α.

For C_2, we have $|z^\alpha - z_\circ^\alpha| < |w^\alpha - z_\circ^\alpha|$. Hence, $\left|\frac{z^\alpha - z_\circ^\alpha}{w^\alpha - z_\circ^\alpha}\right| < 1$. So, on C_2, we have

$$\frac{1}{(w^\alpha - z_\circ^\alpha) - (z^\alpha - z_\circ^\alpha)} = \frac{1}{(w^\alpha - z_\circ^\alpha)\left(1 - \frac{(z^\alpha - z_\circ^\alpha)}{(w^\alpha - z_\circ^\alpha)}\right)}. \tag{3.14}$$

But,

$$\frac{1}{1 - \frac{(z^\alpha - z_\circ^\alpha)}{(w^\alpha - z_\circ^\alpha)}} = \sum_{n=0}^{\infty} \frac{(z^\alpha - z_\circ^\alpha)^n}{(w^\alpha - z_\circ^\alpha)^n}. \tag{3.15}$$

Thus, on C_2,

$$\frac{1}{(w^\alpha - z_\circ^\alpha) - (z^\alpha - z_\circ^\alpha)} = \frac{1}{(w^\alpha - z_\circ^\alpha)\left(1 - \frac{(z^\alpha - z_\circ^\alpha)}{(w^\alpha - z_\circ^\alpha)}\right)} = \sum_{n=0}^{\infty} \frac{(z^\alpha - z_\circ^\alpha)^n}{(w^\alpha - z_\circ^\alpha)^{n+1}} \tag{3.16}$$

Alternatively, on C_1, we have $|w^\alpha - z_\circ^\alpha| < |z^\alpha - z_\circ^\alpha|$. So, $\left|\frac{w^\alpha - z_\circ^\alpha}{z^\alpha - z_\circ^\alpha}\right| < 1$. So, on C_1, we have

$$\frac{1}{(z^\alpha - z_\circ^\alpha) - (w^\alpha - z_\circ^\alpha)} = \frac{1}{(z^\alpha - z_\circ^\alpha)\left(1 - \frac{(w^\alpha - z_\circ^\alpha)}{(z^\alpha - z_\circ^\alpha)}\right)} = \sum_{n=0}^{\infty} \frac{(w^\alpha - z_\circ^\alpha)^n}{(z^\alpha - z_\circ^\alpha)^{n+1}}. \tag{3.17}$$

Therefore, utilizing both (3.16, 3.17), one can invoke the geometric series argument by which integral of uniformly convergent series converges to the sum of the integrals. Also, because both series expansions used in this representation are uniformly convergent on their respective *contours, we are allowed to integrate term by term. Thus, (3.13) can be written as*

$$\begin{aligned} f(z^\alpha) &= \frac{\alpha}{2\pi i} \oint\limits_{C_2 \cup \{-C_1\}} \frac{f(w^\alpha)}{w^\alpha - z^\alpha} dw^\alpha \\ &= \frac{\alpha}{2\pi i} \oint\limits_{C_2} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha) - (z^\alpha - z_\circ^\alpha)} dw^\alpha \\ &+ \frac{\alpha}{2\pi i} \oint\limits_{C_1} \frac{f(w^\alpha)}{(z^\alpha - z_\circ^\alpha) - (w^\alpha - z_\circ^\alpha)} dw \\ &= \frac{\alpha}{2\pi i} \sum_{n=0}^{\infty} \oint\limits_{C_2} \left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha \right) (z^\alpha - z_\circ^\alpha)^n \\ &+ \frac{\alpha}{2\pi i} \sum_{n=0}^{\infty} \oint\limits_{C_1} \left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{-n}} dw^\alpha \right) (z^\alpha - z_\circ^\alpha)^{-(n+1)} . \end{aligned} \tag{3.18}$$

Clearly, the two series in (3.18) *, reveal positive and negative powers of* $(z - z_\circ)$ *. By reindexing the last series of* (3.18)*, one can write* (3.18) *as*

$$\begin{aligned} f(z^\alpha) &= \\ & \frac{\alpha}{2\pi i} \sum_{n=0}^{\infty} \oint\limits_{C_2} \left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha \right) (z^\alpha - z_\circ^\alpha)^n \\ & + \frac{\alpha}{2\pi i} \sum_{n=-\infty}^{-1} \oint\limits_{C_1} \left(\frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha \right) (z^\alpha - z_\circ^\alpha)^n . \end{aligned} \tag{3.19}$$

Hence, (3.19) *becomes*

$$f(z^\alpha) = \sum_{n=0}^{\infty} a_n (z^\alpha - z_\circ^\alpha)^n + \sum_{n=-\infty}^{-1} b_n (z^\alpha - z_\circ^\alpha)^n , \tag{3.20}$$

where

$$a_n = \frac{\alpha}{2\pi i} \oint\limits_{C_2} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha,$$

and

$$b_n = \frac{\alpha}{2\pi i} \oint\limits_{C_1} \frac{f(w^\alpha)}{(w^\alpha - z_\circ^\alpha)^{n+1}} dw^\alpha.$$

Noting that $\sum_{n=0}^{\infty} a_n (z^\alpha - z_\circ^\alpha)^n$ *converges in* $|z^\alpha - z_\circ^\alpha| < r_2$ *and* $\sum_{n=-\infty}^{-1} b_n (z^\alpha - z_\circ^\alpha)^n$ *converges in* $|z^\alpha - z_\circ^\alpha| > r_1$*. So, the series* (3.20) *converges in* $r_1 < |z^\alpha - z_\circ^\alpha| < r_2$*. Hence, the proof is complete.*

4 Conclusions

This work was assigned to present the α*-conformable fractional versions of the Taylor and Laurent series expansions. The procedure was implemented by the use of* α*-conformable fractional analytic functions,* α*-conformable fractional Cauchy integral formula, and establishing the so-called* $x^\alpha y^\alpha$*-fractional plane. The results are in agreement with those for the classical complex analysis forms.*

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