Palestine Journal of Mathematics

Vol 14(4)(2025) , 329–338

Extended energy of the complement of the line graph of a regular graph

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Communicated by: Ambat Vijayakumar

MSC (2020): Primary: 05C50; Secondary: 05C76

Keywords : Graph energy, extended energy, extended adjacency matrix, line graph, complement of a graph.

The authors would like to thank the reviewers and editors for their constructive comments and valuable suggestions that improved the quality of the paper.

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Abstract. The energy of a graph G is denoted by $E(G)$ and defined by $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i, i = 1, 2, ..., n$ are the eigenvalues of G i.e., eigenvalues of the matrix $A(G)$, the adjacency matrix of G. The extended energy of a graph G is given by $\varepsilon_{ex}(G) = \sum_{i=1}^{n} |\eta_i|$, where $\eta_i, i = 1, 2, ..., n$ are the eigenvalues of the extended adjacency matrix A_{ex} of G. In this paper, we give the extended energy of the complement of the line graph of a regular graph, the extended energy of the line graph of the complement of a regular graph, and the extended energy of the complement of the line graph of the complement of a regular graph.

1 Introduction

Let G be a connected graph of order n with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A(G)$ of G is a real symmetric matrix. Therefore, all its eigenvalues are real. Let the eigenvalues of $A(G)$ be $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ listed in decreasing order. The eigenvalues of $A(G)$ are also known as the eigenvalues of G. Sum of absolute values of the eigenvalues of G being the mathematical formulation of π-electron energy, an extensively studied quantum-chemical characteristic of conjugated molecules, is simply known as the energy of G and is denoted by $E(G)$, i.e., $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The graph energy is very attractive due to its importance in mathematical chemistry. Gutman [14] introduced the graph energy in 1978. After that, graph energy, Laplacian energy and Laplacian-energy-like invariant are continued to be researched by Gutman [10, 11, 12, 13], Zhou et al. [31], Li et al. [19], Pirzada et al. [21], Balakrishnan et al. [5, 4], Amin and Nayeem [3] and many others. Also, many researchers examine upper and lower bounds for graph energy in [9, 18, 28, 16] and Sombor energy of some derived graphs of a regular graph in [25, 26]. Two graphs of the same order are said to be equienergetic if their energies are the same. In [17] Indulal and Vijayakumar have constructed a pair of equienergetic graphs on n vertices for $n = 6, 14, 18$ and for all $n \geq 20$. Later, Ramane et al. [22, 23, 24] have given another construction of equienergetic graphs and have proved that there exists a pair of equienergetic graphs for all $n \geq 9$. More studies on equienergetic graphs can be found in [15, 27, 29]. The energy of the complement of regular line graphs has been studied in [1]. McClelland [20] obtained an upper bound for energy of an r-regular graph of order n as follows.

$$E(G) \leq n\sqrt{r}. \tag{1.1}$$

Gutman et al. [12] obtained a lower bound for energy as follows.

$$E(G) \quad \geq \quad n. \tag{1.2}$$

Yan et al. [30] introduced a new matrix called the extended adjacency matrix. The extended adjacency matrix of the graph G is denoted by $A_{ex}(G)$ and is defined as the $n \times n$ matrix whose (i, j)-th entry is equal to $\frac{d_i}{d_j} + \frac{d_j}{d_i}$ if $v_i v_j$ belongs to the edge set of G and 0 otherwise, where d_i is the degree of the vertex v_i. $A_{ex}(G)$ being a real symmetric matrix, all its eigenvalues are real. Let the eigenvalues of $A_{ex}(G)$ be $\eta_1 \geq \eta_2 \geq \cdots \geq \eta_n$ listed in decreasing order. The sum of absolute values of the eigenvalues of $A_{ex}(G)$ is known as the extended energy of G and is denoted by $\varepsilon_{ex}(G)$, i.e., $\varepsilon_{ex}(G) = \sum_{i=1}^{n} |\eta_i|$. For an r-regular graph G, the degree of each vertex is equal to r and hence $A(G) = A_{ex}(G)$, i.e., eigenvalues of $A(G)$ are the same as the eigenvalues of $A_{ex}(G)$. Let $\eta_1, \eta_2, \ldots, \eta_s$ be the eigenvalues of $A_{ex}(G)$ with multiplicity $m_1, m_2, \ldots, m_s$ respectively. Then $m_1 + m_2 + \cdots + m_s = n$ and $\eta_1 m_1 + \eta_2 m_2 + \ldots + \eta_s m_s = 0$. For an r-regular graph G, r is an eigenvalue of G with multiplicity k, where k is the number of components of G. Also, the absolute value of each eigenvalue of G is less than or equal to r. Das et al. [8] obtained various upper bounds and lower bounds for $\eta_1(G)$ and $\varepsilon_{ex}(G)$ by using the definition of extended energy of G and eigenvalues of $A_{ex}(G)$ of G. Also, Adiga [2] obtained upper bounds for the extended energy of graphs and some extended equienergetic graphs. Motivated by these works, in this paper, we give the extended energy of the complement of a regular line graph, the extended energy of the line graph of the complement of a regular graph, and the extended energy of the complement of the line graph of the complement of a regular graph.

Theorem 1.1. *[6] Let λ_1 be an eigenvalue of a connected graph G, then $-\lambda_1$ is an eigenvalue of G if and only if G is bipartite.*

Theorem 1.2. *[7] Let G be a connected r-regular graph of order n with eigenvalues in the array*

$$\begin{pmatrix} r & \lambda_2 & \lambda_3 & \cdots & \lambda_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix},$$

where the integers in the bottom row indicate the multiplicities of the corresponding eigenvalues in the top row. Then $L(G)$, line graph of G is a $(2r-2)$-regular graph with eigenvalues in the array

$$\begin{pmatrix} 2r-2 & \lambda_2+r-2 & \lambda_3+r-2 & \cdots & \lambda_s+r-2 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Theorem 1.3. *[7] Let G be an r-regular graph of order n with eigenvalues in the array*

$$\begin{pmatrix} r & \lambda_2 & \lambda_3 & \cdots & \lambda_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then $\overline{G}$, complement of G is an $(n-r-1)$-regular graph with eigenvalues in the array

$$\begin{pmatrix} n-r-1 & -\lambda_2-1 & -\lambda_3-1 & \cdots & -\lambda_s-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Theorem 1.4. *[7] Let G be a connected r-regular graph of order n and $A_{ex}(G)$ be the extended adjacency matrix of G with eigenvalues in the array*

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then $L(G)$, the line graph of G being a $(2r-2)$-regular graph has the eigenvalues of $A_{ex}(L(G))$, the extended adjacency matrix of $L(G)$ in the array

$$\begin{pmatrix} 2r-2 & \eta_2+r-2 & \eta_3+r-2 & \cdots & \eta_s+r-2 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Theorem 1.5. *[7] Let G be an r-regular graph of order n and $A_{ex}(G)$ be the extended adjacency matrix of G with eigenvalues in the array*

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then $\overline{G}$, the complement of G being an $(n-1-r)$-regular graph has the eigenvalues of $A_{ex}(\overline{G})$, the extended adjacency matrix of $\overline{G}$ in the array

$$\begin{pmatrix} n-1-r & -\eta_2-1 & -\eta_3-1 & \cdots & -\eta_s-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

2 Eigenvalues of the extended adjacency matrix of $\overline{L(G)}$, $L(\overline{G})$ and $\overline{L(\overline{G})}$

Theorem 2.1. *Let G be a connected r-regular graph of order n with eigenvalues of extended adjacency matrix $A_{ex}(G)$ in the array*

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then (i) if G is not bipartite, $\overline{L(G)}$, complement of the line graph of G has the eigenvalues of the extended adjacency matrix $A_{ex}(\overline{L(G)})$ in the array

$$\begin{pmatrix} \frac{nr}{2}-1-(2r-2) & -\eta_2-r+1 & -\eta_3-r+1 & \cdots & -\eta_s-r+1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}, \text{ and}$$

(ii) if G is bipartite, $\overline{L(G)}$, complement of the line graph of G has the eigenvalues of the extended adjacency matrix $A_{ex}(\overline{L(G)})$ in the array

$$\begin{pmatrix} \frac{nr}{2}-1-(2r-2) & -\eta_2-r+1 & -\eta_3-r+1 & \cdots & -\eta_{s-1}-r+1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

Proof. (i) Let G be a non-bipartite r-regular graph with eigenvalues of $A_{ex}(G)$

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

The line graph $L(G)$ is a $(2r-2)$-regular graph. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(G)$ are

$$\begin{pmatrix} 2r-2 & -\eta_2+r-2 & -\eta_3+r-2 & \cdots & -\eta_s+r-2 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Again, $\overline{L(G)}$ is a $\frac{nr-4r+2}{2}$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{L(G)}$ are in the array

$$\begin{pmatrix} \frac{nr}{2}-1-(2r-2) & -\eta_2-r+1 & -\eta_3-r+1 & \cdots & -\eta_s-r+1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

(ii) Let G be a bipartite r-regular graph with the eigenvalues of $A_{ex}(G)$ given in the array

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_{s-1} & -r \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & 1 \end{pmatrix}.$$

The line graph $L(G)$ of G is a $(2r-2)$-regular graph. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(G)$ are in the array

$$\begin{pmatrix} 2r-2 & -\eta_2+r-2 & -\eta_3+r-2 & \cdots & -\eta_{s-1}+r-2 & -2 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

Again, $\overline{L(G)}$ is a $\frac{nr-4r+2}{2}$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{L(G)}$ are in the array

$$\begin{pmatrix} \frac{nr}{2}-1-(2r-2) & -\eta_2-r+1 & -\eta_3-r+1 & \cdots & -\eta_{s-1}-r+1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

□

Theorem 2.2. *Let G be a connected r-regular graph of order n with eigenvalues of extended adjacency matrix $A_{ex}(G)$ in the array*

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then (i) if G is not bipartite, $L(\overline{G})$, the line graph of the complement of G has the eigenvalues of the extended adjacency matrix $A_{ex}(L(\overline{G}))$

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_s+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(n-r-3)}{2} \end{pmatrix}, \text{ and}$$

(ii) if G is bipartite, $L(\overline{G})$, the line graph of the complement of G has the eigenvalues of the extended adjacency matrix $A_{ex}(L(\overline{G}))$

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_{s-1}+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(n-r-3)}{2}+1 \end{pmatrix}.$$

Proof. (i) Let G be a non-bipartite r-regular graph with eigenvalues of $A_{ex}(G)$

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

The complement of $\overline{G}$ is a $(n-1-r)$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{G}$ are

$$\begin{pmatrix} n-1-r & -\eta_2-1 & -\eta_3-1 & \cdots & -\eta_s-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Again, $L(\overline{G})$ is a $(2n-2r-4)$-regular graph. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(\overline{G})$ are

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_s+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(n-r-3)}{2} \end{pmatrix}.$$

(ii) Let G be a bipartite r-regular graph with eigenvalues of $A_{ex}(G)$

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_{s-1} & -r \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & 1 \end{pmatrix}.$$

The complement of G is an $(n-1-r)$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{G}$ are

$$\begin{pmatrix} n-1-r & -\eta_2-1 & -\eta_3-1 & \cdots & -\eta_{s-1}-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

Again, $L(\overline{G})$ is a $(2n-2r-4)$-regular graph. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(\overline{G})$ are

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_{s-1}+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(n-r-3)}{2}+1 \end{pmatrix}.$$

□

Theorem 2.3. *Let G be a connected r-regular graph of order n with eigenvalues of extended adjacency matrix $A_{ex}(G)$*

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

Then (i) if G is not a bipartite, $\overline{L(\overline{G})}$, the complement of the line graph of the complement of G is the graph has the eigenvalues of the extended adjacency matrix $A_{ex}(\overline{L(\overline{G})})$

$$\begin{pmatrix} \frac{n^2-nr-5n+4r+6}{2} & \eta_2-n+r+3 & \eta_3-n+r+3 & \cdots & \eta_s-n+r+3 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(n-r-1)(n-r-3)}{2} \end{pmatrix}.$$

(ii) If G is a bipartite, $\overline{L(\overline{G})}$, the complement of the line graph of the complement of G has the eigenvalues of the extended adjacency matrix $A_{ex}(\overline{L(\overline{G})})$

$$\begin{pmatrix} \frac{n^2-nr-5n+4r+6}{2} & \eta_2-n+r+3 & \eta_3-n+r+3 & \cdots & \eta_{s-1}-n+r+3 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(n-r-1)(n-r-3)}{2}+1 \end{pmatrix}.$$

Proof. (i) Let G be a non bipartite r-regular graph with eigenvalues of $A_{ex}(G)$

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_s \\ 1 & m_2 & m_3 & \cdots & m_s \end{pmatrix}.$$

The complement of G is an $(n-1-r)$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{G}$ are

$$\begin{pmatrix} n-1-r & -\eta_2-1 & -\eta_3-1 & \cdots & -\eta_s-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(r-2)}{2} \end{pmatrix}.$$

Now, $L(\overline{G})$ is a $(2n-2r-4)$-regular graph of order $\frac{n(n-r-1)}{2}$. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(\overline{G})$ are

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_s+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(n-r-3)}{2} \end{pmatrix}.$$

$\overline{L(\overline{G})}$ is a $\frac{n(n-r-1)}{2}-(2n-2r-4)-1$ regular graph, i.e., $\frac{n^2-nr-5n+4r+6}{2}$-regular. Therefore by Theorem 1.5, the eigenvalues of $A_{ex}(\overline{L(\overline{G})})$ are

$$\begin{pmatrix} \frac{n^2-nr-5n+4r+6}{2} & \eta_2-n+r+3 & \eta_3-n+r+3 & \cdots & \eta_s-n+r+3 & 1 \\ 1 & m_2 & m_3 & \cdots & m_s & \frac{n(n-r-1)(n-r-3)}{2} \end{pmatrix}.$$

(ii) Let G be a bipartite r-regular graph with eigenvalues of $A_{ex}(G)$

$$\begin{pmatrix} r & \eta_2 & \eta_3 & \cdots & \eta_{s-1} & -r \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & 1 \end{pmatrix}.$$

The complement of G is a $(n-1-r)$-regular graph. Therefore by Theorem 1.5, the eigenvalues of the extended adjacency matrix of $\overline{G}$ are

$$\begin{pmatrix} n-1-r & -\eta_2-1 & -\eta_3-1 & \cdots & -\eta_{s-1}-1 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(r-2)}{2}+1 \end{pmatrix}.$$

Again, $L(\overline{G})$ is a $(2n-2r-4)$-regular graph of order $\frac{n(n-r-1)}{2}$. Therefore by Theorem 1.4, the eigenvalues of the extended adjacency matrix of $L(\overline{G})$ are

$$\begin{pmatrix} 2n-2r-4 & -\eta_2+n-r-4 & -\eta_3+n-r-4 & \cdots & -\eta_{s-1}+n-r-4 & -2 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(n-r-3)}{2}+1 \end{pmatrix}.$$

As $\overline{L(\overline{G})}$ is a $\frac{n(n-r-1)}{2}-(2n-2r-4)-1$ regular graph, i.e., $\frac{n^2-nr-5n+4r+6}{2}$-regular, therefore by Theorem 1.5, the eigenvalues of $A_{ex}(\overline{L(\overline{G})})$ are

$$\begin{pmatrix} \frac{n^2-nr-5n+4r+6}{2} & \eta_2-n+r+3 & \eta_3-n+r+3 & \cdots & \eta_{s-1}-n+r+3 & 1 \\ 1 & m_2 & m_3 & \cdots & m_{s-1} & \frac{n(n-r-1)(n-r-3)}{2}+1 \end{pmatrix}.$$

□

3 The extended energy of the complement of a regular line graph

Theorem 3.1. *Let G be a connected r-regular graph of order n, and $\overline{L(G)}$ be the complement of the regular line graph G. Then*

(i) $\varepsilon_{ex}(\overline{L(G)}) = (2n-4)(r-1)-2$ *if G is not bipartite and $-r+1 \leq \eta_s$,*

(ii) $\varepsilon_{ex}(\overline{L(G)}) = (2n-2)(r-1)$ *if G is bipartite and $-r+1 \leq \eta_2$.*

Proof. (i) Let G be a non-bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned} \varepsilon_{ex}(\overline{L(G)}) &= \frac{nr}{2}-1-(2r-2)+\frac{n(r-2)}{2}+\sum_{i=2}^{s} m_i|-r-\eta_i+1| \\ &= nr-2r-n+1+\sum_{i=2}^{s} m_i(r+\eta_i-1) \\ &= nr-2r-n+1+(r-1)\sum_{i=2}^{s} m_i+\sum_{i=2}^{s} m_i\eta_i \\ &= nr-2r-n+1+(r-1)(n-1)-r \\ &= (2n-4)(r-1)-2. \end{aligned}$$

(ii) Let G be a bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned} \varepsilon_{ex}(\overline{L(G)}) &= \frac{nr}{2}-1-(2r-2)+\frac{n(r-2)}{2}+1+\sum_{i=2}^{s-1} m_i|-r-\eta_i+1| \\ &= nr-2r-n+2+\sum_{i=2}^{s-1} m_i(r+\eta_i-1) \\ &= nr-2r-n+2+(r-1)\sum_{i=2}^{s-1} m_i+\sum_{i=2}^{s-1} m_i\eta_i \\ &= nr-2r-n+2+(r-1)(n-2)+0 \\ &= (2n-4)(r-1). \end{aligned}$$

□

4 The extended energy of the line graph of the complement of a regular graph

Theorem 4.1. *Let G be a connected r-regular graph of order n, and $L(\overline{G})$ be the line graph of the complement of G. Then* *(i)* $\varepsilon_{ex}(L(\overline{G})) = 2n(n-r-3)$ *if G is not bipartite and* $n-r-4 \geq \eta_2$,

(ii) $\varepsilon_{ex}(L(\overline{G})) = 2n^2 - 2nr - 7n + 8$ *if G is bipartite and* $n-r-4 \geq \eta_2$.

Proof. (i) Let G be a non bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned}
\varepsilon_{ex}(L(\overline{G})) &= (2n-2r-4) + 2.\frac{n(n-r-3)}{2} + \sum_{i=2}^{s} m_i|n-r-\eta_i-4| \\
&= (2n-2r-4) + n(n-r-3) + \sum_{i=2}^{s} m_i(n-r-\eta_i-4) \\
&= (2n-2r-4) + n(n-r-3) \\
&\quad + n\sum_{i=2}^{s} m_i - r\sum_{i=2}^{s} m_i - \sum_{i=2}^{s} \eta_i m_i - 4\sum_{i=2}^{s} m_i \\
&= (2n-2r-4) + n(n-r-3) + n(n-1) - r(n-1) + r - 4(n-1) \\
&= 2n(n-r-3).
\end{aligned}$$

(ii) Let G be a bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned}
\varepsilon_{ex}(L(\overline{G})) &= (2n-2r-4) + 2\left\{\frac{n(n-r-3)}{2} + 1\right\} + \sum_{i=2}^{s-1} m_i|n-r-\eta_i-4| \\
&= (2n-2r-4) + n(n-r-3) + 4 + \sum_{i=2}^{s-1} m_i(n-r-\eta_i-4) \\
&= (2n-2r-4) + n(n-r-3) + 4 \\
&\quad + n\sum_{i=2}^{s-1} m_i - r\sum_{i=2}^{s-1} m_i - \sum_{i=2}^{s-1} \eta_i m_i - 4\sum_{i=2}^{s-1} m_i \\
&= (2n-2r-4) + n(n-r-3) + 4 + (n-r-4)(n-2) \\
&= 2n^2 - 2nr - 7n + 8.
\end{aligned}$$

□

5 The extended energy of the complement of the line graph of the complement of a regular graph

Theorem 5.1. *Let G be a connected* r*-regular graph of order* n*, and* $\overline{L(\overline{G})}$ *be the complement of the line graph of the complement of G. Then*

(i) $\varepsilon_{ex}(\overline{L(\overline{G})}) = \frac{n^3-n^2+nr(r-2n+1)-10n+8r+12}{2}$ *if G is not bipartite and* $n-r-3 \geq \eta_2$,

(ii) $\varepsilon_{ex}(\overline{L(\overline{G})}) = \frac{n^3-n^2-2n^2r+r^2n-12n+8r+20}{2}$ *if G is bipartite* $n-r-3 \geq \eta_2$.

Proof. (i) Let G be a non-bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned}
\varepsilon_{ex}(\overline{L(\overline{G})}) &= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} \\
&\quad + \sum_{i=2}^{s} m_i|-n+r+\eta_i+3| \\
&= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} \\
&\quad + \sum_{i=2}^{s} m_i(n-r-\eta_i-3) \\
&= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} \\
&\quad + (n-r-3)\sum_{i=2}^{s} m_i - \sum_{i=2}^{s} \eta_i m_i \\
&= \frac{n^3 - n^2 + nr(r-2n+1) - 10n + 8r + 12}{2}.
\end{aligned}$$

(ii) Let G be a bipartite r-regular graph. Then, by the definition of extended energy we have,

$$\begin{aligned}
\varepsilon_{ex}(\overline{L(\overline{G})}) &= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} + 1 \\
&\quad + \sum_{i=2}^{s-1} m_i|-n+r+\eta_i+3| \\
&= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} + 1 \\
&\quad + \sum_{i=2}^{s-1} m_i(n-r-\eta_i-3)
\end{aligned}$$

$$\begin{aligned}
&= \frac{n^2 - nr - 5n + 4r + 6}{2} + \frac{n(n-r-3)(n-r-1)}{2} + 1 \\
&\quad + (n-r-3)\sum_{i=2}^{s-1} m_i - \sum_{i=2}^{s} \eta_i m_i \\
&= \frac{n^3 - n^2 - 2n^2 r + r^2 n - 12n + 8r + 20}{2}.
\end{aligned}$$

□

6 Concluding Remarks

In this paper, we have investigated the extended energy of some derived graphs, such as the complement of a regular line graph, the line graph of the complement of a regular graph, and the complement of the line graph of the complement of a regular graph. In the same research direction, the extended energy of some other derived graphs, such as double graph, shadow graph, Mycielskian graph, etc., of a regular graph can be investigated further.

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Received: 2024-12-17
Accepted: 2025-09-27