Commutative Monoid of Minimal Soft Pre-rough Sets and their Lattice Structure

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Abstract Minimal soft pre-rough sets are presented in this work as an improved mathematical framework for managing approximation and uncertainty in data analysis. This strategy maintains decision-making flexibility while improving approximation techniques by fusing concepts from rough set theory and soft set theory. A commutative monoid that enables effective modification and combining of minimum soft pre-rough sets is introduced and its lattice structure is studied.This algebraic framework increases computing efficiency, facilitating better categorisation, data processing, and decision-making. The study aids in the creation of sophisticated mathematical models for handling imprecise data in domains including medical diagnosis, optimisation, and machine learning. This work provides a systematic and algebraically solid method to uncertainty modelling and its real-world applications, laying the groundwork for future developments.

1 Introduction

A mathematical framework for expressing and controlling uncertainty and imprecision was first presented by Lotfi A. Zadeh [27] in 1965 and is known as fuzzy set theory. Fuzzy set theory offers a more sophisticated method of processing data that is not strictly binary, in contrast to traditional set theory, which treats elements as either belonging to a set or not. B. Ahmad1,and Athar Kharal [1] extends fuzzy soft set theory by presenting new fuzzy soft intersection and union features, backed up by instances and counterexamples. Additionally, it contributes to the mathematical development of soft set theory by establishing DeMorgan Laws and defining arbitrary fuzzy soft operations. Zdzisław Pawlak [28] created rough set theory (RS) in the early 1980s as a conceptual mathematical framework to address the lack of clarity in data processing. RS theory makes use of approximations to represent and make sense of imperfect or incomplete information, in contrast to classical set theory, which demands precise definitions. Because of this, it works especially effectively in situations when other information is not available, like membership functions or probability distributions. Rough set theory is a useful tool for feature selection, pattern recognition, and creating reliable classification models, particularly when working with imprecise data, since it excels at extracting important information from big and possibly noisy datasets. Notably, rough sets are useful in many domains, such as medical diagnostics, where they assist in identifying significant signs and enhance the precision of diagnosis. In engineering, management, and finance, they additionally enable data-driven decision-making. Mostafa A. El-Gayar, Radwan Abu-Gdairi [13] emphasizes the integration of lattices and rough sets, highlighting the isomorphism between topology and its extension for improved computational efficiency and data representation. By adding rough X-sub-exact sequences over rough

rings, [24] this study expands on module theory. Wijayanti, I. E., & Faisol, A. investigates their characteristics, especially the way these modules connect the divide between rough set theory and algebra to deal with uncertainty. Using CPNIVFS, M Palanikumar, N Kausar et, al., [14] introduces a unique MADM technique that has been used to the treatment selection of brain tumours and confirmed by algebraic characteristics. Its advantages and dependability over current techniques for precise decision-making are shown by comparative study. M Palanikumar, N Kausar, M Deveci [15] introduces new aggregation operators and an algorithm for MADM in a complex Pythagorean neutrosophic normal interval-valued set approach. Comparative analysis demonstrates its superiority and dependability in the assessment of AI-driven robotics.

Molodtsov [10] (1999) created soft set theory, which provides a broad mathematical framework for handling uncertainty without requiring a membership function as does fuzzy set theory. By associating each parameter to a collection of items, a generalised family of sets known as a soft set offers a comprehensive means of characterising unknown data. In order to deal with uncertainty and vagueness, which frequently arise in fields like artificial intelligence, data analysis, and decision making, Molodtsov developed the mathematical framework known as soft set theory in 1999.In order to solve confusion and scepticism in data analysis, soft rough set theory is a composite mathematical model that combines the notions of rough set theory with soft set theory. Santra, D., Basu, et,al., [23] the context of a modified soft rough (MSR) approximation space, this work presents the idea of rough soft sets and examines its subset, union, and intersection features. It also defines the roughness measure of soft sets, creates an order relation, and looks at lattice theory, illustrating with real-world instances. This method improves the ability to handle vague, ambiguous, and imprecise data in a variety of real-world scenarios. H. Aktas, & N. Cagman [2] compares fuzzy and rough sets to examine soft set theory as a method for managing uncertainty. It presents the idea of soft groups, explains their characteristics, and draws important conclusions from Molodtsov's soft set architecture. Soft rough set theory requires advantage of both the approximation abilities of rough sets and the flexibility of soft sets by combining these two ideas. Within this paradigm, the discourse universe is divided into granules, which are subsequently employed to generate the upper and lower approximations in the soft set context. Riaz, M., Karaaslan, F., Nawaz, I. et al. [21] defines the soft multi-rough topology (SMR-topology) for discussing the topological structure of SMRSs and demonstrates their applicability in real-life scenarios like diagnosing depression and diabetes through numerical examples and comparative analysis.

Decision making in imprecise settings has been increasingly important in recent years. Roy & Maji [22] presents a unique approach to object recognition utilising imprecise multi-observer data, which involves building a Comparison Table from a fuzzy soft set for better decision making. Soft rough sets offer rough set approximations parameterization, enabling the handling of uncertainty to be modified according to specific needs or parameters. Soft rough sets provide an adaptable tool for handling different kinds of data uncertainty by fusing the flexibility of soft sets with the structured approximation techniques of rough sets. A modular approach to information processing and decision-making is made possible by the fundamental idea of granules, or subsets of the discourse universe. introduces a new approach to modifying and generalizing soft rough sets, linking soft set and rough set theories to model uncertain data and address ambiguity. The paper [3] proposed methods are compared with previous ones and demonstrated through a medical application for COVID-19 diagnosis, supported by an algorithm for decision-making. Soft rough set theory has numerous applications in fields like, such as, Decision-Making Optimising decision-making processes through the management of unclear and partial information, Data Mining Providing reliable approaches to handle imprecise and uncertain data, Pattern Recognition Enabling the identification of patterns in datasets that are vague and uncertain, Medical Diagnosis Supporting medical diagnostics by managing uncertain medical data and enhancing diagnostic accuracy. The graphs of lower and upper approximations of a subset in a nearring with respect to an ideal are investigated in this study [8]. Crasta, S.R., and Jagadeesha, B. investigates their relationship to ideal qualities, connectedness, and properties under nearring homomorphisms. Praba B,Gomathi. G [19] uniquely combines hypergraphs with rough set theory, illustrated through a real-life example and validated by a comparative analysis demonstrating its accuracy over existing rough set theory. M Palanikumar, N Kausar et al., [16] present generalised Diophantine fuzzy sets, which expand Diophantine and Pythagorean fuzzy sets, and propose additional operators, distance measurements, and decision-making models. Its superiority and

wider applicability in multi-criteria decision-making are shown by comparative analysis. In order to better manage uncertainty, M palanikumar, N Kausar et al., [17] extends vague and square root fuzzy sets to include logarithmic square root vague sets for multi-attribute decision-making. The suggested aggregation operators' superiority and dependability over current techniques are validated by comparative analysis. In order to improve uncertainty management, M palanikumar, N Kausar et al., [18] enhances vague and q-rung fuzzy sets to develop vague normal sets (VNS) for multi-attribute decision-making. The accuracy and efficacy of the suggested models in medical diagnosis have been established by comparative analysis.

The notion of minimum soft pre-rough sets a particular kind of soft set intended to manage ambiguity and uncertainty in data analysis is explored in this study. A minimal soft pre-rough set lattice structure is established by the investigation.A commutative monoid, a mathematical structure formed by an identity element and a set with an associative binary operation, is also examined in relation to minimal soft pre-rough sets. Determining this monoid's properties aids in emphasising the algebraic properties and purposes of minimum soft pre-rough sets. Methods to regulate uncertainty in data processing, pattern recognition, and decision-making have been significantly refined by the investigation of minimal soft pre-rough sets, lattice structures, and commutative monoids. These mathematical developments allow for an effective structure that successfully addresses ambiguity and uncertainty in data analysis and decision-making. Theoretically-based approaches to managing and interpreting imprecise and imprecise data ultimately provide practical answers.

2 Preliminaries

This section covers the basic definitions, which lay a solid basis for the topics that will be studied later in the study. The section makes sure that readers have an established knowledge of the key concepts and words by providing these fundamental definitions.

Definition 2.1.
Let $\mho$ be the universal sets and For any $\mathcal{D} \subseteq \mho$,define the lower approximation space $\mathfrak{P}_-(\mathcal{D})$, $\ni: \mathfrak{P}_-(\mathcal{D}) = \{q \in \mho/[\mathcal{D}]_{\mathfrak{P}} \subseteq \mathcal{D}\}$ Also define upper approximation space $\mathfrak{P}^-(\mathcal{D})$, $\ni:$ $\mathfrak{P}^-(\mathcal{D}) = \{q \in \mho/[\mathcal{D}]_{\mathfrak{P}} \cap \mathcal{D} \neq \emptyset\}$. Then $\Re S(\mathcal{D}) = (\mathfrak{P}_-(\mathcal{D}), \mathfrak{P}^-(\mathcal{D}))$ is called Rough Set theory.

Definition 2.2.
Consider the universal set $\mho$ and the set of parameters $\aleph$. When $H : \aleph \to P(\mho)$, then a pair $G = (H, \aleph)$ is referred to as a Soft Set theory over $\mho$.

Definition 2.3.
Let $G = (H, \aleph)$ be a soft set on $\mho$. For any $\mathcal{D} \in \mho$ the minimal soft description of an element $\varpi \in \mho$ is defined as

$$\mathfrak{MO}s\, \varpi = \prod_{\mathfrak{k} \,\in\, \aleph} \{\mathcal{F}(\mathfrak{k}) \mid \varpi \in \mathcal{F}(\mathfrak{k})\}$$

Definition 2.4.
Let $\mho$ be the universal set and Let $G = (H, \aleph)$ be a soft set on $\mho$. For any $\varpi \in \mho$. The Minimal soft rough set ϖ denoted by $\mathfrak{MO\Re}S(\varpi)$ is defined by

$$\mathfrak{MO\Re}S(\varpi) = (\mathfrak{MO}_-(\varpi), \mathfrak{MO}^-(\varpi))$$

Where $\mathfrak{MO}_-(\varpi) = \{\mathfrak{Y} \in \mho | \mathfrak{MO}s(\mathfrak{Y}) \subseteq \varpi\}$ and
$\mathfrak{MO}^-(\varpi) = \{\mathfrak{Y} \in \mho | \mathfrak{MO}s(\mathfrak{Y}) \cap \varpi \neq \emptyset\}$

Definition 2.5.
A groupoid is defined as a non-empty set $\mathfrak{E}$ paired with a binary operation $*$, and it is represented as $(\mathfrak{E}, *)$.

Definition 2.6.
A semigroup $(\mathfrak{E}, *)$ is a groupoid in which the binary operation $*$ is associative, meaning that for all $\kappa, \varrho, \varpi \in \mathfrak{E}$, the equation $(\kappa * \varrho) * \varpi = \kappa * (\varrho * \varpi)$ holds.

Definition 2.7.
A semigroup $(\mathfrak{E}, *)$ is called a monoid if it includes an identity element $\mathfrak{e} \in \mathfrak{E}$ such that $\mathfrak{e} * \kappa = \kappa * \mathfrak{e} = \kappa$ for all $\kappa \in \mathfrak{E}$.

Definition 2.8.
A monoid $(\mathfrak{E}, *)$ is called a commutative monoid if it satisfies the commutative property, meaning $\kappa * \varrho = \varrho * \kappa$ for all $\kappa, \varrho \in \mathfrak{E}$.

3 Minimal Soft pre-rough sets

The minimal soft pre-rough set, which integrates the soft pre-rough set with the minimum soft rough set, is covered in this section. We also demonstrate that the collection of all minimal soft pre-rough sets forms a lattice structures.

Definition 3.1.
Let $G = (H, \aleph)$ be a Soft set over $\mho$ and $\varpi \subseteq \mho$. The Minimal soft pre-rough set is defined by,

$$\mathfrak{M\eth R} S_{\mathfrak{P}}(\varpi) = (\mathfrak{M R}_{-\mathfrak{P}}(\varpi), \mathfrak{M\eth}^-_{\mathfrak{R}}(\varpi))$$

, where $\mathfrak{M\eth}_{-\mathfrak{R}}(\varpi) = \{\varpi \subseteq \mho | \mathfrak{M\eth}s(\varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))\}$
$\mathfrak{M\eth}^-_{\mathfrak{R}}(\varpi) = \{\varpi \subseteq \mho | \mathfrak{M\eth}s(\varpi) \bigcup \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\varpi))\}$

Example 7.1
Let $\mho = \{\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4\}$
$\mathfrak{D}(\jmath_1) = \{\mathfrak{W}_1, \mathfrak{W}_3\}; \mathfrak{D}(\jmath_2) = \{\mathfrak{W}_2, \mathfrak{W}_4\}; \mathfrak{D}(\jmath_3) = \{\mathfrak{W}_3, \mathfrak{W}_4\}$
(i) Let $\nu = \{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}$
$\mathfrak{M\eth}s(\nu) = \{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}$
$\mathfrak{M\eth}_-(\nu) = \{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}; \mathfrak{M\eth}^-(\nu) = \mho$
$\mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\nu)) = \mho; \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\nu)) = \mho$
$\mathfrak{M\eth}_{-\mathfrak{R}}(\nu) = \{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}; \mathfrak{M\eth}^-_{\mathfrak{R}}(\nu) = \mho$
$\mathfrak{M\eth R}S_{\mathfrak{R}}(\nu) = (\{\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3\}, \mho)$

Example 7.2
Let $\mho = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5\}$
$\mathfrak{D}(\mathfrak{b}_1) = \{\varsigma_1, \varsigma_2, \varsigma_3\}; \mathfrak{D}(\mathfrak{b}_2) = \{\varsigma_2, \varsigma_4\}; \mathfrak{D}(\mathfrak{b}_3) = \{\varsigma_5\}$
(ii)Let $\tau = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_5\}$
$\mathfrak{M\eth}s(\tau) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_5\}$
$\mathfrak{M\eth}_-(\tau) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_5\}; \mathfrak{M\eth}^-(\tau) = \mho$
$\mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\tau)) = \mho; \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\tau)) = \mho$
$\mathfrak{M\eth R}S_{\mathfrak{R}}(\tau) = (\{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_5\}, \mho)$

3.1 Properties of Minimal soft pre-rough sets

(i) If $\mathfrak{A} \subseteq \mho$, then $\mathfrak{M\eth}_-(\mathfrak{A}) \subseteq \mathfrak{M\eth}s(\mathfrak{A})$.
Proof:
(From Example 7.1)
Let $\mathfrak{A} = \{\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3\}$ then $\mathfrak{M\eth}s(\mathfrak{A}) = \{\mathfrak{W}_1, \mathfrak{W}_2, \mathfrak{W}_3\}$
and $\mathfrak{M\eth}_-(\mathfrak{A}) = \{\mathfrak{W}_1, \mathfrak{W}_3\}$
Hence, $\mathfrak{M\eth}_-(\mathfrak{A}) \subseteq \mathfrak{M\eth}s(\mathfrak{A})$.

(ii) If $\mathfrak{M\eth}_-(\vartheta) = \emptyset$ then $\mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\vartheta)) = \emptyset$.
Proof:
(From Example 7.2)
Let $\vartheta = \{\varsigma_1, \varsigma_4\}$ then $\mathfrak{M\eth}s(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$
$\mathfrak{M\eth}_-(\vartheta) = \emptyset; \mathfrak{M\eth}^-(\vartheta) = \{\varsigma_1, \varsigma_3, \varsigma_4\} \Rightarrow \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\vartheta)) = \emptyset$.
Hence, $\mathfrak{M\eth}_-(\vartheta) = \emptyset \Rightarrow \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\vartheta)) = \emptyset$.

(iii) If $\mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\vartheta)) \subseteq \mathfrak{M\eth}s(\vartheta)$, then $\mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \mathfrak{M\eth}s(\vartheta)$.

Proof:

(From Example 7.1)

Let $\vartheta = \{\mathfrak{W}_2, \mathfrak{W}_3\}$ then $\mathfrak{M\eth}s(\vartheta) = \{\mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4\}$

$\mathfrak{M\eth}_-(\vartheta) = \{\mathfrak{W}_2\}; \mathfrak{M\eth}^-(\vartheta) = \mho$

$\mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\vartheta)) = \{\mathfrak{W}_1, \mathfrak{W}_2\} \Rightarrow \mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \mho$

Hence, $\mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \mathfrak{M\eth}s(\vartheta)$.

(iv) If $\mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\vartheta)) = \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\vartheta)) = \mathfrak{M\eth}s(\mathfrak{A})$,
then $\mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \mathfrak{M\eth}_{\mathfrak{R}}^-(\vartheta)$.

Proof:

(From Example 7.2)

Let $\vartheta = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$ then $\mathfrak{M\eth}s(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$

$\mathfrak{M\eth}_-(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}; \mathfrak{M\eth}^-(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$

$\Rightarrow \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\vartheta)) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}; \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\vartheta)) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$

$\mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}; \mathfrak{M\eth}_{\mathfrak{R}}^-(\vartheta) = \{\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4\}$

Hence, $\mathfrak{M\eth}_{-\mathfrak{R}}(\vartheta) = \mathfrak{M\eth}_{\mathfrak{R}}^-(\vartheta)$.

4 Set of Minimal Soft Pre-Rough Sets with Lattice Structure

In this section,We define Lattice on Minimal soft pre-rough sets $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$ is associated with Soft set $G = (H, \aleph)$ on $\mho$.

Definition 4.1.

Let $G_P = \{\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) | \mathfrak{A} \subseteq \mho\}$. We define relation $\approx$ on G_P by $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \approx \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$ if and only if $\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\varpi)$ and $\mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{\mathfrak{R}}^-(\varpi)$

Theorem 4.2.

A partial ordering relation on G_P is the relation $\approx$.

Theorem 4.3.

Let $\mho$ be the universal set and let $\mathfrak{A}$ & $\varpi \subseteq \mho$, then, $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ is the LUB of $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A})$ or $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

Proof:

Consider $\mathfrak{A}\&\varpi \subseteq \mho$, then we have to prove that $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ & $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

First we have to prove that $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

That is to pove that $\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ and $\mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi)$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi)$ & $\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$(\because \mathfrak{M\eth}s(\mathfrak{A}) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi)$ & $\mathfrak{D}(\jmath) \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigvee \varpi))$

$(\because \mathfrak{M\eth}^-(\mathfrak{A}) \subseteq \mathfrak{M\eth}^-(\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

Hence, $\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

Similarly, we can prove $\mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A}) \subseteq \mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi)$

Hence, $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

Similarly, we can prove $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

Hence $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ is an upper bound of $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A})\&\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$.

Next we have to prove $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ is the LUB of $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A})\&\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$.

Let $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varrho)$ is the LUB of $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varrho)\&\mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

TP: $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\varrho)$

That is to prove that $(\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi), \mathfrak{M\eth}_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi)) \subseteq (\mathfrak{M\eth}_{-\mathfrak{R}}(\varrho), \mathfrak{M\eth}_{\mathfrak{R}}^-(\varrho))$

Let $\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A})) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \cap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \cap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A})or \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\varpi)$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-\mathfrak{R}}(\varrho)$

Hence, $\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\varrho)$

Let $\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi) \bigcup \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigvee \varpi) \, or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A} \bigvee \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A})) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A})) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \, or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \cup \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A})) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \cup \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-_{\mathfrak{R}}(\varpi)$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M\eth}^-_{\mathfrak{R}}(\varrho)$

$\Rightarrow \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) \subseteq \mathfrak{M\eth}^-_{\mathfrak{R}}(\varrho)$

Hence $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$ *is the LUB of* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

Example7.3

From example 7.1

Let $\mathfrak{A} = \mathfrak{W}_3; \varpi = \{\mathfrak{W}_1, \mathfrak{W}_3\}$

$\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = (\{\mathfrak{W}_1, \mathfrak{W}_3\}, \{\mathfrak{W}_1, \mathfrak{W}_3\})$

Theorem 4.4.

Let $\mho$ *be the Universal set and let* $\mathfrak{A} \& \varpi \subseteq \mho$, *then,* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$ *is the GLB of* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

Proof:

Consider $\mathfrak{A} \& \varpi \subseteq \mho$, *then we have to prove that* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \&$ $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

First we have to prove that $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A})$

That is to prove that $(\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi), \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)) \subseteq (\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}), \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A}))$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigwedge \varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigwedge \varpi) \& \mathfrak{D}(\jmath) \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A})) \, \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \& \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \& \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

Hence, $\mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-\mathfrak{R}}(\varpi)$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigwedge \varpi) \bigcup \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigwedge \varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}^-(\mathfrak{M\eth}_-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A})) \, \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \& \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \& \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A}) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A}))$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\varpi))$

Hence, $\mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-P}(\mathfrak{A}) \& \mathfrak{D}(\jmath) \in \mathfrak{M\eth}_{-P}(\varpi)$

$\therefore \mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$ *is the lower bound of* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$.

Next to prove that $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$ *is the GLB of* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$

Let $\mathfrak{M\eth R}S_{\mathfrak{R}}(\varrho)$ *is the lower bound of* $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{M\eth R}S_{\mathfrak{R}}(\varpi)$.

TP: $\mathfrak{M\eth R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{M\eth R}S_{\mathfrak{R}}(\varrho)$

That is to prove that $(\mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi), \mathfrak{M\eth}^-_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)) \subseteq (\mathfrak{M\eth}_{-\mathfrak{R}}(\varrho), \mathfrak{M\eth}^-_{\mathfrak{R}}(\varrho))$

Let $\mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\eth}s(\mathfrak{A} \bigwedge \varpi) \bigcap \mathfrak{M\eth}_-(\mathfrak{M\eth}^-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A} \bigwedge \varpi) \, \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_-(\mathfrak{MO}^-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_-(\mathfrak{MO}^-(\mathfrak{A})) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_-(\mathfrak{MO}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_-(\mathfrak{MO}^-(\mathfrak{A})) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_-(\mathfrak{MO}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) \cap \mathfrak{MO}_-(\mathfrak{MO}^-(\mathfrak{A})) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) \cap \mathfrak{MO}_-(\mathfrak{MO}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_{-\mathfrak{R}}(\mathfrak{A}) \& \subseteq \mathfrak{MO}_{-\mathfrak{R}}(\varpi)$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{MO}_{-\mathfrak{R}}(\varrho)$

Hence, $\mathfrak{MO}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{MO}_{-\mathfrak{R}}(\varrho)$

Let $\mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_{\mathfrak{R}}^-(\mathfrak{A} \bigwedge \varpi) \Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A} \bigwedge \varpi) \bigcup \mathfrak{MO}^-(\mathfrak{MO}_-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A} \bigwedge \varpi) \, or \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}^-(\mathfrak{MO}_-(\mathfrak{A} \bigwedge \varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MO}^-(\mathfrak{MO}_-(\mathfrak{A})) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}^-(\mathfrak{MO}_-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}^-(\mathfrak{MO}_-(\mathfrak{A})) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) \, or \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}^-(\mathfrak{MO}_-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\mathfrak{A}) \cup \mathfrak{MO}_-(\mathfrak{MO}^-(\mathfrak{A})) \&$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{MOs}(\varpi) \cup \mathfrak{MO}_-(\mathfrak{MO}^-(\varpi))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_{\mathfrak{R}}^-(\mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{MO}_{\mathfrak{R}}^-(\varpi)$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{MO}_{\mathfrak{R}}^-(\varrho)$

$\Rightarrow \mathfrak{MO}_{\mathfrak{R}}^-(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{MO}_{\mathfrak{R}}^-(\varrho)$

Hence $\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$ *is the GLB of* $\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A}) \& \mathfrak{MORS}_{\mathfrak{R}}(\varpi)$

Example7.4
From Example 7.2

Let $\mathfrak{A} = \{\varsigma_1, \varsigma_3\}; \varpi = \{\varsigma_2, \varsigma_3\}$

$\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = (\varsigma_3, \mho)$

Theorem 4.5.
If $\mathfrak{A}$ *is the subsets of* $\mho$, *then* $G_P = \{\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A}) \mid \mathfrak{A} \subseteq \mho\}$ *is a lattice.*

Proof:-

This can be proved by theorem 9 & 10.

Note:
$LUB(\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A}), \mathfrak{MORS}_{\mathfrak{R}}(\varpi)) = \mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

$GLB(\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A}), \mathfrak{MORS}_{\mathfrak{R}}(\varpi)) = \mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$

This lattice $(G_P, \approx)$ is called a Minimal soft-pre rough lattice.

Theorem 4.6.
The lattice G_P *has a maximal & minimal elements.*

Proof:

$\mathfrak{MORS}_{\mathfrak{R}}(\emptyset) = (\emptyset, \emptyset)$ *is the minimum elements.*

$\mathfrak{MORS}_{\mathfrak{R}}(\mho) = (\mho, \mho)$ *is the maximum elements.*

Example 7.5
From example 7.1

Let $\mathfrak{A} = \{\mathfrak{W}_4\}, \varpi = \{\mathfrak{W}_3\}$

$\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = (\emptyset, \emptyset)$

Let $\mathfrak{A} = \{\mathfrak{W}_1\}, \varpi = \{\mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4\}$

$\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = (\mho, \mho)$

For each subset $\mathfrak{A} \subseteq \mho$, there is an associated Soft pre-rough sets $\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A})$. The set of all Soft Per-rough sets corresponding to each subset $\mathfrak{A} \subseteq \mho$, is the $G_P = \{\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A}) | \mathfrak{A} \subseteq \mho\}$.

From Example 7.1,

$G_P = \{(\emptyset, \emptyset), (\emptyset, \{\mathfrak{W}_1, \mathfrak{W}_3\}), (\emptyset, \{\mathfrak{W}_2, \mathfrak{W}_4\}), (\mathfrak{W}_3, \{\mathfrak{W}_1, \mathfrak{W}_3\}), (\mathfrak{W}_4, \mho), (\mathfrak{W}_4, \{\mathfrak{W}_2, \mathfrak{W}_4\}), (\mathfrak{W}_3, \mho),$

$(\emptyset, \mho), (\{\mathfrak{W}_1, \mathfrak{W}_3\}, \{\mathfrak{W}_1, \mathfrak{W}_3\}), (\{\mathfrak{W}_2, \mathfrak{W}_4\}, \{\mathfrak{W}_2, \mathfrak{W}_4\}), (\{\mathfrak{W}_2, \mathfrak{W}_3, \mathfrak{W}_4\}, \mho), (\{\mathfrak{W}_3, \mathfrak{W}_4\}, \{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}),$

$(\{\mathfrak{W}_1, \mathfrak{W}_3\}, \mho), (\{\mathfrak{W}_2, \mathfrak{W}_4\}, \mho), (\{\mathfrak{W}_1, \mathfrak{W}_3, \mathfrak{W}_4\}, \mho), (\mho, \mho)\}$

The example 1 lattice construction is shown in figure 1.
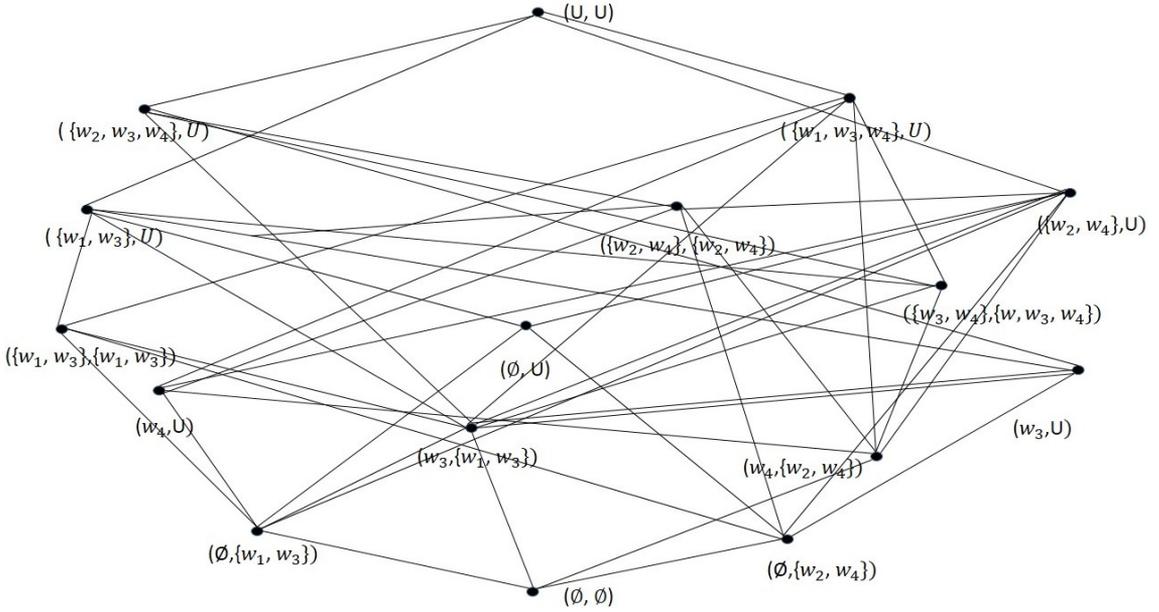
Figure 1. Minimal soft pre-rough lattice Example 7.1

5   Commutative Monoid of Minimal soft pre-rough sets

In this section we discusses the commutative monoid of minimal soft pre-rough sets

Definition 5.1.
Let G_P be the collection of rough sets and $\bigvee : G_P \times G_P \to G_P$ such that $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi)$

Theorem 5.2.
If $\mathfrak{A}, \varpi \& \varrho$ are the subsets of $\mho$ then, $(G_P, \bigvee)$ is a Commutative monoid.

Proof.
For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho$ and Let $\mathfrak{A} \bigvee \varpi = \varrho \subseteq \mho$
Closure Property
For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho \Rightarrow \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varrho) \in G_P$
Associative Property
For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho$ & $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}), \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \& \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varrho) \in G_P$
TP:$\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$
That is to prove that $(\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}) \bigvee(\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varrho))) = ((\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi)) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varrho))$
Claim (i) $\mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) = \mathfrak{M}\eth_{-\mathfrak{R}}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$
(ii)$\mathfrak{M}\eth_{\mathfrak{R}}^{-}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) = \mathfrak{M}\eth_{\mathfrak{R}}^{-}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$
(i)Let $\mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho))$
$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) \bigcap \mathfrak{M}\eth_{-}(\mathfrak{M}\eth^{-}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)))$
$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_{-}(\mathfrak{M}\eth^{-}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)))$
$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \varrho) \&$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_{-}(\mathfrak{M}\eth^{-}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$
$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varrho) \&$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_{-}(\mathfrak{M}\eth^{-}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$
$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varrho) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_{-}(\mathfrak{M}\eth^{-}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s((\mathfrak{A} \bigvee \varpi) \bigvee \varrho) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_- (\mathfrak{M}\eth^-((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s((\mathfrak{A} \bigvee \varpi) \bigvee \varrho) \bigcap \mathfrak{M}\eth_- (\mathfrak{M}\eth^-((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{-\mathfrak{R}}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$

Hence $\mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) \subseteq \mathfrak{M}\eth_{-\mathfrak{R}}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee(\varpi \bigvee \varrho))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) \bigcup \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\mathfrak{A} \bigvee(\varpi \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\mathfrak{A} \bigvee(\varpi \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \varrho) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- ((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varrho) or$

$\mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- ((\mathfrak{A} \bigvee \varpi) \bigvee \varrho))))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varrho) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- ((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s((\mathfrak{A} \bigvee \varpi) \bigvee \varrho) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- ((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s((\mathfrak{A} \bigvee \varpi) \bigvee \varrho) \bigcup \mathfrak{M}\eth^- (\mathfrak{M}\eth_- ((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)))$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{\mathfrak{R}}^-((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$

Hence $\mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) \subseteq \mathfrak{M}\eth_{\mathfrak{R}}^-((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$

Similarly we can prove $\mathfrak{M}\eth_{\mathfrak{R}}^-((\mathfrak{A} \bigvee \varpi) \bigvee \varrho) \subseteq \mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee(\varpi \bigvee \varrho))$

$\therefore \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee(\varpi \bigvee \varrho)) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}((\mathfrak{A} \bigvee \varpi) \bigvee \varrho)$

Hence, $(G_P, \bigvee)$ is a Semigroup. This Semigroup is called a Minimal soft pre-rough semigroup.

Identity Property

For $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}) \in G_P$ & $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\emptyset) \in G_P$

$\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\emptyset) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \emptyset)$

$\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \bigvee \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\emptyset) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi)$

Hence, $(G_P, \bigvee)$ is a Monoid.

This Monoid is called Minimal soft pre-rough Monoid.

Commutative Property

For $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \& \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi) \in G_P$

TP: $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi \bigvee \mathfrak{A})$

Claim (i) $\mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth_{-\mathfrak{R}}(\varpi \bigvee \mathfrak{A})$

(ii) $\mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth_{\mathfrak{R}}^-(\varpi \bigvee \mathfrak{A})$

(i) Let $\mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) \bigcap \mathfrak{M}\eth_- (\mathfrak{M}\eth^- (\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_- (\mathfrak{M}\eth^- (\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth_- (\mathfrak{M}\eth^- (\varpi \bigvee \mathfrak{A})$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \mathfrak{A}) \bigcap \mathfrak{M}\eth_- (\mathfrak{M}\eth^- (\varpi \bigvee \mathfrak{A})$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{-\mathfrak{R}}(\varpi \bigvee \mathfrak{A})$

Similarly we can prove $\mathfrak{M}\eth_{-\mathfrak{R}}(\varpi \bigvee \mathfrak{A}) \subseteq \mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi)$

Hence $\mathfrak{M}\eth_{-\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth_{-\mathfrak{R}}(\varpi \bigvee \mathfrak{A})$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) \bigcup \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\mathfrak{A} \bigvee \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\mathfrak{A} \bigvee \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\varpi \bigvee \mathfrak{A})$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M}\eth s(\varpi \bigvee \mathfrak{A}) \bigcup \mathfrak{M}\eth^- (\mathfrak{M}\eth_- (\varpi \bigvee \mathfrak{A})$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M}\eth_{\mathfrak{R}}^-(\varpi \bigvee \mathfrak{A})$

Hence $\mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi) \subseteq \mathfrak{M}\eth_{\mathfrak{R}}^-(\varpi \bigvee \mathfrak{A})$

Similarly we can prove $\mathfrak{M}\eth_{\mathfrak{R}}^-(\varpi \bigvee \mathfrak{A}) \subseteq \mathfrak{M}\eth_{\mathfrak{R}}^-(\mathfrak{A} \bigvee \varpi)$

$\therefore \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigvee \varpi) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi \bigvee \mathfrak{A})$

Hence $(G_P, \bigvee)$ is Commutative Monoid. $\qquad\qquad\square$

Definition 5.3.

Let G_P be the collection of rough sets and $\bigwedge : G_P \times G_P \to G_P$ such that $\mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\mathfrak{A}) \bigwedge \mathfrak{M}\eth\mathfrak{R}S_{\mathfrak{R}}(\varpi)$

Theorem 5.4.

If $\mathfrak{A}, \varpi \& \varrho$ are the subsets of $\mho$ then, $(G_P, \bigwedge)$ is a Commutative monoid.

Proof.
For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho$ and Let $\mathfrak{A} \bigwedge \varpi = \varrho \subseteq \mho$

Closure Property

For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho \Rightarrow \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varrho) \in G_P$

Associative Property

For $\mathfrak{A}, \varpi \& \varrho \subseteq \mho \ \& \ \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A}), \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \& \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varrho) \in G_P$

TP:$\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

That is to prove that $(\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A}) \bigwedge (\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \bigwedge \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varrho))) =$
$((\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A}) \bigwedge \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi)) \bigwedge \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varrho))$

Claim (i) $\mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) = \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

(ii)$\mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) = \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

(i)Let $\mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) \bigcap \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi \bigwedge \varrho) \&$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{D}) \&$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varrho) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho) \bigcap \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

Hence $\mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) \subseteq \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) \bigcup \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi \bigwedge \varrho) or$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varrho) or$
$\mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho))))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varrho) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho) or \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho) \bigcup \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)))$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

Hence $\mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) \subseteq \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

Similarly we can prove $\mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho) \subseteq \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho))$

$\therefore \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge (\varpi \bigwedge \varrho)) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}((\mathfrak{A} \bigwedge \varpi) \bigwedge \varrho)$

Hence, $(G_P, \bigwedge)$ is a Semigroup. This Semigroup is called a Minimal soft pre-rough semigroup.

Identity Property

For $\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A}) \in G_P \ \& \ \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\emptyset) \in G_P$

$\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \bigwedge \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\emptyset) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \emptyset)$

$\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \bigwedge \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\emptyset) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi)$

Hence, $(G_P, \bigwedge)$ is a Monoid.

This Monoid is called Minimal soft pre-rough Monoid.

Commutative Property

For $\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \& \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi) \in G_P$

TP:$\mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M\mathfrak{O}RS}_{\mathfrak{R}}(\varpi \bigwedge \mathfrak{A})$

Claim (i) $\mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\varpi \bigwedge \mathfrak{A})$

(ii)$\mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\varpi \bigwedge \mathfrak{A})$

(i)Let $\mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge \varpi) \bigcap \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge \varpi) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi \bigwedge \mathfrak{A}) \& \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\varpi \bigwedge \mathfrak{A})$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\varpi \bigwedge \mathfrak{A}) \bigcap \mathfrak{M\mathfrak{O}}_{-}(\mathfrak{M\mathfrak{O}}^{-}(\varpi \bigwedge \mathfrak{A})$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\varpi \bigwedge \mathfrak{A})$

Similarly we can prove $\mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\varpi \bigwedge \mathfrak{A}) \subseteq \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi)$

Hence $\mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{M\mathfrak{O}}_{-\mathfrak{R}}(\varpi \bigwedge \mathfrak{A})$

Let $\mathfrak{D}(\jmath) \in \mathfrak{M\mathfrak{O}}_{\mathfrak{R}}^{-}(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{M\mathfrak{O}}s(\mathfrak{A} \bigwedge \varpi) \bigcup \mathfrak{M\mathfrak{O}}^{-}(\mathfrak{M\mathfrak{O}}_{-}(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MDs}(\mathfrak{A} \bigwedge \varpi) or \mathfrak{D}(\jmath) \subseteq \mathfrak{MD}^-(\mathfrak{MD}_-(\mathfrak{A} \bigwedge \varpi)$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MDs}(\varpi \bigwedge \mathfrak{A}) or \mathfrak{D}(\jmath) \subseteq \mathfrak{MD}^-(\mathfrak{MD}_-(\varpi \bigwedge \mathfrak{A})$

$\Rightarrow \mathfrak{D}(\jmath) \subseteq \mathfrak{MDs}(\varpi \bigwedge \mathfrak{A}) \bigcup \mathfrak{MD}^-(\mathfrak{MD}_-(\varpi \bigwedge \mathfrak{A})$

$\therefore \mathfrak{D}(\jmath) \in \mathfrak{MD}_{\mathfrak{R}}^-(\varpi \bigwedge \mathfrak{A})$

Hence $\mathfrak{MD}_{\mathfrak{R}}^-(\mathfrak{A} \bigwedge \varpi) \subseteq \mathfrak{MD}_{\mathfrak{R}}^-(\varpi \bigwedge \mathfrak{A})$

Similarly we can prove $\mathfrak{MD}_{\mathfrak{R}}^-(\varpi \bigwedge \mathfrak{A}) \subseteq \mathfrak{MD}_{\mathfrak{R}}^-(\mathfrak{A} \bigwedge \varpi)$

$\therefore \mathfrak{MDRS}_{\mathfrak{R}}(\mathfrak{A} \bigwedge \varpi) = \mathfrak{MDRS}_{\mathfrak{R}}(\varpi \bigwedge \mathfrak{A})$

Hence $(G_P, \bigwedge)$ is Commutative Monoid. □

Operations on sets, such as intersection ($\bigwedge$) and union ($\bigvee$), are represented by a Cayley table 1 and table 2. It provides a systematic way to display the results of applying these operations to all possible pairs of elements within a given set. It is also possible to confirm if the structure forms a commutative monoid using this table 1 and table 2. A set containing an identity member and an associative binary operation that is commutative is called a commutative monoid. One may determine if an operation is commutative, associative, or contains an identity element through a peek at the Cayley table. The structure is a commutative monoid if and only if these requirements are satisfied.

5.1 Weather Prediction & Climate Analysis Using Minimal Soft Pre-Rough Sets

Giving accurate forecasts is challenging because weather forecasting entails managing uncertainty brought on by changing climatic circumstances. Predictions are improved by minimal soft pre-rough sets, that categorise weather conditions according to several overlapping characteristics, including temperature, humidity, wind speed, and air pressure.

Let $\mathfrak{U} = \{\varrho_1, \varrho_2, \varrho_3, \varrho_4, \varrho_5\}$ be the five days of weather prediction.

Let $\mathfrak{D}(\mathfrak{b}_1) = \{\varrho_1, \varrho_2, \varrho_3\}$ be high temperature, $\mathfrak{D}(\mathfrak{b}_2) = \{\varrho_2, \varrho_4\}$ be high humidity and $\mathfrak{D}(\mathfrak{b}_3) = \{\varrho_5\}$ be strong wind

(ii)Let Consider, $\tau = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$ a subset of days likely to experience extreme weather

$\mathfrak{MDs}(\tau) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}$

$\mathfrak{MD}_-(\tau) = \{\varrho_1, \varrho_2, \varrho_3, \varrho_5\}; \mathfrak{MD}^-(\tau) = \mathfrak{U}$

$\mathfrak{MD}_-(\mathfrak{MD}^-(\tau)) = \mathfrak{U}; \mathfrak{MD}^-(\mathfrak{MD}_-(\tau)) = \mathfrak{U}$

$\mathfrak{MDRS}_{\mathfrak{R}}(\tau) = (\{\varrho_1, \varrho_2, \varrho_3, \varsigma_5\}, \mathfrak{U})$

The chosen days are fully included in the lower approximation; these are confirmed extreme weather days, and appropriate precautions, like heat advisories or storm warnings, should be issued. By using minimal soft pre-rough sets, meteorologists can improve climate risk assessment, reduce uncertainty, and improve weather forecasts, resulting in more accurate early warning systems for extreme weather events.

5.2 Case study: Analyzing Uncertainty in Data Using Minimal Soft Pre-Rough Sets

A methodical technique to managing uncertainty, approximation, and data classification in complicated datasets is laid out by minimal soft pre-rough sets. Equivalence relations, that form the foundation of traditional rough set models, can be constrictive for working with multi-valued, overlapping, or ambiguous characteristics. Minimal soft pre-rough sets, on the other hand, provide a commutative monoid and a lattice structure, enabling more accurate approximations in data categorisation. For instance, typical rough sets could not adequately reflect the intricacy of relationships in a dataset if objects share many memberships (for instance, a client may belong to multiple market groups or a person may display symptoms of multiple illnesses). By establishing lower and higher approximations that adjust to the degree of uncertainty in the data, the minimal soft pre-rough technique allows for a more accurate and flexible categorisation. Furthermore, by preserving associativity and commutativity, minimal soft pre-rough sets enhance computing efficiency and are more scalable for large-scale and high-dimensional datasets. The flaws of traditional rough set models in managing ambiguous and changing data are addressed by this structured method, it guarantees improved decision-making, data aggregation, and classification accuracy.

5.3 Novelty and Limitation:

Minimal soft pre-rough sets stand out because they combine soft set theory with rough set approximations while incorporating a commutative monoid and lattice structure. This unique feature strengthens algebraic operations, making it easier to manage uncertainty. Unlike traditional rough sets, this approach provides more flexibility in decision-making by integrating minimal soft rough approximations, which is particularly useful for data analysis and classification tasks. This approach presents a unique set of difficulties. It can be levy to work with big datasets, and it might not function properly if the data is noisy or missing. Furthermore, its practical uses may be restricted due to its reliance on particular mathematical presumptions. Researchers could look at combining it with methods like deep learning, machine learning, or fuzzy logic to increase its efficacy. Including pertinent references would also help to clarify the way it stacks up against other approaches now in use, pointing out both its advantages and disadvantages.

6 Conclusion

Minimal soft pre-rough sets represent a significant advance in data analysis, especially in handling approximation and uncertainty, and by investigating their lattice structure and basic properties, this study provides a logical and systematic approach to uncertainty management. The addition of a commutative monoid improves the algebraic characterisation of these sets, enabling them an effective mathematical basis for practical applications. The results show that minimal soft pre-rough sets facilitate decision-making by providing more accurate and dependable ways to handle ambiguous data. The theoretical and practical contributions of this study increase its potential applicability in a variety of sectors that demand accurate data processing. By exploring novel properties and hybrid models and utilising $\bigvee$ and $\bigwedge$ operations inside the lattice framework to improve uncertainty modelling and decision support systems, future research can further expand and improve these ideas.

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$\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A})$

$\bigvee$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\emptyset$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$
$\mathfrak{W}_2$	$\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_2$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_3$	$\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_3$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mho$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$
$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mho$	$\mho$	$\mho$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mho$	$\mho$	$\mho$
$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$
$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$	$\mho$	$\mho$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$	$\mho$

Table 1. Cayle Table for Commutative monoid of $\bigvee$

$\mathfrak{MORS}_{\mathfrak{R}}(\mathfrak{A})$

$\wedge$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathfrak{W}_1$	$\emptyset$	$\mathfrak{W}_1$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\emptyset$	$\mathfrak{W}_1$
$\mathfrak{W}_2$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_2$
$\mathfrak{W}_3$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathfrak{W}_3$	$\emptyset$	$\emptyset$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_3$	$\mathfrak{W}_3$
$\mathfrak{W}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathfrak{W}_4$	$\emptyset$	$\emptyset$	$\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_4$
$\mathfrak{W}_1\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\emptyset$	$\emptyset$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2$
$\mathfrak{W}_1\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_1$	$\emptyset$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_3$
$\mathfrak{W}_1\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_1$	$\emptyset$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_1$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$
$\mathfrak{W}_2\mathfrak{W}_3$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$
$\mathfrak{W}_2\mathfrak{W}_4$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_2$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_2$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$
$\mathfrak{W}_3\mathfrak{W}_4$	$\emptyset$	$\emptyset$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$
$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\emptyset$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$
$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\emptyset$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$
. $\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\emptyset$	$\mathfrak{W}_1$	$\emptyset$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_1$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$
$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\emptyset$	$\emptyset$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$
$\mho$	$\emptyset$	$\mathfrak{W}_1$	$\mathfrak{W}_2$	$\mathfrak{W}_3$	$\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2$	$\mathfrak{W}_1\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_3$	$\mathfrak{W}_1\mathfrak{W}_2\mathfrak{W}_4$	$\mathfrak{W}_1\mathfrak{W}_3\mathfrak{W}_4$	$\mathfrak{W}_2\mathfrak{W}_3\mathfrak{W}_4$	$\mho$

Table 2. Cayle Table for Commutative monoid of $\wedge$