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Existence results for fractional integro differential equations involving two boundary conditions with ψ-Caputo fractional derivative

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Abstract In this article, we discuss the existence and uniqueness of solutions for a fractional integro-differential equation involving the ψ-Caputo fractional derivative with boundary conditions. The primary analysis is based on Hölder's inequality and Mönch fixed-point theorem. An example is provided to demonstrate the applicability of the results.

1 Introduction

Fractional calculus has recently emerged as a powerful tool for modeling various processes in engineering, physics, and economics. Fractional-order models have been shown to be more suitable than integer-order models for many real-world scenarios, as fractional derivatives effectively capture memory and hereditary properties of materials and processes. This theory plays a significant role in numerous fields, including material science, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid dynamics, thermodynamics, mechanics, geology, astrophysics, economics, and control theory (see [1, 2, 9, 10, 12, 25, 26, 31]). This ability to describe such phenomena is a key advantage of fractional differential equations over traditional integer-order models.

Many definitions of fractional derivatives and fractional integrals, including those by Riemann-Liouville, Caputo, Caputo–Hadamard, Katugampola, and Hilfer are included in the same line. Fractional differential equations have recently attracted a lot of attention. For example, boundary value problems for nonlinear fractional differential equations can be used to model and describe non-homogeneous physical processes that occur in their form.

Nonlinear fractional boundary value problems are increasingly recognized for their broad applications in areas such as economics, financial mathematics, and various applied sciences. In recent years, there has been a notable theoretical focus on exploring fractional differential and integro-differential equations with boundary conditions. Researchers have employed techniques from nonlinear analysis, including the Banach fixed-point theorem and Leray-Schauder theory, to obtain significant results in this field. For further examples, refer to [2, 3, 4, 5, 30, 31, 37] and related works.

The measure of noncompactness is a function defined on the collection of all nonempty, bounded subsets of a given metric space. It is characterized by being zero for all relatively compact sets. This concept was first introduced by Kuratowski in 1930.

The Kuratowski and Hausdorff measures are among the most notable measures of noncompactness. The Hausdorff measure, in particular, is widely applied in various areas of nonlinear analysis and its applications, as evidenced by references [3, 4, 5, 17, 30] and related works. This paper focuses on measures of noncompactness, a widely used method in nonlinear analysis. This approach has facilitated the development of different types of integral equations, as demonstrated in [4, 5, 17, 30].

In [6], Almeida et al. investigated models for population growth and gross domestic product using the ψ-Caputo derivative, proving existence and uniqueness results through a fixed-point theorem. Their method is highly versatile, making it well-suited for analyzing and developing complex dynamical systems that exhibit non-local properties and memory effects. Compared to the classical Caputo derivative, the ψ-Caputo derivative provides greater flexibility and improved accuracy, especially for higher orders. This derivative has attracted significant attention in various scientific and engineering disciplines. For instance, [8] presents a numerical analysis of non-integer-order relaxation–oscillation equations employing the ψ-Caputo derivative.

Karthikeyan et al.[25] discussed the existence and uniqueness of fractional integro differential equations with boundary value conditions of the form

$$^{C}D^{\alpha_1}_{a+}v(\sigma) = f(\sigma, v(\sigma), (Rv)(\sigma)), \sigma \in J = [0, \hbar], \alpha_1 \in (0, 1),$$

$$a\eth(0) + b\eth(\hbar) = c,$$

where $^{C}D^{\alpha_1}_{a+}$ is the Caputo fractional derivative of order α_1, $f : \mathbb{J} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, where X is the Banach spaces and a,b,c are real constant with a+b $\neq$ 0 and R is a nonlinear integral operator given by $(Rv)(\sigma) = \int_0^\sigma h(\sigma, \wp)\eth(\wp)d\wp$, where $h \in \mathbb{C}(\mathbb{J} \times \mathbb{J}, \mathbb{R}+)$.

Thabet et al.[33] studied the existence and uniqueness of Caputo fractional differential equation with two boundary condition.

$$^{C}D^{\alpha_1}_{a+}v(\sigma) = f(\sigma, v(\sigma), (Rv)(\sigma)), 1 < \alpha_1 \leq 2, \sigma \in \Im; = [0, \hbar],$$

$$v(0) = v_0, v^{'}(\hbar) = v_\hbar,$$

where $^{C}D^{\alpha_1}_{a+}$ is the Caputo fractional derivative of order α_1, $f : \mathbb{J} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a continuous function and $v_0, v_\hbar$ are elements of $\mathbb{X}$ and R is a nonlinear integral operator given by $(Rv)(\sigma)$=$\int_0^\sigma h(\sigma, \wp, \eth(\wp))d\sigma$, where $h \in \mathbb{C}(\mathbb{J} \times \mathbb{J} \times \mathbb{X}, \mathbb{X})$.

Sun et al.[35] discussed the existence and uniqueness result of the boundary value problem for a fractional differential equation including Caputo fractional derivative.

$$^{C}D^{\alpha_1}_{a+}v(\sigma) = f(\sigma, v(\sigma)), \sigma \in [a, b],$$

$$v^k(a) = v_k, k = 0, 1, ...n - 2; v^{n-1}(b) = y_b,$$

where $^{C}D^{\alpha_1}_{a+}$ is the Caputo fractional derivative of order $n - 1 < \alpha_1 \leq n$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is the continuous functions and $v_0, v_1, v_3.....v_{n-2}, v_b$ are the real constants.

Almeida et al.[6] discussed the existence and uniqueness of ψ-Caputo fractional differential equation with boundary condition.

$$^{C}D^{\alpha_1;\psi}_{a+}v(\sigma) = f(\sigma, v(\sigma)), \sigma \in [a, b],$$

$$v(a) = v_a, v^{[k]}_\psi(a) = v^k_a, k = 1, 2, 3...n - 1,$$

where $^{C}D^{\alpha_1;\psi}_{a+}$ is the ψ-Caputo fractional derivative of order $n - 1 < \alpha_1 < n$, $v_a, v^k_a \in \mathbb{R}$, $\sigma \in [a, b]$, $v \in \mathbb{C}^{n-1}[a, b]$ and $f : [a, b] \times \mathbb{R} \to \mathbb{R}$.

Dhaigude et al.[17] discussed the existence and uniqueness of ψ-Caputo fractional differential equation with boundary condition.

$$^{C}D^{\alpha_1;\psi}_{a+}v(\sigma) = F(\sigma, v), 0 < \sigma \leq \hbar,$$

$$G(v(0), v(\hbar)) = 0,$$

where $^{C}D^{\alpha_1;\psi}_{a+}$ is the ψ-Caputo fractional derivative of order α_1, $0 < \alpha_1 \leq 1$ and
$F(\sigma, v) : [0, \hbar] \times \mathbb{R} \to \mathbb{R}$ is real valued continuous function.

Boutiara et al.[14] discussed the existence and uniqueness of Implicit ψ-Caputo fractional differential equation with boundary condition.

$$^{C}D_{a+}^{\alpha_1;\psi}v(\sigma) = f(\sigma, v(\sigma), ^{C}D_{a+}^{\alpha_1;\psi}v(\sigma)), \sigma \in \Im; = [0, \hbar],$$

$$v(\hbar) = \lambda v(\delta),$$

where $^{C}D_{a+}^{\alpha_1;\psi}$ is the ψ-Caputo fractional derivative of order α_1, $0 < \alpha_1 \leq 1$, $f : [a, \hbar] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. λ is a real constant and $\delta \in (a, \hbar)$.

Mfadel et al.[27] studied the existence and uniqueness of ψ-Caputo fractional differential equation with boundary condition.

$$^{C}D_{a+}^{\alpha_1;\psi}v(\sigma) = f(\sigma, v(\sigma)), \sigma \in \Im; = [0, \hbar],$$

$$v(0) = v^{'}(0) = 0, v(\hbar) = v_{\hbar},$$

where $^{C}D_{a+}^{\alpha_1\psi}$ is the ψ-Caputo fractional derivative of order α_1 , $2 < \alpha_1 < 3$, $f : [0, \hbar] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function.

Inspired by the above works, we consider the ψ-Caputo fractional integro-differential equation with two boundary condition of the form

$$^{C}D_{a+}^{\alpha_1;\psi}v(\sigma) = f(\sigma, v(\sigma), (Rv)(\sigma)), 2 < \alpha_1 \leq 3, \sigma \in \Im; = [0, \hbar], \tag{1.1}$$

$$v(0) = v^{'}(0) = 0, v(\hbar) = v_{\hbar}, \tag{1.2}$$

where $^{C}D_{a+}^{\alpha_1;\psi}$ is the Caputo fractional derivative of order α_1, $f : \mathbb{J} \times \mathbb{X} \times \mathbb{X} \to \mathbb{X}$ is a continuous function and $v_0, v_{\hbar}$ are elements of $\mathbb{X}$ and R is a nonlinear integral operator given by $(Rv)(\sigma)$=$\int_0^{\sigma} h(\sigma, s, v(s))d\sigma$, where $h \in \mathbb{C}(\mathbb{J} \times \mathbb{J} \times \mathbb{X}, \mathbb{X})$.

This paper has the following structure: In Section 2, provides the basic definitions and fundamental results. In section 3, the existence results by using the Mönch fixed-point theorem via Hausdorff's measure of noncompactness are discussed . In Section 4, presents an illustrative example

2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus and present preliminary results needed in our proofs later.

Let $C(\Im, \mathbb{R})$ the space of real and continuous functions with the norm

$$\|v\|_{\infty} = \sup\{\|v(\sigma)\| : \quad \sigma \in \Im\}.$$

Let $L^1(\Im, \mathbb{R})$ be the Banach space of Lebesgue integrable functions $u : \Im \to \mathbb{R}$, equipped with the norm

$$\|v\|_{L^1} = \int_{\Im} |v(\sigma)|d\sigma.$$

Definition 2.1. ([14]). The Riemann-Liouville fractional integral of order $\alpha_1 > 0$ and $\alpha_1 \in \mathbb{R}$ of a the function v is defined by

$$I_{a^+}^{\alpha_1}v(\sigma) = \int_{a_1}^{\sigma} \frac{(\sigma - \eta)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} v(\eta)d\eta.$$

Definition 2.2. ([26]). Let v:$(0, \infty) \to R$ be the Caputo fractional, then the Riemann-Liouville fractional derivative of order $\alpha_1 > 0$, $m = [\alpha_1] + 1$, $[\alpha_1]$ denote the integer part of α_1 and v defined by

$$(D_{a^+}^{\alpha_1}v)(\sigma) = \left(\frac{d}{d\sigma}\right)^m \int_a^{\sigma} \frac{(\sigma - \eta)^{m - \alpha_1 - 1}}{\Gamma(m - \alpha_1)} v(\eta)d\eta.$$

Definition 2.3. ([14]). Let $v:(0,\infty) \to R$ be the Caputo fractional, then the Caputo fractional derivative of order $\alpha_1 > 0$, $m = [\alpha_1] + 1$, $[\alpha_1]$ denote the integer part of α_1 and v defined by

$$({}^C\mathcal{D}^{\alpha_1}_{a^+}v)(\sigma) = \int_a^\sigma \frac{(\sigma-\eta)^{m-\alpha_1-1}}{\Gamma(\alpha_1)} v^{(n)}(\eta)d\eta.$$

Definition 2.4. ([14]). The left-sided ψ-Riemann-Liouville fractional integral of order $\alpha_1 > 0$ for the integral function defined as

$$\mathcal{I}^{\alpha_1;\psi}_{a^+}v(\sigma) = \int_a^\sigma \frac{\psi'(\eta)(\psi(\sigma)-\psi(\eta))^{\alpha_1-1}}{\Gamma(\alpha_1)} v(\eta)\mathrm{d}\eta.$$

Definition 2.5. ([14]). The left-sided ψ Riemann-Liouville fractional derivative of order $\alpha_1 > 0$, $m = [\alpha_1] + 1$ is defined by

$$\begin{aligned}\mathcal{D}^{\alpha_1;\psi}_{a^+}v(\sigma) &= \beta^{(m)}_\psi \mathcal{I}^{m-\alpha_1;\psi}_{a^+}v(\sigma),\\ &= \beta^{(m)}_\psi \int_a^t \psi'(\eta)\frac{(\psi(\sigma)-\psi(\eta))^{m-\alpha_1-1}}{\Gamma(m-\alpha_1)} v(\eta)\mathrm{d}\eta,\end{aligned}$$

where $\beta^{(m)}_\psi = \left(\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}\right)^m$.

Definition 2.6. ([14]). The left-sided ψ-Caputo fractional derivative of v of order α_1, $m = [\alpha_1] + 1$ for $\alpha_1 \notin \mathbb{N}, m = \alpha_1$ for $\alpha_1 \in \mathbb{N}$ is defined by

$${}^C\mathcal{D}^{\alpha_1;\psi}_{a^+}v(\sigma) = \mathcal{I}^{m-\alpha_1;\psi}_{a^+}\beta^{(m)}_\psi v(\sigma),$$

where $\beta^{(m)}_\psi = \left(\frac{1}{\psi'(\sigma)}\frac{d}{d\sigma}\right)^m$.

From the definition, it is clear that

$${}^C\mathcal{D}^{\alpha_1;\psi}_{a^+}v(\sigma) = \begin{cases} \int_a^\sigma \frac{\psi'(\eta)(\psi(\sigma)-\psi(\eta))^{m-\alpha_1-1}}{\Gamma(m-\alpha_1)} v^{[m]}_\psi(\eta)d\eta, & \text{if } \alpha_1 \notin \mathbb{N},\\ u^{[m]}_\psi(\sigma) & \text{if } \alpha_1 \in \mathbb{N}.\end{cases}$$

Here α_1 is the order of ψ-Caputo fractional derivative, if $v \in \mathbb{C}^n(\Im, \mathbb{R})$ then

$${}^C\mathcal{D}^{\alpha_1;\psi}_{a^+}v(\sigma) = \mathcal{D}^{\alpha_1;\psi}_{a^+}\left[v(\sigma) - \sum_{z=0}^{m-1}(\psi(\sigma)-\psi(a))^z \frac{\beta^{(z)}_\psi(a)}{z!}\right].$$

Lemma 2.7. *([33]). Let* $\alpha_1 > 0$, *then* ${}^C\mathcal{D}^{\alpha_1;\psi}_{a^+}v(\sigma) = 0$, *has the following solution*

$$v(\sigma) = c_0 + c_1 t + c_2 t^2 + ... + c_{m-1}t^{m-1}, \tag{2.1}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$, *where* $m = [\alpha_1] + 1$.

Lemma 2.8. *([33]). Let* $\alpha_1 > 0$, *then*

$$I^{y;\psi}_{a^+};{}^C D^{y;\psi}_{a^+}v(\sigma) = v(\sigma)+c_0+c_1(\psi(\sigma)-\psi(0))+c_2(\psi(\sigma)-\psi(0))^2+...+c_{m-1}(\psi(\sigma)-\psi(0))^{m-1}, \tag{2.2}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, ..., m-1$, *where* $m = [\alpha_1] + 1$.

Lemma 2.9. *([14]). Let* $y, z > 0$, *and* $v \in L^1(\Im, \mathbb{R})$. *Then*

$$\mathcal{I}^{y;\psi}_{a^+}\mathcal{I}^{z;\psi}_{a^+}v(\sigma) = \mathcal{I}^{y+z;\psi}_{a^+}v(\sigma),\ a.e.\ \sigma \in \Im,$$

specifically, if $v \in \mathcal{C}(\Im, \mathbb{R})$, *then* $\mathcal{I}^{y;\psi}_{a^+}\mathcal{I}^{z;\psi}_{a^+}v(\sigma) = \mathcal{I}^{y+z;\psi}_{a^+}v(\sigma), \sigma \in \Im$.

Lemma 2.10. *([14]). Let $y > 0$, the subsequent holds: If $v \in \mathcal{C}(\Im, \mathbb{R})$, then*

$${}^{C}\mathcal{D}_{a^+}^{y;\psi}\mathcal{I}_{a^+}^{y;\psi}v(\sigma) = v(\sigma), \sigma \in \Im,$$

suppose $m-1 < y < n$, $v \in C^m(\Im, \mathbb{R})$, then

$$\mathcal{I}_{a^+}^{y;\psi}\,{}^{C}\mathcal{D}_{a^+}^{y;\psi}v(\sigma) = v(\sigma) - \sum_{z=0}^{m-1}(\psi(\sigma) - \psi(a))^z \frac{\beta_{\psi}^{(z)}(a)}{z!}, \quad \sigma \in \Im.$$

Lemma 2.11. *([14]). Let $\sigma > a$, $y, z \geq 0$, and by assuming $(Q_b(\sigma)) = \psi(\sigma) - \psi(a)$. Then*

- $\mathcal{I}_{a^+}^{y;\psi}(Q_b(\sigma)^{z-1} = \frac{\Gamma(z)}{\Gamma(z+y)}(Q_b(\sigma))^{z+y-1}$.
- ${}^{C}\mathcal{D}_{a^+}^{y;\psi}(Q_b(\sigma))^{z-1} = \frac{\Gamma(z)}{\Gamma(z-y)}(Q_b(\sigma))^{z-y-1}$.
- ${}^{C}\mathcal{D}_{a^+}^{y;\psi}(Q_b(\sigma))^{k} = 0,$ *for everyone $k \in \{0, \ldots, m-1\}, m \in \mathbb{N}$.*

Definition 2.12. ([33]). Let D^+ be the non negative cone of an ordered Banach space $(D, \leq)$. A function $\Re$ defined on the X with value in D^+ is called a Measure of Non-Compactness (MNC) on X iff $\Re(\overline{conv}(\omega)) = \Re(\omega)$ for all $\omega \in X$.
The Measure of noncompactness is considered as:
(i) Monotone iff $(\omega_1 \subseteq \omega_2) \Rightarrow \Re(\omega_1) \leq \Re(\omega_2)$ for every ω_1, ω_2 of X,
(ii)Non singular if $\Re((a) \cup \omega)$=$\Re(\omega)$ for every non empty subset $\omega \subseteq X$, $\forall a \in X$,
(iii) Regular if $\Re(\omega) = 0 \Leftrightarrow \omega$ is relative compact in X.

The Hausdorff measure of noncompactness of ρ is given by $\rho(\omega) = \inf r > 0$: ω be the finite number of balls with radii r.

It is established that Measure of noncompactness ρ possess the aforementioned properties and additional properties for all bounded subsets $\omega, \omega_1, \omega_2$ of X.
(iv)$\rho(\omega_1 + \omega_2) \leq \rho(\omega_1) + \rho(\omega_2)$, where $\omega_1 + \omega_2 = (x + y; x \in \omega_1, y \in \omega_2)$,
(v) $\rho(\omega_1) \cup (\omega_2) \leq \max(\rho(\omega_1), \rho(\omega_2))$,
(vi)$\rho(\lambda\omega) = |\lambda|\,\rho(\omega)$ for any $\lambda \in R$,
(vii)If the map $U : D(U) \subseteq x \to Y$ Lipschitz continuous, k is constant then $\rho_z(U(\omega)) \leq k\rho(\omega)$ for any bounded subset $\omega \subseteq D(U)$, as Y is the Banach space.

Definition 2.13. (Dominated convergent theorem [33]). Let $(X, \mathcal{A}, \mu)$ be a measure space. Suppose $\{f_n\}$ is a sequence of measurable functions that converges pointwise to a function f almost everywhere on X. If there exists an integrable function $g : X \to \mathbb{R}$ such that

$$|f_n(x)| \leq g(x) \quad \text{for all } n \text{ and for almost every } x \in X,$$

then,

$$\lim_{n\to\infty}\int_X f_n\,d\mu = \int_X f\,d\mu.$$

Definition 2.14. (Hölder's inequality [33]). Measurable functions f and g on a measure space $(X, \mathcal{A}, \mu)$ and for $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we have:

$$\int_X |f(x)g(x)|\,d\mu(x) \leq \left(\int_X |f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}}\left(\int_X |g(x)|^q\,d\mu(x)\right)^{\frac{1}{q}}.$$

Lemma 2.15. *([33]). Let $\rho(W(\sigma))$ is continuous for $\sigma \in [a, b]$, if $W \subseteq C([a, b], X)$ is bounded and equicontinuous, then*

$$\rho(W) = \sup(\rho(W(\sigma)),\ \sigma \in [a, b]),$$

where

$$W(\sigma) = x(\sigma); x \in W \subseteq X.$$

Lemma 2.16. *([33]). The sequence $\{u_n\}_{n=1}^{\infty}$ is a Bochner integrable function from $J \to X$ with the estimation $\|u_n(\sigma)\| \leq \mu(\sigma)$ for almost all $\sigma \in J$ and every $n \geq 1$, where $\mu \in L^1(J, R)$ then the function $\psi(\sigma) = \rho(u_n(\sigma) : n \geq 1)$ belongs to $L^1(J, R)$ and satisfies*

$$\rho\Big(\int_0^{\sigma} u_n(s)ds : n \geq 1\Big) \leq 2\int_0^{\sigma} \psi(s)ds.$$

Lemma 2.17. *(Monch fixed point theorem [33]). Let H be a closed convex subset of a Banach space X and $0 \in H$, the continuous map $G : H \to X$, satisfy Monch's condition. i.e., if $M \subseteq H$ is denumerable and $M \subseteq conv(0 \cup G(M)) \Rightarrow \overline{M}$ is compact. Then G has a fixed point in D.*

Lemma 2.18. *([14]). Let $\omega \in C(\Im, \mathbb{R})$, $2 < \alpha_1 < 3$ and $\rho > 0$, then the linear antiperiodic boundary value problem*

$$\begin{aligned} &{}^{C}D^{\alpha_1;\psi} v(\sigma) = \kappa(\sigma), \sigma \in \Im, \\ &v(0) = v^{'}(0) = 0, v(\hbar) = v_{\hbar}, \end{aligned} \tag{2.3}$$

has the unique solution is given by

$$\begin{aligned} v(\sigma) = &\int_0^{\sigma} \frac{[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta \\ &- \Big[\frac{(\psi(\sigma) - \psi(0))}{\psi(\hbar) - \psi(0)}\Big]^2 \Bigg[\int_a^{\hbar} \frac{[\psi(\hbar) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta - v_{\hbar}\Bigg]. \end{aligned} \tag{2.4}$$

Proof. Let v satisfies equation (2.3), then lemma (2.8) implies that

$$v(\sigma) = c_0 + c_1(\psi(\sigma) - \psi(0)) + c_2(\psi(\sigma) - \psi(0))^2 + \int_0^{\sigma} \frac{[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta,$$

where $c_0, c_1, c_2 \in \mathbb{R}$,

$$\begin{aligned} v(\sigma) &= \int_0^{\sigma} \frac{[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta + c_0 + c_1(\psi(\sigma) - \psi(0)) + c_2(\psi(\sigma) - \psi(0))^2. \\ v'(\sigma) &= \int_a^{\sigma} \frac{(\alpha_1 - 1)[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 2}\psi'(\sigma)\psi'(\eta)\kappa(\sigma) + [\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1}\kappa'(\sigma)}{\Gamma(\alpha_1)}d\eta + c_1\psi'(\sigma) \\ &\quad + 2c_2(\psi(\sigma) - \psi(0))\psi^{'}(\sigma). \end{aligned}$$

By applying the boundary conditions, $v(0) = 0$ and $v^{'}(0) = 0$, then $c_0 = c_1 = 0$. Then

$$v(\sigma) = c_2(\psi(\sigma) - \psi(0))^2 + \int_0^{\sigma} \frac{[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta.$$

By using the condition $v(\hbar) = v_{\hbar}$, we get

$$v(\hbar) = c_2(\psi(\hbar) - \psi(0))^2 + \int_0^{\hbar} \frac{[\psi(\hbar) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta = v_{\hbar},$$

then we obtain

$$c_2 = \Big[\frac{1}{\psi(\hbar) - \psi(0)}\Big]^2 \Bigg[v_{\hbar} - \int_a^{\hbar} \frac{[\psi(\hbar) - \psi(\eta)]^{\alpha_1 - 1}\psi'(\eta)}{\Gamma(\alpha_1)}\kappa(\sigma)d\eta\Bigg]$$

Finally we obtain the solution in the equation (2.4). □

Lemma 2.19. *([13]). Let f be the continuous function, the function $v(\sigma)$ solve the problem (1.1) if and only if it is a fixed point of the operator $\rho : C(\Im, \Re) \to C(\Im, \Re)$, then*

$$\rho v(\sigma) = \int_0^{\sigma} \psi_{(\sigma)}^{(\eta)} f(\eta, v(\eta), Rv(\eta))d\eta - Q_{(\hbar)}^{(\sigma)} \left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, v(\eta), Rv(\eta))d\eta + v_{\hbar} \right]$$

$where$

$$\psi_{(\sigma)}^{(\eta)} = \frac{[\psi(\sigma) - \psi(\eta)]^{\alpha_1 - 1} \psi'(\eta)}{\Gamma(\alpha_1)},$$

$$\psi_{(\hbar)}^{(\eta)} = \frac{[\psi(\hbar) - \psi(\eta)]^{\alpha_1 - 1} \psi'(\eta)}{\Gamma(\alpha_1)}$$

$$Q_{(\hbar)}^{(\sigma)} = \left[\frac{\psi(\sigma) - \psi(0)}{\psi(\hbar) - \psi(0)} \right]^2.$$

3 Main Result

To prove the main results, we enumerate the subsequent hypotheses;

(H_1).The function $f : \Im \times X \times X \to X$ is satisfies the caratheodory-type conditions; i.e.,$f(\cdot, x, y)$ is measurable for all $x, y \in X$ and $f(\sigma, \cdot, \cdot)$ is continuous for a.e $\sigma \in [0, \hbar]$.

(H_2).The function $h : \Im \times \Im \times X \to X$ is satisfies the caratheodory-type conditions; i.e., $h(\cdot, \cdot, x)$ is measurable for all $x \in X$ and $h(\sigma, s, \cdot)$ is continuous for a.e $s \in [0, \sigma]$, $\sigma \in \Im$.

(H_3).There exist a constants $L_f, L_k > 0$, ensuring that

$$\|f(\sigma, v(\sigma), Rv(\sigma))\| \leq L_f(1 + \|v(\sigma)\| + \|Rv(\sigma)\|),$$

$$\|h(\sigma, s, v(s))\| \leq L_k(1 + \|v(s)\|),$$

for every $s \in [0, \sigma]$, $\sigma \in \Im$ and all $v \in X$.

(H_4).There exist a constant $\alpha \in (0, \alpha_1)$ and real valued functions $n_1(\sigma), n_2(\sigma) \in L^{\frac{1}{\alpha_1}}(\Im, R)$ ensuring that

$$\rho(f(\sigma, E, F)) \leq n_1(\sigma)(\rho(E) + \rho(F)),$$

$$\rho(h(\sigma, s, F)) \leq n_2(\sigma)\rho(C),$$

for each bounded sets $E, F \subset X$ a.e $\sigma \in \Im$.
To be brief , Assume $N = \|n_1 + 2Tn_1 n_2\|_{L^{\frac{1}{\alpha_1}}(\Im, R)}$

(H_5).There exist a positive constants L_f, L_k, we have

$$\frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} L_f(1 + L_k T) \leq 1.$$

Now , we are going to state and prove our main result.

Theorem 3.1. *Assume that hypotheses $(H_1) - (H_5)$ are holds. If*

$$L = \frac{2N(Q_b(\hbar))^{\alpha_1 - \alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1 - \alpha}{1 - \alpha})^{1 - \alpha}} + \frac{2NQ_{(\hbar)}^{(\sigma)}(Q_b(\hbar))^{\alpha_1 - \alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1 - \alpha}{1 - \alpha})^{1 - \alpha}} < 1. \tag{3.1}$$

Then, the equation (1.1) has at least one solution on $\Im$.

Proof. First we have to prove the existence solution of the problem (1.1) transform it into a fixed point problem.
Define the mapping $G : C(\Im, \mathbb{X}) \to C(\Im, \mathbb{X})$ as follows;

$$Gv(\sigma) = \int_0^{\sigma} \psi_{(\sigma)}^{(\eta)} f(\eta, v(\eta), Rv(\eta))d\eta - Q_{(\hbar)}^{(\sigma)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, v(\eta), Rv(\eta))d\eta + Q_{(\hbar)}^{(\sigma)} v_{\hbar}. \tag{3.2}$$

Let $M_r = x \in C(\Im, \mathbb{X}), \|x\|_\infty \leq r$ for some $r > 0$.
We shall show that G fulfills the assumptions of lemma (2.17). The proof will be the following steps.
Step:1. There exist, $r > 0$ ensuring that $G(M_r) \subseteq M_r$.
If this is false, then for each nonnegative number r, there exist a function $p^r \in M_r$.
But $G(p^r)$ does not belong to M_r.
ie)., $\|G(p^r)\| > r$ for some $\sigma \in J$ using (H_3). We have

$$\begin{aligned}
r &< \|G(p^r)(\sigma)\| \\
&= \| \int_0^\sigma \psi_{(\sigma)}^{(\eta)} f(\eta, p^r(\eta), (Rp^r)(\eta)) - Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\hbar)}^{(\eta)} f(\eta, p^r(\eta), (Rp^r)(\eta)) + Q_{(\hbar)}^{(\sigma)} v_\hbar \| \\
&\leq \int_0^\sigma \psi_{(\sigma)}^{(\eta)} \|f(\eta, p^r(\eta), (Rp^r)(\eta))\| \, d\eta \\
&+ Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\hbar)}^{(\eta)} \|f(\eta, p^r(\eta), (Rp^r)(\eta))\| \, d\eta + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\| \\
&\leq \int_0^\sigma \psi_{(\sigma)}^{(\eta)} L_f(1 + \|p^r(\eta)\| + \|(Rp^r)(\eta)\|) d\eta \\
&+ Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\sigma)}^{(\eta)} L_f(1 + \|p^r(\eta)\| + \|(Rp^r)(\eta)\|) d\eta + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\| \\
&\leq \int_0^\sigma \psi_{(\sigma)}^{(\eta)} L_f(1 + \|p^r\|_\infty + L_k(1 + \|p^r\|_\infty)) T d\eta \\
&+ Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\hbar)}^{(\eta)} L_f(1 + \|p^r\|_\infty + L_k(1 + \|p^r\|_\infty)) T d\eta + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\| \\
&\leq \int_0^\sigma \psi_{(\sigma)}^{(\eta)} L_f(1 + r + L_k(1 + r)) T d\eta \\
&+ Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\hbar)}^{(\eta)} L_f(1 + r + L_k(1 + r)) T d\eta + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\| \\
&\leq L_f(1 + r)(1 + L_k T) \int_0^\sigma \psi_{(\sigma)}^{(\eta)} d\eta \\
&+ L_f(1 + r)(1 + L_k T) Q_{(\hbar)}^{(\sigma)} \int_a^\hbar \psi_{(\hbar)}^{(\eta)} d\eta + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\| \\
&\leq L_f(1 + r)(1 + L_k T) \frac{(\psi(\sigma) - \psi(\eta))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \\
&+ L_f(1 + r)(1 + L_k T) Q_{(\hbar)}^{(\sigma)} \frac{(\psi(\sigma) - \psi(\eta))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + Q_{(\hbar)}^{(\sigma)} \|v_\hbar\|,
\end{aligned}$$

while dividing the above equation by $'r'$,

$$1 < \frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} L_f(\frac{1}{r} + 1)(1 + L_k T) + \frac{1}{r}(\|v_\hbar\|) \tag{3.3}$$

Now taking lim as $r \to \infty$,

$$1 < \frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} L_f(1 + L_k T), \tag{3.4}$$

which goes against the hypothesis (H_5). Hence for some positive number r.

$$G(M_r) \subseteq M_r \ \ that \ \ is \ \ G : M_r \to M_r.$$

Step:2. G is continuous operator on M_r.
Let p_n be sequence ensuring that $p_n \to p$ in M_r as $n \to \infty$.

Given that f fulfills (H_1) and (H_2) for almost every $\sigma \in J$, we get

$$f(\eta, p_n(\eta), (Rp_n)(\eta)) \to f(\eta, p(\eta), (Rp)(\eta)).$$

Here assume $\Lambda = f(\eta, p(\eta), (Rp)(\eta))$ and $\Lambda_n = f(\eta, p_n(\eta), (Rp_n)(\eta))$.
By the definition (2.13), for each $\sigma \in \Im$,

$$\begin{aligned}
\|G(p_n)(\sigma) - G(p)(\sigma)\| \leq \| & \int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} f(\eta, p_n(\eta), (Rp_n)(\eta)) d\eta \\
& - Q_{(\hbar)}^{(\sigma)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, p_n(\eta), (Rp_n)(\eta)) d\eta + Q_{(\hbar)}^{(\sigma)} v_{\hbar} \\
& - \int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta \\
& + Q_{(\hbar)}^{(\sigma)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta - Q_{(\hbar)}^{(\sigma)} v_{\hbar} \| \\
& \leq \int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} \|\Lambda_n - \Lambda\| \, d\eta \\
& + Q_{(\hbar)}^{(\sigma)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} \|\Lambda_n - \Lambda\| \, d\eta \\
& \leq \|f(., p_n(.), (Rp_n)(.)) - f(., p(.), (Rp)(.))\| \\
& \Big[\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} d\eta + \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} d\eta \Big] \\
& \leq \frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \|f(., p_n(.), R(p_n)(.)) - f(., p(.), R(p)(.))\|_{\infty}
\end{aligned}$$

So by taking supremum, to obtain

$$\|Gp_n - Gp\|_{\infty} \leq \frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \|f(., p_n(.), R(p_n)(.)) - f(., p(.), R(p)(.))\|_{\infty}$$

Therefore,

$$\|Gp_n - Gp\|_{\infty} \to 0 \;\; as \;\; n \to \infty.$$

Step:3. $G(M_r)$ is equicontinuous on $\Im$. Indeed for $0 \leq \sigma_1 \leq \sigma_2 \leq \hbar$, $\mathfrak{x} \in \mathfrak{w}_r$ and (H_3), we have

$$\begin{aligned}
& \|G(p)(\sigma_2) - G(p)(\sigma_1)\| \\
& = \| \int_a^{\sigma_2} \psi_{(\sigma_2)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta \\
& - Q_{(\hbar)}^{(\sigma_2)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta + Q_{(\hbar)}^{(\sigma_2)} v_{\hbar} \\
& - \int_a^{\sigma_1} \psi_{(\sigma_1)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta \\
& + Q_{(\hbar)}^{(\sigma_1)} \int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} f(\eta, p(\eta), (Rp)(\eta)) d\eta - Q_{(\hbar)}^{(\sigma_1)} v_{\hbar} \|
\end{aligned}$$

$$\le L_f(1+r)(1+L_kT)\int_0^{\sigma_1}(\psi^{(\eta)}_{(\sigma_2)}-\psi^{(\eta)}_{(\sigma_1)})d\eta$$
$$+L_f(1+r)(1+L_kT)\int_{\sigma_1}^{\sigma_2}(\psi^{(\eta)}_{(\sigma_2)}d\eta$$
$$+L_f(1+r)(1+L_kT)(Q^{(\sigma_2)}_{(\hbar)}-Q^{(\sigma_1)}_{(\hbar)})\int_0^{\hbar}\psi^{(\eta)}_{(\hbar)}d\eta+\|v_\hbar\|(Q^{(\sigma_2)}_{(\hbar)}-Q^{(\sigma_1)}_{(\hbar)})$$
$$\le\frac{L_f(1+r)(1+L_kT)}{\Gamma(\alpha_1+1)}(\sigma_2^\alpha-\sigma_1^\alpha)+\frac{L_f(1+r)(1+L_kT)}{\Gamma(\alpha_1+1)}(\sigma_2-\sigma_1)(Q_b(\hbar))^{\alpha_1}$$
$$+\|v_\hbar\|(Q^{(\sigma_2)}_{(\hbar)}-Q^{(\sigma_1)}_{(\hbar)}).$$

Since $\sigma_2\to\sigma_1$ the aforementioned inequality's right hand side tends to 0.
Since p is an arbitrary in $\mathbb{M}_r$. We infer that $G(\mathbb{M}_r)$ is equicontinuous.
Step:4. The Monch condition holds.
Assuming $Z\subseteq\mathbb{M}_r$ is denumerable and $Z\subseteq conv(0\cup G(Z))$.
To prove that $\zeta(Z)=0$, where ζ is Hausdorff Measure of noncompactness.
Suppose that Z = $(p_k)_{k=1}^{\infty}$. To show that $\zeta(Z(\sigma))$ is relatively compact in X, $\forall\sigma\in\Im$,
Let $\Pi=f(\eta,(p_k(\eta))_{k=1}^{\infty},(R(p_k(\eta))_{k=1}^{\infty}))$.
By using lemma (2.15), (2.16), hypotheses (H_4) and Holder inequality, we have

$$\zeta(F(p_k(\eta))_{k=1}^{\infty}(\sigma))=\zeta(\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}(\Pi)d\eta-Q^{(\sigma)}_{(\hbar)}\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}(\Pi)d\eta+Q^{(\sigma)}_{(\hbar)}v_\hbar)$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}\zeta(\Pi)d\eta-2Q^{(\sigma)}_{(\hbar)}\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}\zeta(\Pi)d\eta$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+\zeta(R((p_k(\eta))_{k=1}^{\infty}))\right]d\eta$$
$$-2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+\zeta(R(p_k(\eta))_{k=1}^{\infty})\right]\right]d\eta$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+\zeta(\int_0^{\eta}k((\eta,\theta,(p_k(\theta))_{k=1}^{\infty})d\theta))\right]d\eta$$
$$-2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+\zeta(\int_0^{\eta}k(\eta,\theta,(p_k(\theta))_{k=1}^{\infty})d\theta)\right]d\eta\right]$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+2\int_0^{\eta}\zeta(k(\eta,\theta,(p_k(\theta))_{k=1}^{\infty})d\theta)\right]d\eta$$
$$-2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\psi^{(\eta)}_{(\eta)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+2\int_0^{\eta}\zeta(k(\eta,\theta,(p_k(\theta))_{k=1}^{\infty})d\theta)\right]d\eta\right]$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+2\int_0^{\eta}n_2(\eta)\zeta((p_k(\theta))_{k=1}^{\infty})d\theta)\right]d\eta$$
$$-2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}n_1(\eta)\left[\zeta((p_k(\eta))_{k=1}^{\infty})+2\int_0^{\eta}n_2(\eta)\zeta((p_k(\theta))_{k=1}^{\infty})d\theta\right]d\eta\right]$$
$$\le 2\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}n_1(\eta)(\zeta(Z)+2Tn_2(\eta)\zeta(Z))d\eta$$
$$-2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\psi^{(\eta)}_{(\hbar)}n_1(\eta)(\zeta(Z)+2Tn_2(\eta)\zeta(Z))d\eta\right]$$
$$\le 2\zeta(Z)\int_a^{\sigma}\psi^{(\eta)}_{(\sigma)}(n_1(\eta)+2Tn_1(\eta)n_2(\eta))d\eta$$

$$
\begin{aligned}
&- 2Q^{(\sigma)}_{(\hbar)}\zeta(Z)\left[\int_a^{\hbar} \psi^{(\eta)}_{(\hbar)}(n_1(\eta)+2Tn_1(\eta)n_2(\eta))d\eta\right] \\
&\leq \Big[2\int_a^{\sigma}\Big(\Big(\psi^{(\eta)}_{(\sigma)}\Big)^{\frac{1}{1-\alpha}}d\eta\Big)^{1-\alpha}\Big(\int_a^{\sigma}(n_1(\eta)+2Tn_1(\eta)n_2(\eta))^{\frac{1}{\alpha}}d\eta\Big)^{\alpha} \\
&- 2Q^{(\sigma)}_{(\hbar)}\left[\int_a^{\hbar}\Big(\Big(\psi^{(\eta)}_{(\hbar)}\Big)^{\frac{1}{1-\alpha}}d\eta\Big)^{1-\alpha}\Big(\int_a^{\hbar}(n_1(\eta)+2Tn_1(\eta)n_2(\eta))^{\frac{1}{\alpha}}d\eta\Big)^{\alpha}\right]\zeta(Z) \\
&\leq \Big[\frac{2N(\psi(\sigma)-\psi(a))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}}+\frac{2NQ^{(\sigma)}_{(\hbar)}(\psi(\hbar)-\psi(\eta))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}}\Big]\zeta(Z) \\
&\zeta(\phi(Z)) \leq \Big[\frac{2N(Q_b(\hbar))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}}+\frac{2NQ^{(\sigma)}_{(\hbar)}(Q_b(\hbar))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}}\Big]\zeta(Z) \\
&\zeta(\phi(Z)) \leq L\zeta(Z).
\end{aligned}
$$

Thus from Monch condition, we have

$$\zeta(Z) < \zeta(conv(0\cup(\mathbb{G}(Z)))) = \zeta(\mathbb{G}(Z)) \leq L\zeta(Z),$$

which implice that $\zeta(Z) = 0$. Hence Z is relatively compact.
Therefore, according to lemma (2.16), There exists a fixed point x of the operator $\mathbb{G}$ on M_r.
Which is a solution of the equation (1.1).

□

Next we will prove the result of Theorem 3.2 in the case there is not restricted hypothesis (H_5) via another Measure of noncompactness.

Let $\Re$ be the Measure of noncompactness in the Banach space $\mathbb{C}(\Im, X)$ determined by

$$\Re(\omega) = \max_{D\in\Delta(\omega)}(\aleph(D)mod_e(D), \tag{3.5}$$

for all bounded subsets ω of $C(\Im, X)$, where $\Delta(\omega)$ is the set of denumerable subsets of $\omega, \aleph$ is the real Measure of noncompactness define by

$$\aleph(D) = \sup_{L\in[0,b]} e^{-L\sigma}\zeta(D(\sigma)),$$

$D(\sigma) = (v(\sigma) : v \in D)$, $\sigma \in \Im$, L is a constant that we shall choose conveniently and $mod_c(D)$ is the modulus of equi continuity of the function set D given by the formula,

$$mod_c(D) = (\lim_{\delta\to 0})(\sup_{x\in D})(\max_{|\sigma_2-\sigma_1|\leq\delta})\|x(\sigma_2)-x(\sigma_1)\|,$$

that $\Re$ is properly defined (i.e., there is $D_0 \in \Delta(\omega)$ which achieves the maximum in equation (3.5) and is nonsingular, monotone and regular Measure of noncompactness.

Theorem 3.2. *Assume that $(H_1)-(H_4)$ holds, then the fractional BVP (1.1) has at least one solution on $\Im$.*

Proof. It is sufficient to prove that the function $\mathbb{G} : C(\Im, \mathbb{X}) \to C(\Im, \mathbb{X})$ definited by equation (3.2) satisfies the Mönch condition.
Let $Z \in \mathfrak{M}_r$ be denumerable set and $Z \subseteq conv((0)\cup G(Z))$, by the definition (2.12) Z is relatively compact. It sufficient to prove that $\Re(Z)$ =$(0, 0)$. Since $\Re(G(Z))$ is a maximum.
Let $(p_n)_{n=1}^{\infty} \subseteq G(Z)$ be the denumerable set which arrive its maximum.
Obviously, There exists a set $(x_n)_{n=1}^{\infty} \subseteq \mathbb{Z}$, such that $p_n = (Gx_n)(\sigma)$ for all $\sigma \in \Im$, $n \geq 1$.

Now we need to give the value for $\aleph((p_n)_{n=1}^{\infty})$,

$$\zeta((p_n)_{n=1}^{\infty}) = \zeta(((\mathbb{G}x_n)(\sigma))_{n=1}^{\infty})$$
$$\leq 2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} n_1(\eta)((\zeta(p_k(\eta))_{k=1}^{\infty}) + 2\int_0^{\eta} n_2(\eta)\zeta((p_k(\theta))_{k=1}^{\infty})d\theta)d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} n_1(\eta)((\zeta(p_k(\eta))_{k=1}^{\infty}) + 2\int_0^{\eta} n_2(\eta)\zeta((p_k(\theta))_{k=1}^{\infty})d\theta)d\eta\right]$$
$$\leq 2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} n_1(\eta)(\sup_{\sigma\in(a,\hbar)}(\zeta(p_k(\eta))_{k=1}^{\infty}) + 2Tn_2(\eta)\sup_{\sigma\in(a,\hbar)}\zeta(p_k(\sigma)_{k=1}^{\infty}))d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} n_1(\eta)(\sup_{\sigma\in(a,\hbar)}(\zeta(p_k(\eta))_{k=1}^{\infty}) + 2Tn_2(\eta)\sup_{\sigma\in(a,\hbar)}\zeta(p_k(\sigma)_{k=1}^{\infty}))d\eta\right]$$
$$\leq 2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)} e^{L\eta} n_1(\eta) * (\sup_{\sigma\in(a,T)} e^{-L\sigma}\zeta(p_k(\eta))_{k=1}^{\infty} + 2Tn_2(\eta)\sup_{\sigma\in(a,\hbar)} e^{-L\sigma}\zeta((p_k(\sigma)_{k=1}^{\infty}))d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)} e^{L\eta} n_1(\eta) * (\sup_{\sigma\in(a,T)} e^{-L\sigma}\zeta(p_k(\eta))_{k=1}^{\infty} + 2Tn_2(\eta)\sup_{\sigma\in(a,\hbar)} e^{-L\sigma}\zeta((p_k(\sigma)_{k=1}^{\infty})))d\eta\right]$$
$$\leq 2\beta(p_k(\eta))_{k=1}^{\infty}\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)}(e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta))d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\beta(p_k(\eta))_{k=1}^{\infty}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)}(e^{L\eta}n_1 + 2Te^{L\eta}n_1(\eta)n_2(eta))d\eta\right]$$
$$\leq (2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\Big[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta\Big])\beta((p_k(\eta))_{k=1}^{\infty})$$

Hence,

$$\aleph((p_k(\eta))_{k=1}^{\infty}) \leq \sup_{\sigma\in(a,\hbar)} e^{L\sigma}[2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta\right]\beta((p_k(\eta))_{k=1}^{\infty}).$$

set

$$L' = \sup_{\sigma\in(a,\hbar)} e^{L\sigma}[2\int_a^{\sigma} \psi_{(\sigma)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta$$
$$- 2Q_{(\hbar)}^{(\sigma)}\left[\int_a^{\hbar} \psi_{(\hbar)}^{(\eta)}[e^{L\eta}n_1(\eta) + 2Te^{L\eta}n_1(\eta)n_2(\eta)]d\eta\right].$$

Then, $\aleph(p_k(\eta))_{k=1}^{\infty} \leq L'\aleph(p_k(\eta))_{k=1}^{\infty}$. It is not difficult to choose a suitable constant $0 < L' < 1$
.
Therefore,

$$\aleph((p_k(\eta))_{k=1}^{\infty}) \leq \aleph(Z) \leq \aleph(conv(0 \cup G(Z))) = \aleph(((p_k(\eta))_{k=1}^{\infty}) \leq L'\aleph(p_k(\eta))_{k=1}^{\infty})$$

$\implies$ that $\aleph((p_k(\eta))_{k=1}^{\infty}) = 0$ and hence $\aleph((p_k(\eta))_{k=1}^{\infty}) = 0$,
by the theorem (3.1), the set $(p_k(\eta))_{k=1}^{\infty}$ is equicontinuous on $\Im$. Thus, by definition of $\Re$, we can see that $\Re(p_k(\eta))_{k=1}^{\infty} = (0, 0)$. Since $\Re$ is monotone, nonsingular and regular measures of noncompactness.

Hence,
$\Re(Z) \leq \Re(conv(0 \cup \mathbb{G}(Z))) \leq \Re(\mathbb{G}(Z))$, where $\Re(G(Z)) = \Re(p_k(\eta))_{k=1}^{\infty} = (0,0)$.
As a result, Z is relatively compact. This completes the proof. □

4 Example

Consider the followimg ψ-Caputo fractional integrodifferential equation,

$$\begin{aligned} {}^{C}D^{\frac{5}{2};\psi}_{a+} v(\sigma) &= \Big(\frac{e^{-\sigma}}{9+e^{\sigma}} \frac{|v(\sigma)|}{1+|v(\sigma)|} \int_0^{\sigma} (e^{\frac{-1}{4}})^{\sigma} d\sigma\Big), \sigma \in \Im; = [0,\hbar], \\ v(0) &= v^{'}(0) = 0, v(1) = \frac{1}{4}. \end{aligned} \tag{4.1}$$

Set

$$f(\sigma, v(\sigma), (Rv)(\sigma)) = f\Big(\frac{e^{-\sigma}}{9+e^{\sigma}}, \frac{|v(\sigma)|}{1+|v(\sigma)|}, \int_0^{\sigma} (e^{\frac{-1}{4}})^{\sigma} d\sigma\Big),$$

where $\alpha_1 = \frac{5}{2}$, $n_1(\sigma) = \frac{e^{-\sigma}}{9+e^{\sigma}}$, $n_2(\sigma) = e^{\frac{-1}{4}}$, $\psi(t) = t$, $T = 1$, $\hbar = 1$

for each bounded sets and $\sigma \in \Im$, we get

$$\begin{aligned} N &= (n_1(\sigma) + 2Tn_1(\sigma)n_2(\sigma)) \\ N &= 0.0804 \end{aligned}$$

Hence the hypotheses (H_1)-(H_4) is holds, and $L_f = \frac{6}{10}$, $L_k = \frac{3}{10}$, T=1, α_1= $\frac{5}{2}$, then,

$$\frac{2(Q_b(\hbar))^{\alpha_1}}{\Gamma(\alpha_1+1)} L_f(1+L_kT) \approx 0.4695 \leq 1.$$

which is the existence result for equation (4.1).
Hence the hypothesies $(H_1) - (H_5)$ holds, then

$$L = \frac{2N(Q_b(\hbar))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}} + \frac{2NQ^{(\sigma)}_{(\hbar)}(Q_b(\hbar))^{\alpha_1-\alpha}}{\Gamma(\alpha_1)(\frac{\alpha_1-\alpha}{1-\alpha})^{1-\alpha}} \approx 0.1614 < 1$$

Therefore the above condition fulfilled by hypothesis $(H_1) - (H_5)$ and theorem (3.2) and show that the problem (4.1) has atleast one solution.

5 Conclusions

By applying Monch's fixed-point theorem, along with the Hausdorff measure of non-compactness and Carathéodory's condition, we have demonstrated the existence of a solution to the ψ-Caputo fractional differential equations. We conclude the study by solving an example based on the theoretical findings. Future research could extend this work by employing other fractional derivatives, such as the ψ-Hilfer and Caputo-Hadamard derivatives, to further validate the results obtained.

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