

A NOTE ON WEAKLY n -SUBMODULES

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Communicated by: Nasreen Kausar

MSC 2020 Classifications: Primary 13C13, 13C99.

Keywords and phrases: r -submodule, n -ideal, weakly n -ideal, n -submodule, weakly n -submodule.

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Abstract. Let M be a nonzero unital module over a commutative ring R with a nonzero identity. A proper submodule N of M is said to be a weakly n -submodule if whenever $0 \neq am \in N$ for some $a \in R$ and $m \in M$, then $a \in \sqrt{\text{ann}(M)}$ or $m \in N$. We examine the relations between weakly n -submodules and classical submodules such as prime, weakly prime, weakly primary and r -submodules. Also, we characterize modules over which every nonzero submodule is secondary by using weakly n -submodules.

1 Introduction

In module theory, various generalizations of prime and primary submodules have been proposed to capture a wider range of algebraic behaviors under scalar multiplication. While classical definitions such as prime and primary submodules offer strong structural insights, they may prove overly rigid in the presence of zero divisors or complicated annihilator structures. To overcome these limitations, alternative notions such as weakly prime, weakly primary, and r -submodules have been introduced. Moreover, generalizations like n -absorbing primary submodules and almost n -absorbing submodules have been proposed to capture a broader range of containment behavior in modules [1], [2].

Throughout this paper, let R denote a commutative ring with identity $1 \neq 0$, and let M be a nonzero unital R -module. A proper submodule P of M is called a prime submodule if $rm \in P$ implies $r \in (P : M)$ or $m \in P$ [3]. This notion was weakened by Atani in [4], who introduced the concept of a weakly prime submodule, where $0 \neq rm \in N$ implies $m \in N$ or $r \in (N : M)$. This modification enables the study of a broader class of submodules, particularly when modules contain zero divisors. The idea was further extended in [5] to define weakly primary submodules, where $0 \neq rm \in N$ implies $m \in N$ or $r \in \sqrt{(N : M)}$, allowing the scalar to act nilpotently modulo the submodule.

In a related direction, Koç and Tekir [6] introduced the concept of r -submodules, inspired by the idea of r -ideals in ring theory, with the aim of generalizing containment through scalar action. A proper submodule N of M is said to be an r -submodule of M if $am \in N$ with $\text{ann}_M(a) = 0$, then $m \in N$, where $\text{ann}_M(a) = \{m \in M : am = 0\}$. These extensions reveal the need to interpolate between weak and power-based submodule conditions. While such developments have broadened the theoretical landscape, a conceptual gap remains between the weakly defined and power-based notions of submodules.

To address this gap, Tekir et al. [7] introduced the concept of n -ideals, which generalize weakly prime ideals. A proper ideal I of R is an n -ideal if whenever $ab \in I$ with $a \notin \sqrt{0}$, then $b \in I$ for every $a, b \in R$. Building upon this, Ahmadi and Moghaderi [8] defined n -submodules where a proper submodule N of M satisfies: if $rm \in N$, then $r \in \sqrt{\text{ann}(M)}$ or $m \in N$, transferring the idea into module-theoretic language by incorporating the radical of the annihilator. Further generalizations, such as $(n - 1, n)$ - ϕ -second submodules, have been studied in [9], which explores the structural depth of submodule containment.

In this paper, we introduce and investigate the notion of weakly n -submodules, a new class

that bridges the gap between weakly prime submodules and n -submodules. A proper submodule N of M is said to be a weakly n -submodule of M if whenever $0 \neq am \in N$ for some $a \in R$ and $m \in M$, then $a \in \sqrt{\text{ann}(M)}$ or $m \in N$. We analyze their fundamental properties and relationships with classical concepts such as prime, weakly prime, weakly primary, r -submodules, and secondary submodules. We also provide structural conditions under which weakly n -submodules coincide with previously known classes. Our findings demonstrate that this new class provides a rich framework for studying submodule behavior that extends beyond the scope of existing notions. Furthermore, we explore the behavior of weakly n -submodules in the context of multiplication modules, focusing specifically on their preservation under submodules and factor modules. Our results show that weakly n -submodules not only generalize multiple existing concepts but also provide new structural insights that are not apparent from previously established definitions.

2 Characterization of Weakly n -submodules

Definition 2.1. Let M be an R -module and N be a proper submodule of M . N is said to be a weakly n -submodule of M if whenever $0 \neq am \in N$ for some $a \in R$ and $m \in M$, then $a \in \sqrt{\text{ann}(M)}$ or $m \in N$.

Example 2.2. (i) Every n -submodule is also a weakly n -submodule. However, the converse is not always true. For example, (0) is a weakly n -submodule of \mathbb{Z} -module \mathbb{Z}_6 , which is not an n -submodule.

(ii) Suppose that R is a local ring with a unique maximal ideal $\sqrt{\text{ann}(M)}$. Then every proper submodule is a weakly n -submodule. To see this, choose $x \in R$ and $m \in M$ such that $0 \neq xm \in N$. If x is a unit, then $m \in N$. If x is not a unit, then $x \in \sqrt{\text{ann}(M)}$, as R has a unique maximal ideal $\sqrt{\text{ann}(M)}$. Thus, N is a weakly n -submodule of M .

Lemma 2.3. Let N be a weakly n -submodule of M . Then $(N : M) \subseteq \sqrt{\text{ann}(M)}$. In this case $\sqrt{(N : M)} = \sqrt{\text{ann}(M)}$.

Proof. Let $x \in (N : M) - \sqrt{\text{ann}(M)}$. Then we have $xM \subseteq N$. Now, choose $m \in M$ such that $xm \neq 0$. Since $0 \neq xm \in N$ and $x \notin \sqrt{\text{ann}(M)}$, we get $m \in N$. This gives $M - \text{ann}_M(x) \subseteq N$, and so we have $M = \text{ann}_M(x) \cup N$. By [3], we have $M = \text{ann}_M(x)$ or $M = N$. Since N is proper, we get $M = \text{ann}_M(x)$, that is, $x \in \text{ann}(M) \subseteq \sqrt{\text{ann}(M)}$, which is a contradiction. Therefore, $(N : M) \subseteq \sqrt{\text{ann}(M)}$. The rest is clear. \square

For a submodule N of M and a nonempty set J of R , the residual of N by J is defined as $(N :_M J) = \{m \in M : Jm \subseteq N\}$ [10]. In particular If $J = \{a\}$ for some $a \in R$, we use $(N :_M a)$ instead of $(N :_M \{a\})$.

Proposition 2.4. Let M be an R -module and N be a proper submodule of M . The following statements are equivalent:

- (i) N is a weakly n -submodule of M .
- (ii) For every $a \notin \sqrt{\text{ann}(M)}$, we have $(N :_M a) = N \cup \text{ann}_M(a)$.
- (iii) For every $a \notin \sqrt{\text{ann}(M)}$, we have $(N :_M a) = N$ or $(N :_M a) = \text{ann}_M(a)$.
- (iv) For every ideal I of R and submodule K of M , if $0 \neq IK \subseteq N$, then either $I \subseteq \sqrt{\text{ann}(M)}$ or $K \subseteq N$.

Proof. (i) \Rightarrow (ii) : Let $x \in (N :_M a)$, that is, $ax \in N$. If $ax = 0$, then $x \in \text{ann}_M(a)$. So assume that $ax \neq 0$. Since N is a weakly n -submodule of M and $a \notin \sqrt{\text{ann}(M)}$, we get $x \in N$. This implies that $(N :_M a) \subseteq N \cup \text{ann}_M(a)$. Since the reverse inclusion is always true, we have the equality $(N :_M a) = N \cup \text{ann}_M(a)$.

(ii) \Rightarrow (iii) : Follows from the fact that if a submodule is a union of two submodules, then it must be equal one of them.

(iii) \Rightarrow (iv) : Let $0 \neq IK \subseteq N$ and $I \not\subseteq \sqrt{\text{ann}(M)}$ for some ideal I of R and some submodule K of M . Then there exist $a \in I - \sqrt{\text{ann}(M)}$ and $aK \subseteq N$. This implies that $K \subseteq (N :_M a)$. Then by assumption $N = (N :_M a)$ or $(N :_M a) = \text{ann}_M(a)$. If $aK \neq 0$,

then we have $K \subseteq (N :_M a) = N$ which completes the proof. So assume that $aK = 0$. Since $0 \neq IK$, there exists $x \in I$ such that $xK \neq 0$. Since $xK \subseteq N$, we have $x \in \sqrt{\text{ann}(M)}$ or by (iii), $K \subseteq (N :_M x) = N$. So we may assume that $x \in \sqrt{\text{ann}(M)}$. This implies that $a + x \notin \sqrt{\text{ann}(M)}$. Then we have $0 \neq (x + a)K = xK \subseteq N$, by assumption $K \subseteq (N :_M x + a) = N$ which completes the proof.

(iv) \Rightarrow (i) : Let $0 \neq ax \in N$ for some $a \in R$ and $x \in M$ with $a \notin \sqrt{\text{ann}(M)}$. Define $I = (a)$ and $K = Rx$. Note that $I \not\subseteq \sqrt{\text{ann}(M)}$ and $0 \neq IK \subseteq N$. By (iv), we get $K \subseteq N$, so $x \in N$. \square

Proposition 2.5. *The intersection of any family of weakly n -submodules is also a weakly n -submodule.*

Proof. Let $\{N_i\}_{i \in I}$ be a nonempty set of weakly n -submodules of R -module M and $N = \bigcap_{i \in I} N_i$. Suppose $a \in R$ and $x \in M$ with $a \notin \sqrt{\text{ann}(M)}$ and $0 \neq ax \in N$. Then for each $i \in I$, we have $0 \neq ax \in N_i$. Since N_i is a weakly n -submodule, it follows that $x \in N_i$ for each i , so $x \in \bigcap_{i \in I} N_i = N$. \square

Proposition 2.6. *Suppose that I is an ideal of R such that $I \not\subseteq \sqrt{\text{ann}(M)}$. The following statements are satisfied:*

- (i) *If K_1, K_2 are weakly n -submodules of M with $0 \neq IK_1 = IK_2$, then $K_1 = K_2$.*
- (ii) *If $0 \neq IK$ is a weakly n -submodule of M , then $IK = K$.*

Proof. (i) Since K_1 is a weakly n -submodule of M and $0 \neq IK_2 \subseteq K_1$ by Proposition 2.4, we get $K_2 \subseteq K_1$. Similarly, we can deduce that $K_1 \subseteq K_2$, which implies $K_1 = K_2$.

(ii) Since IK is a weakly n -submodule of M and $0 \neq IK \subseteq IK$ by Proposition 2.4, we get $K \subseteq IK \subseteq K$, which completes the proof. \square

Proposition 2.7. *Let M be an R -module such that $\sqrt{\text{ann}(M)}$ is a maximal ideal of R . Then every proper submodule of M is a weakly n -submodule of M .*

Proof. Let $0 \neq xm \in N$ where $x \notin \sqrt{\text{ann}(M)}$. Since $\sqrt{\text{ann}(M)}$ is a maximal ideal of R , there exists $a \in R$ such that $1 - xa \in \sqrt{\text{ann}(M)}$. Then there exists $n \in \mathbb{N}$ such that $(1 - xa)^n = 1 - xr \in \text{ann}(M)$ for some $r \in R$. This gives $(1 - xr)m = 0$ which implies that $m = rxm \in N$. Thus, N is a weakly n -submodule of M . \square

Lemma 2.8. *Let M be a multiplication module such that $\text{ann}(M)$ is a semiprime ideal. Then, zero submodule is the only weakly n -submodule of M .*

Proof. Let N be a weakly n -submodule of M , then by the previous lemma, $(N : M) \subseteq \sqrt{\text{ann}(M)} = \text{ann}(M)$. Since M is a multiplication module, we get $N = (N : M)M \subseteq \text{ann}(M)M = 0$, that is, $N = 0$. \square

Corollary 2.9. *Let M be a faithful multiplication module over a reduced ring R . Then zero submodule is the only weakly n -submodule of M .*

Proof. Since M is faithful and R is a reduced ring, $\sqrt{\text{ann}(M)} = \sqrt{0} = 0 = \text{ann}(M)$. The rest follows from the previous lemma. \square

Let I be an ideal of R , then we say that I is a regular ideal with respect to M if $I \not\subseteq Z_R(M)$, where $Z_R(M) = \{a \in R : \text{ann}_M(a) \neq 0\}$, or equivalently, there exists $a \in I$ such that $\text{ann}_M(a) = 0$.

Proposition 2.10. *Let M be an R -module and I be a regular ideal with respect to M . If N is a weakly n -submodule of M such that $I \not\subseteq (N : M)$, then $(N :_M I)$ is a weakly n -submodule of M .*

Proof. Assume that $0 \neq rm \in (N :_M I)$. Then $r \text{Im} \subseteq N$. Because I is a regular ideal with respect to M , $r \text{Im} \neq 0$. Suppose that $r \notin \sqrt{\text{ann}(M)}$. Since N is a weakly n -submodule of M and $0 \neq r(\text{Im}) \subseteq N$, by Proposition 2.4, it follows that $\text{Im} \subseteq N$. Thus $m \in (n :_M I)$. \square

Proposition 2.11. *Let N be a proper submodule of M . Then N is a weakly n -submodule of M if and only if for every $m \in M$, either $(N : m) = R$ or $(N : m) \subseteq \sqrt{\text{ann}(M)}$ or $(N : m) \subseteq \text{ann}(m)$.*

Proof. (\Rightarrow) : Let N be a weakly n -submodule of M and $m \in M$. If $m \in N$, then clearly $(N : m) = R$. Suppose that $m \notin N$ and let $r \in (N : m)$, so $rm \in N$. If $rm = 0$, then $r \in \text{ann}(m)$. If $rm \neq 0$, since N is a weakly n -submodule of M , we have $r \in \sqrt{\text{ann}(M)}$. Therefore $(N : m) \subseteq \text{ann}(m) \cup \sqrt{\text{ann}(M)}$, which means $(N : m) \subseteq \sqrt{\text{ann}(M)}$ or $(N : m) \subseteq \text{ann}(m)$.

(\Leftarrow) : Clear. \square

Definition 2.12. Let N be a weakly n -submodule of M . If there is no weakly n -submodule of M containing N , then we say that N is a maximal weakly n -submodule of M .

Theorem 2.13. *Let N be a maximal weakly n -submodule of M such that $Z_R(M) \subseteq (N : M)$. Then N is a prime submodule of M .*

Proof. Let $am \in N$ with $a \notin (N : M)$. Since $Z_R(M) \subseteq (N : M)$, we get $\text{ann}_M(a) = 0$. If $am = 0$, then $m = 0 \in N$. So assume that $am \neq 0$. Since N is a weakly n -submodule of M and (a) is a regular ideal with respect to M , by Proposition 2.10, we get $(N :_M a)$ is also a weakly n -submodule of M . By the maximality of N , we have $m \in (N :_M a) = N$. \square

Recall from [6] that a proper submodule N of M is said to be an r -submodule of M if $am \in N$ with $\text{ann}_M(a) = 0$, then $m \in N$. With this definition in mind, we now show a further relationship regarding weakly n -submodules by presenting the following proposition.

Proposition 2.14. *Every maximal weakly n -submodule is also an r -submodule.*

Proof. Suppose that N is a maximal weakly n -submodule of M . Let $am \in N$ with $\text{ann}_M(a) = 0$. Then note that (a) is a regular ideal with respect to M , by Proposition 2.10 and the maximality of N , we have $m \in (N :_M a) = N$. \square

Remark 2.15. Ahmadi and Moghaderi showed that every n -submodule is an r -submodule in [8, Proposition 2.21]. It is important to note, however, that a maximal weakly n -submodule need not be an n -submodule. For instance, \mathbb{Z} -module \mathbb{Z}_6 has only weakly n -submodule which is zero. However \mathbb{Z} -module \mathbb{Z}_6 has no n -submodule.

Proposition 2.16. *Let N be a prime submodule of M . The following statements are equivalent.*

- (i) N is a weakly n -submodule of M .
- (ii) N is an n -submodule of M .
- (iii) $(N : M) = \sqrt{\text{ann}(M)}$.

Proof. $(ii) \Leftrightarrow (iii)$: Follows from [8, Proposition 2.13].

$(ii) \Rightarrow (i)$: Clear.

$(i) \Rightarrow (iii)$: Suppose that N is a weakly n -submodule of M . Then by Lemma 2.3, $(N : M) \subseteq \sqrt{\text{ann}(M)}$. Since N is a prime submodule, $(N : M)$ is a prime ideal. This gives $\sqrt{\text{ann}(M)} \subseteq \sqrt{(N : M)} = (N : M)$. Then we obtain $(N : M) = \sqrt{\text{ann}(M)}$. \square

Recall from [4] that a proper submodule N of an R -module M is said to be a weakly prime submodule if $0 \neq rm \in N$ implies $m \in N$ or $rM \subseteq N$ for every $r \in R$ and $m \in M$.

For a submodule N of M , if there exists a prime submodule P of M containing N , then $\text{rad}(N)$ is defined to be the intersection of all prime submodules of M containing N . If there is no such prime submodule, we accept that $\text{rad}(N) = M$. In particular, $\text{rad}(0)$ is the intersection of all prime submodules of M [11].

Proposition 2.17. *Let M be a finitely generated R -module. Then $\text{rad}(0)$ is a weakly n -submodule of M if and only if $\text{rad}(0)$ is a weakly prime submodule of M .*

Proof. By [11, Theorem 4.4] we know that $(rad(0) : M) = \sqrt{ann(M)}$ since M is a finitely generated module. Let $rad(0)$ be a weakly n -submodule of M and $0 \neq am \in rad(0)$ for some $a \in R$ and $m \in M$. We will show that $a \in (rad(0) : M)$ or $m \in rad(0)$. First, assume that $a \notin (rad(0) : M) = \sqrt{ann(M)}$. Since $rad(0)$ is a weakly n -submodule of M , it follows that $m \in rad(0)$. This demonstrates that $rad(0)$ is a weakly prime submodule. Conversely, assume that $rad(0)$ is a weakly prime submodule. Let $0 \neq am \in rad(0)$ with $a \notin \sqrt{ann(M)} = (rad(0) : M)$. Since $rad(0)$ is a weakly prime submodule, we get $m \in rad(0)$, which completes the proof. \square

Proposition 2.18. *Let N be a proper submodule of M with $(N : M) = \sqrt{ann(M)}$. Then N is a weakly n -submodule of M if and only if N is a weakly prime submodule of M .*

Proof. This statement follows from applying a similar argument to the proof of the previous Proposition, ensuring that each logical step is analogous. \square

Lemma 2.19. *Suppose that M is a faithful module and N is a weakly n -submodule of M . Then $(N : M)$ is a weakly n -ideal of R .*

Proof. Let $0 \neq ab \in (N : M)$ with $a \notin \sqrt{0}$. Since M is faithful, we have $0 \neq abM \subseteq N$ and $a \notin \sqrt{ann(M)}$. By Proposition 2.4, it then follows that $bM \subseteq N$, that is, $b \in (N : M)$. \square

Example 2.20. To illustrate the necessity of the faithfulness condition in the previous Lemma, consider the \mathbb{Z} -module \mathbb{Z}_{p^n} , where p is a prime number and $n \geq 2$. Note that $\sqrt{ann(\mathbb{Z}_{p^n})} = p\mathbb{Z}$ is a maximal ideal of \mathbb{Z} . Furthermore, by Proposition 2.7, all submodules of \mathbb{Z}_{p^n} are weakly n -submodules. In particular (\bar{p}) is a weakly n -submodule. However \mathbb{Z}_{p^n} is not a faithful module and $((\bar{p}) : \mathbb{Z}_{p^n}) = p^{n-1}\mathbb{Z}$ is not a weakly n -ideal of \mathbb{Z} .

Recall from [12] that an R -module M is said to be a multiplication module if for every submodule N of M , there exists an ideal I of R such that $N = IM$. In this case $N = (N : M)M$ see, [12]. For more details about multiplication modules, the reader may consult [13] and [14].

Proposition 2.21. *Let M be a multiplication module and N be a submodule of M such that $(N : M)$ is a weakly n -ideal of R . Then N is a weakly n -submodule of M .*

Proof. Let $0 \neq IK \subseteq N$ for some ideal I of R and some submodule K of M . Suppose that $I \not\subseteq \sqrt{ann(M)}$. Since M is a multiplication module, $K = (K : M)M$ and so $0 \neq I(K : M) \subseteq (N : M)$. Since $(N : M)$ is a weakly n -ideal of R , we have $(K : M) \subseteq (N : M)$ by Proposition 2.4. Then we get $K = (K : M)M \subseteq (N : M)M = N$. \square

Corollary 2.22. *Let M be a faithful multiplication module and N be a submodule of M . Then N is a weakly n -submodule of M if and only if $(N : M)$ is a weakly n -ideal of R .*

Recall from [5] that a proper submodule N of M is said to be a weakly primary submodule if $0 \neq rm \in N$ for some $r \in R$, $m \in M$ implies either $m \in N$ or $r^n m \subseteq N$ for some positive integer n .

Theorem 2.23. *Let N be a submodule of M such that $(N : M) \subseteq \sqrt{ann(M)}$. Then the followings are equivalent.*

- (i) N is a weakly n -submodule of M .
- (ii) N is a weakly primary submodule of M .

Proof. (i) \Rightarrow (ii) : Let $0 \neq am \in N$ for some $a \in R$ and $m \in M$. Suppose that $a \notin \sqrt{(N : M)}$. Then, by assumption $a \notin \sqrt{ann(M)}$. Since N is a weakly n -submodule of M , we get $m \in N$.

(ii) \Rightarrow (i) : Let $0 \neq am \in N$ with $a \notin \sqrt{ann(M)}$. Then, by assumption, we know that $\sqrt{(N : M)} = \sqrt{ann(M)}$. Since N is a weakly primary submodule of M , it follows that $m \in N$, which completes the proof. \square

Proposition 2.24. *Let $f : M \rightarrow L$ be a homomorphism of R -modules.*

(i) If f is surjective and N is a weakly n -submodule of M containing $\text{Ker}(f)$, then $f(N)$ is a weakly n -submodule of L .

(ii) If f is monic and K is a weakly n -submodule of L , then $f^{-1}(K) = M$ or $f^{-1}(K)$ is a weakly n -submodule of M .

Proof. (i) : Let $0 \neq ax \in f(N)$ for some $a \in R$ and $x \in L$. Since f is surjective, we can write $f(m) = x$, and so $0 \neq af(m) = f(am) \in f(N)$. As N contains $\text{Ker}(f)$, we have $0 \neq am \in N$. Since N is a weakly n -submodule of M , we get $a \in \sqrt{\text{ann}(M)}$ or $m \in N$. Furthermore, as f is surjective, we conclude that $\sqrt{\text{ann}(M)} \subseteq \sqrt{\text{ann}(L)}$ which implies that $a \in \sqrt{\text{ann}(L)}$ or $x = f(m) \in f(N)$. Thus, $f(N)$ is a weakly n -submodule of L .

(ii) : Assume that $f^{-1}(K) \neq M$ for some weakly n -submodule K of L . Let $0 \neq am \in f^{-1}(K)$ for some $a \in R$ and $m \in M$. Since f is monic, we get $0 \neq f(am) = af(m) \in K$. As K is a weakly n -submodule of L , we have $a \in \sqrt{\text{ann}(L)}$ or $f(m) \in K$. As f is monic, we get $\sqrt{\text{ann}(L)} \subseteq \sqrt{\text{ann}(M)}$ which implies that $a \in \sqrt{\text{ann}(M)}$ or $m \in f^{-1}(K)$. Thus, $f^{-1}(K)$ is a weakly n -submodule of M . \square

Corollary 2.25. Let M be an R -module and L, N be submodules of M .

(i) If N is a weakly n -submodule of M with $L \subseteq N$, then N/L is a weakly n -submodule of M/L .

(ii) If L is a weakly n -submodule of M with $L \subseteq N$ and N/L is a weakly n -submodule of M/L , then N is a weakly n -submodule of M .

(iii) If N is a weakly n -submodule of M with $L \not\subseteq N$, then $N \cap L$ is a weakly n -submodule of L .

Proof. (i) : Consider the natural homomorphism $\pi : M \rightarrow M/L$ defined by $\pi(m) = m + L$ for each $m \in M$. Since π is surjective and N is a weakly n -submodule of M with $L \subseteq N$, by the previous proposition, $\pi(N) = N/L$ is a weakly n -submodule of M/L .

(ii) : Suppose that $0 \neq am \in N$ for some $a \in R$ and $m \in M$. If $0 \neq am \in L$, then we have $a \in \sqrt{\text{ann}(M)}$ or $m \in L \subseteq N$. Now, assume that $am \notin L$. Then we have $0_{M/L} \neq a(m + L) \in N/L$. Since N/L is a weakly n -submodule, we conclude that $a \in \sqrt{\text{ann}(M/L)} = \sqrt{(L : M)}$ or $m + L \in N/L$. As L is a weakly n -submodule of M , we have $(L : M) \subseteq \sqrt{\text{ann}(M)}$, which implies that $a \in \sqrt{\text{ann}(M)}$ or $m \in N$. Thus, N is a weakly n -submodule of M .

(iii) : Consider the injection $i : L \rightarrow M$ defined by $i(m) = m$ for every $m \in L$. Since i is monic and N is a weakly n -submodule of M with $L \not\subseteq N$, by the previous proposition, we have $i^{-1}(N) = N \cap L$ is a weakly n -submodule of L . \square

Let M be an R -module and $S \subseteq R$ be a multiplicatively closed set. Then, $S^{-1}M = \{\frac{m}{s} : m \in M \text{ and } s \in S\}$ is an $S^{-1}R$ -module which is called the quotient module over the quotient ring $S^{-1}R$. If N is a submodule of M , then $S^{-1}N$ is a submodule of $S^{-1}M$ [10]. For any submodule N of M (any ideal I of R), we denote the set $\{a \in R : am \in N \text{ for some } m \in M - N\}$ ($\{a \in R : ab \in I \text{ for some } b \in R - I\}$) by $Zd_R(N)$ ($Zd_R(I)$). Note that $Zd_R((0_M))$ is equal to $Z_R(M)$.

Theorem 2.26. Let M be an R -module, $S \subseteq R$ be a multiplicatively closed set, and N be a submodule of M .

(i) If N is a weakly n -submodule of M , then $S^{-1}N = S^{-1}M$ or $S^{-1}N$ is a weakly n -submodule of $S^{-1}M$.

(ii) If $S^{-1}N$ is a weakly n -submodule of $S^{-1}M$, M is finitely generated R -module and $S \cap Zd_R(M) = \emptyset = S \cap Zd_R(N) = S \cap Zd_R(\sqrt{\text{ann}(M)})$, then N is a weakly n -submodule of M .

Proof. (i) : Suppose that $S^{-1}N \neq S^{-1}M$ and $0 \neq \frac{a}{s} \frac{m}{t} \in S^{-1}N$. Then there exists $u \in S$ such that $0 \neq uam = a(um) \in N$. As N is a weakly n -submodule of M , we conclude that $a \in \sqrt{\text{ann}(M)}$ or $um \in N$. This implies that $\frac{a}{s} \in \sqrt{\text{ann}(S^{-1}M)}$ or $\frac{m}{t} = \frac{um}{ut} \in S^{-1}N$. Thus, $S^{-1}N$ is a weakly n -submodule of $S^{-1}M$.

(ii) : Suppose that M is finitely generated R -module and $S \cap Zd_R(M) = \emptyset$. Let $S^{-1}N$ be a weakly n -submodule of $S^{-1}M$. Choose $a \in R$ and $m \in M$ with $0 \neq am \in N$. Since $S \cap Zd_R(M) = \emptyset$, we have $0 \neq \frac{a}{1} \frac{m}{1} \in S^{-1}N$. As M is a finitely generated and $S^{-1}N$ is a weakly

n -submodule of $S^{-1}M$, we have $\frac{a}{1} \in \sqrt{\text{ann}(S^{-1}M)} = \sqrt{S^{-1}(\text{ann}(M))} = S^{-1}(\sqrt{\text{ann}(M)})$ or $\frac{m}{1} \in S^{-1}N$. Then there exists $u \in S$ such that $ua \in \sqrt{\text{ann}(M)}$ or $um \in N$. Since $S \cap Z_{d_R}(N) = \emptyset = S \cap Z_{d_R}(\sqrt{\text{ann}(M)})$, we conclude that $a \in \sqrt{\text{ann}(M)}$ or $m \in N$, which completes the proof. \square

Let M_i be an R_i -module for each $i = 1, 2$, $M = M_1 \times M_2$, and $R = R_1 \times R_2$. Then note that M is an R -module with componentwise addition and scalar multiplication, and all submodules of M have the form $N = N_1 \times N_2$ where N_i is a submodule of M_i for each $i = 1, 2$.

Proposition 2.27. *Let M_i be an R_i -module for each $i = 1, 2$, $M = M_1 \times M_2$, and $R = R_1 \times R_2$. Suppose that $N = N_1 \times N_2$ for some submodule N_i of M_i , where $i = 1, 2$. Then N is a weakly n -submodule of M if and only if $N = \{(0, 0)\}$. Thus, M has no nonzero weakly n -submodule.*

Proof. (\Rightarrow) : Suppose that N is a weakly n -submodule of M . First we show that $N_1 = 0$ or $N_2 = 0$. Assume both N_1 and N_2 are nonzero submodules. Since N is proper, without loss of generality, assume $N_1 \neq M_1$. Choose $m_1 \in M_1 - N_1$ and $0 \neq m_2 \in N_2$. Observe that $(0, 0) \neq (0, 1)(m_1, m_2) = (0, m_2) \in N$. As N is a weakly n -submodule of M , $(0, 1) \in \sqrt{\text{ann}(M)} = \sqrt{\text{ann}(M_1)} \times \sqrt{\text{ann}(M_2)}$ or $(m_1, m_2) \in N$. The second option is impossible since $m_1 \notin N_1$. Therefore, $(0, 1) \in \sqrt{\text{ann}(M_1)} \times \sqrt{\text{ann}(M_2)}$ which implies $M_2 = 0$, a contradiction. Hence, $N_1 = 0$ or $N_2 = 0$. Without loss of generality, assume $N_1 = 0$. If $N_2 \neq 0$, choose $0 \neq m_2 \in N_2$ and $0 \neq m_1 \in M_1$. As before, $(0, 0) \neq (0, 1)(m_1, m_2) = (0, m_2) \in N$, so $(0, 1) \in \sqrt{\text{ann}(M_1)} \times \sqrt{\text{ann}(M_2)}$ or $(m_1, m_2) \in N$, both yielding contradictions. Thus, $N_2 = 0$.

(\Leftarrow) : Clear. \square

Let M be an R -module. The idealization $R \times M = R \oplus M$ is a commutative ring with identity $(1, 0)$ under componentwise addition and the following multiplication $(a, m)(b, n) = (ab, an + bm)$ for each $a, b \in R$ and $m, n \in M$. If I is an ideal of R and N is a submodule of M , then $I \times N$ is an ideal of $R \times M$ if and only if $IM \subseteq N$ [15] and [16].

Theorem 2.28. *Let M be an R -module, I be an ideal of R , and N be a submodule of M with $IM \subseteq N$.*

(i) *Suppose that $N = M$. Then $I \times N$ is a weakly n -ideal of $R \times M$ if and only if I is a weakly n -ideal of R and whenever $ab = 0$ for some $a, b \in R$ with $a \notin \sqrt{0}$ and $b \notin I$, then $a, b \in \text{ann}(M)$.*

(ii) *Suppose that $N \neq M$. If $I \times N$ is a weakly n -ideal of $R \times M$, then I is a weakly n -ideal of R and N is a weakly n -submodule of M and $ab = 0$ for some $a \notin \sqrt{0}$ and $b \notin I$ imply that $a, b \in \text{ann}(N)$.*

(iii) *Suppose that $N \neq M$ and $\sqrt{\text{ann}(M)} = \sqrt{0}$. If I is a weakly n -ideal of R , N is an n -submodule of M and $ab = 0$ for some $a \notin \sqrt{0}$ and $b \notin I$ imply that $a, b \in \text{ann}(M)$, then $I \times N$ is a weakly n -ideal of $R \times M$.*

Proof. (i) : Follows from [17, Theorem 7].

(ii) : Suppose that $N \neq M$ and $I \times N$ is a weakly n -ideal of $R \times M$. First, we demonstrate that I is a weakly n -ideal of R . Let $0 \neq ab \in I$ for some $a, b \in R$. Then $(0, 0) \neq (a, 0)(b, 0) = (ab, 0) \in I \times N$. Since $I \times N$ is a weakly n -ideal of $R \times M$, it follows that $(a, 0) \in \sqrt{0_{R \times M}} = \sqrt{0} \times M$ or $(b, 0) \in I \times N$ which implies that $a \in \sqrt{0}$ or $b \in I$. Thus, I is a weakly n -ideal of R . Let $0 \neq am \in N$ with $a \notin \sqrt{\text{ann}(M)}$. Then we have $(a, 0) \notin \sqrt{0_{R \times M}}$ and $(0, 0) \neq (a, 0)(0, m) = (0, am) \in I \times N$. This implies that $(0, m) \in I \times N$, that is, $m \in N$. Therefore, N is a weakly n -submodule of M . Now, let $ab = 0$ for some $a, b \in R$ with $a \notin \sqrt{0}$ and $b \notin I$. We will prove that $a, b \in \text{ann}(N)$. Assume that $a \notin \text{ann}(N)$, that is, $an \neq 0$ for some $n \in N$. Thus, $(0, 0) \neq (a, 0)(b, n) = (0, an) \in I \times N$. Since $(a, 0) \notin \sqrt{0_{R \times M}}$ and $(b, n) \notin I \times N$, it follows that $I \times N$ is not a weakly n -ideal of $R \times M$ which is a contradiction. Thus, we have $a \in \text{ann}(N)$. Similarly, it can be shown that $b \in \text{ann}(N)$.

(iii) : Let $(0, 0) \neq (a, m)(b, n) \in I \times N$ for some $a, b \in R$ and $m, n \in M$. Then we have $ab \in I$ and $an + bm \in N$. We have two cases. **Case 1:** Let $0 \neq ab \in I$. In this case, we have either $a \in \sqrt{0}$ or $b \in I$. If $a \in \sqrt{0}$, then we get $(a, m) \in \sqrt{0_{R \times M}}$, which completes the proof. So assume that $a \notin \sqrt{0}$, that is, $b \in I$. This implies that $bm \in IM \subseteq N$, and

so we obtain $an = (an + bm) - bm \in N$. As N is an n -submodule of M , $an \in N$ and $a \notin \sqrt{0} = \sqrt{\text{ann}(M)}$, we conclude that $n \in N$. This gives $(b, n) \in I \times N$, which completes the proof. **Case 2:** Let $ab = 0$. If $a \notin \sqrt{0}$ and $b \notin I$, then by the assumption, we get $a, b \in \text{ann}(M)$, which implies that $an + bm = 0$ and so $(a, m)(b, n) = (0, 0)$. This is a contradiction. So we have $a \in \sqrt{0}$ or $b \in I$. By above argument, one can show that $I \times N$ is a weakly n -ideal of $R \times M$. \square

Recall from [18] that a nonzero submodule N of M is said to be a secondary if the homothety $N \xrightarrow{a} N$ is either surjective or nilpotent, or equivalently, $aN = N$ for every $a \notin \sqrt{\text{ann}(N)}$. Now, we characterize modules over which every proper submodule is a weakly n -submodule.

Theorem 2.29. Let M be an R -module. The following statements are satisfied.

(i) If every proper submodule of M is a weakly n -submodule, then every nonzero submodule is secondary.

(ii) Every proper submodule N of M is a weakly n -submodule of M if and only if $aN = 0$ or $aN = N$ for every $a \notin \sqrt{\text{ann}(M)}$ and nonzero submodule N of M .

Proof. (i) : Suppose that every proper submodule of M is a weakly n -submodule. Let N be a nonzero submodule of M . Assume $a \notin \sqrt{\text{ann}(N)}$. Then we have $a \notin \sqrt{\text{ann}(M)}$. If $aN = M$, then $N \subseteq M = aN \subseteq N$, that is, $aN = N$. Now, assume that $aN \neq N$. Since $0 \neq aN \subseteq aN$ and aN is a weakly n -submodule of M , Proposition 2.4 implies $N \subseteq aN \subseteq N$, so $N = aN$. Thus N is a secondary submodule of M .

(ii) : (\Rightarrow) Suppose every proper submodule of M is weakly n -submodule of M . Let $a \notin \sqrt{\text{ann}(M)}$ and N be a nonzero submodule of M . If $N = M$, then by (i), $N = M$ is secondary and thus $aN = aM = M$. If N is a proper submodule of M and $aN \neq 0$, since $0 \neq aN \subseteq aN$ and aN is a weakly n -submodule of M , Proposition 2.4 again gives $N \subseteq aN \subseteq N$, so $N = aN$.

(\Leftarrow) Suppose $aN = 0$ or $aN = N$ for every $a \notin \sqrt{\text{ann}(M)}$ and nonzero submodule N of M . Let N be a nonzero proper submodule of M . Suppose $0 \neq IK \subseteq N$ for some ideal I of R with $I \not\subseteq \sqrt{\text{ann}(M)}$ and some submodule K of M . Since $I \not\subseteq \sqrt{\text{ann}(M)}$, there exists $a \in I - \sqrt{\text{ann}(M)}$. If $aK \neq 0$, then by assumption $0 \neq K = aK \subseteq N$. So assume $aK = 0$ for all $a \in I - \sqrt{\text{ann}(M)}$. Since $IK \neq 0$, there exists $x \in I$ such that $xK \neq 0$. If $x \notin \sqrt{\text{ann}(M)}$, by assumption again, $0 \neq K = xK \subseteq N$. So assume that $x \in \sqrt{\text{ann}(M)}$. Then we conclude that $a + x \in I - \sqrt{\text{ann}(M)}$ and $0 \neq xK = (a + x)K$. Again by assumption, $0 \neq xK = (a + x)K = K \subseteq N$. Then Proposition 2.4 implies N is a weakly n -submodule of M . \square

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