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Existence of Multiple Solutions to $p\left(x\right)-$biharmonic Equation

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Abstract In this paper, we study the fourth order nonlinear elliptic problem with $p\left(x\right)-$biharmonic operator,

$$\begin{cases} \Delta^2_{p(x)}u + a\left(x\right)\left|u\right|^{p(x)-2}u = \lambda f\left(x\right)\left|u\right|^{\alpha(x)-2}u + g\left(x\right)\left|u\right|^{\beta(x)-2}u, & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω a bounded domain of $\mathbb{R}^n\ (n \geq 2)$, $p \in C\left(\overline{\Omega}\right)$, with $p(x) > 1$, $x \in \overline{\Omega}$, $a, f, g \in C\left(\overline{\Omega}\right)$, are nonnegative functions and satisfying certain conditions which will be stated later, and $\lambda > 0$ a small real parameter, $\Delta^2_{p(x)}u := \Delta\left(\left|\Delta u\right|^{p(x)-2}\Delta u\right)$ is the $p(x)$-biharmonic operator. Using a technique developed in [3, 11], we prove the the existence of at least two distinct weak solutions appeared in a suitable manifold called Nehari manifold.

1 Introduction

In this paper, we study a pertinent question of the multiplicity results of nontrivial weak solutions of the following nonlinear elliptic problem:

$$\begin{cases} \Delta^2_{p(x)}u + a\left(x\right)\left|u\right|^{p(x)-2}u = \lambda f\left(x\right)\left|u\right|^{\alpha(x)-2}u + g\left(x\right)\left|u\right|^{\beta(x)-2}u, & \text{in } \Omega, \\ \Delta u = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n\ (n \geq 2)$ is a bounded domain with smooth boundary, $p \in C\left(\overline{\Omega}\right)$, with $p(x) > 1$, $x \in \overline{\Omega}$, $a, f, g \in C\left(\overline{\Omega}\right)$, are nonnegative functions and satisfying conditions which will be stated later, and $\lambda > 0$ a small real parameter, $\Delta^2_{p(x)}u := \Delta\left(\left|\Delta u\right|^{p(x)-2}\Delta u\right)$ is the so called $p(x)$-biharmonic operator, which is not homogeneous and related to variable Lebesgue space $L^{p(x)}$ and Sobolev spaces $W^{1,p(x)}$ and $W^{2,p(x)}$. These facts imply many difficulties, for example maximum principle theorem, theorems of regularity, and the Lagrange multiplier theorem cannot be applied in this situation.

In 1920, Bingham was intrigued to find that certain substances did not exhibit flow behavior similar to that of honey. Through his investigations, he identified and characterized an anomalous rheological phenomenon. Specifically, he observed that some fluids would initially begin to flow under stress but would subsequently cease flowing spontaneously, highlighting a complex and non-Newtonian flow behavior. The study of differential equations and variational problems with nonstandard $p(x)-$growth conditions has received more and more interest in recent years. It possesses a solid background in physics and originates from the study on electrorheological fluids by Kovacik in [10] and elastic mechanics by Zhikov in [13, 14, 15]. It also has wide applications in different research fields, such as image processing model see for examples [12], stationary thermorheological viscous flows see [10] and the mathematical description of the processes filtration of an idea barotropic gas through a porous medium see [8]. In [1] Ayoujil and Amrouss studied the existence of solution for the following $p(x)-$biharmonic problem,

$$\begin{cases} \Delta^2_{p(x)} u(x) = \lambda |u(x)|^{q(x)-2} u(x), & x \in \Omega, \\ \Delta u(x) = u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Recently, Z. Zhou in [16], considered he following $p(x)-$biharmonic problem with potential,

$$\begin{cases} \Delta^2_{p(x)} u(x) = \lambda V(x) |u(x)|^{q(x)-2} u(x), & x \in \Omega, \\ \Delta u(x) = u(x) = 0, & x \in \partial\Omega. \end{cases}$$

They obtained the existence of solution under some conditions on the functions $V(x), q(x)$ and the domain Ω.

Motivated by the papers stated above and [3, 11], our main goal is to consider the elliptic problem with three variables with different exponents which present interesting difficulties in PDE. The key argument in our main result is to use an appropriate manifold called in mathematical literature Nehari manifold, it is a suitable manifold and has a pertinent property to prove the existence of two nontrivial solutions with different signs energy. To our best knowledge, the present paper's results are not covered in the literature.

Next, we assume to work under the following assumptions:

(H1). We suppose that the functions α, β and $p \in C(\overline{\Omega})$ such that $1 < \frac{3}{2}p(x) < \beta(x) < p^*(x)$ and $1 < \alpha(x) < \beta(x)$,for all $x \in \Omega$.

(H2). We suppose that the functions α, β and p are satisfying

$$1 < \alpha^- < \alpha^+ < p^- < \frac{3}{2}p^- < \frac{3}{2}p^+ < \beta^- < \beta^+.$$

(H3). We assume f and $g \in C(\overline{\Omega})$ are nonnegative measurable functions with compact support in Ω.

(H4). The function $a \in C(\overline{\Omega})$ is nonnegative such that $a^- > 0$.

The energy functional J_λ corresponding to the problem (1.1) is defined for each $u \in X$,

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx$$
$$-\lambda \int_\Omega \frac{1}{\alpha(x)} f(x) |u|^{\alpha(x)} dx - \int_\Omega \frac{1}{\beta(x)} g(x) |u|^{\beta(x)} dx.$$

At this point, let us define the functionals $J_\lambda, \phi, \phi_\lambda : X \to \mathbb{R}$ by

$$\phi(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx,$$

$$\phi_\lambda(u) = \lambda \int_\Omega \frac{1}{\alpha(x)} f(x) |u|^{\alpha(x)} dx + \int_\Omega \frac{1}{\beta(x)} g(x) |u|^{\beta(x)} dx,$$

and

$$J_\lambda(u) = \phi(u) - \phi_\lambda(u).$$

During the last ten years, several authors used the Nehari manifold to solve the problem of multiplicity. It is well known that the solutions of the equation (1.1) are the critical points of the energy functional J_λ. Moreover, we consider the Nehari minimization problem

$$m_\lambda := \inf_{u \in \mathcal{N}_\lambda} J_\lambda(u),$$

where

$$\mathcal{N}_\lambda = \{u \in X \backslash \{0\} : \Phi_\lambda(u) = 0\}.$$

Note that $\mathcal{N}_\lambda$ contains every nontrivial solution of the problem (1.1).
We split $\mathcal{N}_\lambda$ in three parts

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u \rangle > 0\},$$

$$\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u\rangle < 0\},$$

and

$$\mathcal{N}_\lambda^o = \{u \in \mathcal{N}_\lambda : \langle \Phi'_\lambda(u), u\rangle = 0\}.$$

The main result of this paper is given by the following theorem.

Theorem 1.1. *Suppose that the conditions **(H1), (H2), (H3)** and **(H4)** hold, then there exists $\lambda^* > 0$, such that, for any $\lambda \in (0, \lambda^*)$, the problem (1.1) has two nontrivial weak solutions u_2 and u_1 such that*

$$J_\lambda(u_1) < 0 < J_\lambda(u_2).$$

This paper is organized as follows. In Section 2, we briefly review the properties of generalized Lebesgue–Sobolev spaces. In Section 3, we prove the necessary lemmas. In Section 4, we prove the existence of minimums for the functional energy J_λ in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$. Finally, in Section 5, we present the proof of our main result.

2 Generalized Lebesgue-Sobolev Spaces

In order to discuss the problem (1.1), we recall some necessary definitions concerning the generalized Lebesgue-Sobolev spaces and introduce some useful notations used below.

Set

$$C^+(\Omega) := \left\{p \in C(\bar{\Omega}) : \inf_{x\in\Omega} p(x) > 1\right\}.$$

For every $p \in C^+(\Omega)$, we denote by

$$p^- := \inf_{x\in\Omega} p(x), p^+ := \sup_{x\in\Omega} p(x).$$

For any $p \in C^+(\Omega)$,

$$p_k^*(x) := \begin{cases} \frac{np(x)}{n-kp(x)} & \text{if } kp(x) < n, \\ +\infty, & \text{if } kp(x) \geq n. \end{cases}$$

Denote by $S(\Omega)$ the set of measurable real-valued functions defined on Ω. For $p(.) \in C^+(\Omega)$, define the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ by

$$L^{p(x)}(\Omega) := \left\{u \in S(\Omega) : \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\},$$

equipped with the so called Luxemburg norm, defined by the formula:

$$\|u\|_{p(x)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Define the variable exponent Sobolev space $W^{k,p(x)}(\Omega)$ by

$$W^{k,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\right\}.$$

Then, equipped with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} := \sum_{|\alpha|\leq k} |D^\alpha u|_{L^{p(x)}(\Omega)},$$

$W^{k,p(x)}(\Omega)$ is also a separable and reflexive Banach space. Taking into account the particularity of the problem (1.1), the following representation of the norm might be the best:

$$\|u\|_a := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{\Delta u(x)}{\mu}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\},$$

for all $u \in W^{2,p(x)}(\Omega)$.

Put for all $u \in W^{2,p(x)}(\Omega)$,

$$\rho_{p(x),a(x)}(u) := \int_{\Omega} |\Delta u(x)|^{p(x)} + a(x)|u(x)|^{p(x)} dx.$$

Proposition 2.1. ([16]) *For all $u \in W^{2,p(x)}(\Omega)$, we have:*

(i) $\|u\|_a < 1$ $(resp. = 1$ *or* $> 1) \Longleftrightarrow \rho_{p(x),a(x)}(u) < 1$ $(resp. = 1$ *or* $> 1)$,

(ii) $\|u\|_b < 1 \Longrightarrow \|u\|_a^{p^+} \leq \rho_{p(x),a(x)}(u) \leq \|u\|_a^{p^-}$,

(iii) $\|u\|_a > 1 \Longrightarrow \|u\|_a^{p^-} \leq \rho_{p(x),a(x)}(u) \leq \|u\|_a^{p^+}$,

(iv) $\|u\|_a = \lambda \Leftrightarrow \rho_{p(x),a(x)}\left(\frac{u}{\lambda}\right) = 1,$ *for* $u \neq 0$ *and* $\lambda > 0$.

Proposition 2.2. *For $p, r \in C^+(\Omega)$ and we assuming that $r(x) \leq p_k^*(x)$ for all $x \in \Omega$. Then there exists a continuous embedding*

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Also the embedding is compact when $r(x) < p_k^*(x)$ a.e in Ω.

Proposition 2.3. ([16]) *Let α and β are two measurable functions such that $\alpha \in L^\infty(\Omega)$ and $1 \leq \alpha(x)\beta(x) \leq \infty$ a.e in Ω. If $u \in L^{\alpha(x)}(\Omega), u \neq 0$, then*

(i) If $\|u\|_{\alpha(x)\beta(x)} \leq 1 \Longrightarrow \|u\|_{\alpha(x)\beta(x)}^{\alpha^-} \leq \left\| |u|^{\alpha(x)} \right\|_{\beta(x)} \leq \|u\|_{\alpha(x)\beta(x)}^{\alpha^+}$,

(ii) If $\|u\|_{\alpha(x)\beta(x)} \geq 1 \Longrightarrow \|u\|_{\alpha(x)\beta(x)}^{\alpha^+} \leq \left\| |u|^{\alpha(x)} \right\|_{\beta(x)} \leq \|u\|_{\alpha(x)\beta(x)}^{\alpha^-}$.

In particular, if $\alpha(x) = \alpha$ is constant, then

$$\||u|^\alpha\|_{\beta(x)} = \|u\|_{\beta(x)}^\alpha.$$

We also consider the weighted variable exponent Lebesgue space $L_{c(x)}^{p(x)}(\Omega)$. Let $c: \Omega \to \mathbb{R}$ be a measurable function such that $c(x) > 0$ a.e in $x \in \Omega$.

We define

$$L_{c(x)}^{p(x)}(\Omega) := \left\{ u \in S(\Omega) : \int_{\Omega} c(x)|u(x)|^{p(x)} d < \infty \right\},$$

with the norm

$$\|u\|_{p(x),c(x)} = \|u\|_{p(x)c(x)} := \inf\left\{ \tau > 0 : \int_{\Omega} c(x) \left| \frac{u(x)}{\tau} \right|^{p(x)} dx \leq 1 \right\}.$$

Then, the space $\left(L_{c(x)}^{p(x)}(\Omega), \|.\|_{p(x),c(x)}\right)$ becomes a Banach space. The modular of this space is given by

$$\delta_{p(x),c(x)} : \quad L_{c(x)}^{p(x)}(\Omega) \to \mathbb{R},$$

such that

$$\delta_{p(x),c(x)}(u) := \int_{\Omega} c(x)|u(x)|^{p(x)} dx.$$

In the sequel, we let

$$X := W_o^{1,p(x)}(\Omega) \cap W^{2,p(x)}(\Omega),$$

and we define the norm of X by

$$\|u\|_a := \inf\left\{ \mu > 0 : \int_{\Omega} \left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} + a(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\},$$

for all $u \in X$.

3 Some Necessary Lemmas

In this section, we firstly give the definition of weak solutions to the problem (1.1).

Definition 3.1. We say that $u \in X$ is called a weak solution of the problem (1.1), if the following identity holds: $\forall v \in C_0^\infty(\Omega)$:

$$\int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x) |u|^{p(x)-2} uv \right) dx =$$

$$\lambda \int_\Omega f(x) |u|^{\alpha(x)-2} uv dx + \int_\Omega g(x) |u|^{\beta(x)-2} uv dx.$$

Next, we investigate some properties of the functional J_λ.

Proposition 3.2. ([16])*The functional ϕ is sequentially weakly lower semicontinuous, $\phi \in C^1(X, \mathbb{R})$,and its Gâteaux derivative ϕ' at $u \in X$ is given by*

$$\langle \phi'(u), v \rangle = \int_\Omega \left(|\Delta u|^{p(x)-2} \Delta u \Delta v + a(x) |u|^{p(x)-2} uv \right) dx, \text{ for all } v \in X.$$

Using the previous proposition, the following result can be obtained easily.

Proposition 3.3. ([16])*The functional J_λ is well defined and $J_\lambda \in C^1(X, \mathbb{R})$, and its Gâteaux derivative J'_λ at $u \in X$ is given by*

$$\langle J'_\lambda(u), v \rangle = \int_\Omega \left(|\Delta u|^{p(x)-2} + a(x) |u|^{p(x)-2} \right) uv dx$$

$$-\lambda \int_\Omega f(x) |u|^{\alpha(x)-2} uv dx - \int_\Omega g(x) |u|^{\beta(x)-2} uv dx,$$

for all $v \in X$.

Proposition 3.4. *Under the conditions of the Theorem 2.1 in [10], let $u \in W^{2,p(x)}(\Omega)$, then there exist four positive constants c_3, c_4, c_5 and $c_6 > 0$ such that*

$$\int_\Omega f(x) |u|^{\alpha(x)} dx \leq \begin{cases} c_3 \|u\|_a^{\alpha^-} & if\ \|u\|_a < 1, \\ c_4 \|u\|_a^{\alpha^+} & if\ \|u\|_a > 1, \end{cases}$$

and

$$\int_\Omega g(x) |u|^{\beta(x)} dx \leq \begin{cases} c_5 \|u\|_a^{\beta^-} & if\ \|u\|_a < 1, \\ c_6 \|u\|_a^{\beta^+} & if\ \|u\|_a > 1. \end{cases}$$

Define:

$$\Phi_\lambda(u) := \langle J'_\lambda(u), u \rangle$$

$$\Phi_\lambda(u) = \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx -$$

$$\lambda \int_\Omega f(x) |u|^{\alpha(x)} dx - \int_\Omega g(x) |u|^{\beta(x)} dx$$

and

$$\langle \Phi'_\lambda(u), u \rangle = \int_\Omega p(x) \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx$$

$$-\lambda \int_\Omega \alpha(x) f(x) |u|^{\alpha(x)} dx - \int_\Omega \beta(x) g(x) |u|^{\beta(x)} dx.$$

To state our main result, we now present some important properties of $\mathcal{N}_\lambda^+, \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda$. Let

$$\lambda_1 := \left[\frac{(\beta^- - p^+)}{(\beta^- - \alpha^-) c_4} \right] \left[\frac{(p^- - \alpha^+)}{(\beta^+ - \alpha^+) c_6} \right]^{\frac{\alpha^+ - p^-}{\beta^+ - p^-}}.$$

The following lemma shows that the minimizers on $\mathcal{N}_\lambda$ are "usually" critical points for J_λ.

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. Then, the set $\mathcal{N}_\lambda^o = \emptyset$, for $\lambda \in (0, \lambda_1)$.*

Proof. Arguing by contradiction, we assume that there exists $u \in \mathcal{N}_\lambda^o$ for all $\lambda \in (0, \lambda_1)$ i.e.

$$\langle \nabla \Phi_\lambda(u), u \rangle = 0.$$

Then,

$$\int_\Omega p(x) \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \lambda \int_\Omega \alpha(x) f(x) |u|^{\alpha(x)} dx - \int_\Omega \beta(x) g(x) |u|^{\beta(x)} dx = 0 \tag{3.1}$$

Hence,

$$0 \geq p^- \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \lambda \alpha^+ \int_\Omega f(x) |u|^{\alpha(x)} dx - \beta^+ \int_\Omega g(x) |u|^{\beta(x)} dx$$

and

$$\Phi_\lambda(u) = \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \lambda \int_\Omega f(x) |u|^{\alpha(x)} dx - \int_\Omega g(x) |u|^{\beta(x)} dx = 0 \tag{3.2}$$

Thus,

$$\lambda \int_\Omega f(x) |u|^{\alpha(x)} dx = \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \int_\Omega g(x) |u|^{\beta(x)} dx \tag{3.3}$$

Moreover, by (3.1) and (3.2), we obtain

$$0 \geq p^- \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx$$

$$-\alpha^+ \left(\int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \int_\Omega g(x) |u|^{\beta(x)} dx \right) - \beta^+ \int_\Omega g(x) |u|^{\beta(x)} dx$$

Then,

$$0 \geq \left(p^- - \alpha^+\right) \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx + \left(\alpha^+ - \beta^+\right) \int_\Omega g(x) |u|^{\beta(x)} dx$$

Since $\|u\|_a > 1, p^- - \alpha^+ > 0$ and $\alpha^+ - \beta^+ < 0$, we have

$$0 \geq \left(p^- - \alpha^+\right) \|u\|_a^{p^-} + \left(\alpha^+ - \beta^+\right) c_6 \|u\|_a^{\beta^+}.$$

Since $\beta^+ > p^-$, we have

$$0 \geq \|u\|_a^{p^-} \left[\left(p^- - \alpha^+\right) + \left(\alpha^+ - \beta^+\right) c_6 \|u\|_a^{\beta^+ - p^-} \right].$$

Then,

$$\left[\frac{\left(p^- - \alpha^+\right)}{\left(\beta^+ - \alpha^+\right) c_6} \right]^{\frac{1}{\beta^+ - p^-}} < \|u\|_a.$$

Similarly

$$0 \leq p^+ \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \alpha^- \lambda \int_\Omega f(x) |u|^{\alpha(x)} dx - \beta^- \int_\Omega g(x) |u|^{\beta(x)} dx$$

and

$$\int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \lambda \int_\Omega f(x) |u|^{\alpha(x)} dx = \int_\Omega g(x) |u|^{\beta(x)} dx. \tag{3.4}$$

Moreover, by (3.3) and (3.4), we obtain:

$$0 \leq p^+ \int_\Omega \left(|\Delta u|^{p(x)} + a(x) |u|^{p(x)} \right) dx - \alpha^- \lambda \int_\Omega f(x) |u|^{\alpha(x)} dx$$

$$-\beta^{-}\left(\int_{\Omega}\left(|\Delta u|^{p(x)}+a\left(x\right)|u|^{p(x)}\right)dx-\lambda\int_{\Omega}f\left(x\right)|u|^{\alpha(x)}\,dx\right).$$

Then,

$$0\leq\left(p^{+}-\beta^{-}\right)\int_{\Omega}\left(|\Delta u|^{p(x)}+a\left(x\right)|u|^{p(x)}\right)dx+\lambda\left(\beta^{-}-\alpha^{-}\right)\int_{\Omega}f\left(x\right)|u|^{\alpha(x)}\,dx$$

Since $\|u\|_{a}>1, p^{+}-\beta^{-}<0$ and $\beta^{-}-\alpha^{-}>0$, we have

$$0\leq\left(p^{+}-\beta^{-}\right)\|u\|_{a}^{p^{-}}+\lambda\left(\beta^{-}-\alpha^{-}\right)c_{4}\|u\|_{a}^{\alpha^{+}}.$$

Since $p^{-}>\alpha^{+}$, we obtain

$$0\leq\|u\|_{a}^{\alpha^{+}}\left[\left(p^{+}-\beta^{-}\right)\|u\|_{a}^{p^{-}-\alpha^{+}}+\lambda\left(\beta^{-}-\alpha^{-}\right)c_{4}\right]$$

Then,

$$\|u\|_{a}<\left[\frac{\lambda\left(\beta^{-}-\alpha^{-}\right)c_{4}}{\left(\beta^{-}-p^{+}\right)}\right]^{\frac{1}{p^{-}-\alpha^{+}}}.$$

Thus,

$$\left[\frac{\left(p^{-}-\alpha^{+}\right)}{\left(\beta^{+}-\alpha^{+}\right)c_{6}}\right]^{\frac{1}{\beta^{+}-p^{-}}}\leq\|u\|_{a}<\left[\frac{\lambda\left(\beta^{-}-\alpha^{-}\right)c_{4}}{\left(\beta^{-}-p^{+}\right)}\right]^{\frac{1}{p^{-}-\alpha^{+}}}.$$

If λ is sufficiently small, we choose $\lambda\in(0,\lambda_{1})$, we get $\|u\|_{a}<1$ which contradicts.
Thus,

$$\mathcal{N}_{\lambda}^{o}=\emptyset.$$

□

Lemma 3.6. *Let $\Omega\subset\mathbb{R}^{n}$ $(n\geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. Then, if v is a local minimizer for J_{λ}on $\mathcal{N}_{\lambda}$, then, $J_{\lambda}'(v)=0$ on $\mathcal{N}_{\lambda}$, for $\lambda\in(0,\lambda_{1})$.*

Proof. If v is a local minimizer for J_{λ} on $\mathcal{N}_{\lambda}$, then by Lagrange's multipliers, there exists a $\mu\in\mathbb{R}$ such that for any $\varphi\in\mathcal{N}_{\lambda}$:

$$\langle J_{\lambda}'(v),\varphi\rangle=\mu\,\langle\Phi_{\lambda}'(v),\varphi\rangle=0.$$

If $\mu=0$, then the lemma is proved. If not, we take $\varphi=v$ and using the assumption that $v\in\mathcal{N}_{\lambda}$,

$$\langle J_{\lambda}'(v),v\rangle=\mu\,\langle\Phi_{\lambda}'(v),v\rangle=0.$$

Then,

$$\langle\Phi_{\lambda}'(v),v\rangle=0.$$

Which contradicts that $v\notin\mathcal{N}_{\lambda}^{o}$. □

Lemma 3.7. *The energy functional J_{λ} is coercive and bounded below on $\mathcal{N}_{\lambda}$, for $\lambda\in(0,\lambda_{1})$.*

Proof. Let $u\in\mathcal{N}_{\lambda}$ with $\|u\|_{a}>1$, the functional J_{λ} writes

$$J_{\lambda}(u)=\phi(u)-\phi_{\lambda}\left(u\right),$$

where

$$\phi(u)=\int_{\Omega}\frac{1}{p\left(x\right)}\left(|\Delta u|^{p(x)}+a\left(x\right)|u|^{p(x)}\right)dx,$$

and

$$\phi_{\lambda}(u)=\lambda\int_{\Omega}\frac{1}{\alpha\left(x\right)}f\left(x\right)|u|^{\alpha(x)}\,dx+\int_{\Omega}\frac{1}{\beta\left(x\right)}g\left(x\right)|u|^{\beta(x)}\,dx.$$

Using the Proposition 2, we have

$$\phi(u)\geq\frac{1}{p^{+}}\int_{\Omega}\left(|\Delta u|^{p(x)}+a\left(x\right)|u|^{p(x)}\right)dx,$$

and

$$\phi_\lambda(u) \leq \frac{\lambda}{\alpha^-}\int_\Omega f(x)|u|^{\alpha(x)}\,dx + \frac{1}{\beta^-}\int_\Omega g(x)|u|^{\beta(x)}\,dx.$$

Then,

$$J_\lambda(u) \geq \frac{1}{p^+}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$
$$-\frac{\lambda}{\alpha^-}\int_\Omega f(x)|u|^{\alpha(x)}\,dx - \frac{1}{\beta^-}\int_\Omega g(x)|u|^{\beta(x)}\,dx.$$

Since $u \in \mathcal{N}_\lambda$, then

$$\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx = \lambda\int_\Omega f(x)|u|^{\alpha(x)}\,dx + \int_\Omega g(x)|u|^{\beta(x)}\,dx.$$

Hence,

$$J_\lambda(u) \geq \frac{\beta^- - p^+}{\beta^- p^+}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx + \lambda\frac{\alpha^- - \beta^-}{\alpha^-\beta^-}\int_\Omega f(x)|u|^{\alpha(x)}\,dx.$$

$$J_\lambda(u) \geq \frac{\beta^- - p^+}{\beta^- p^+}\|u\|_a^{p^-} + \lambda\frac{\alpha^- - \beta^-}{\alpha^-\beta^-}c_4\|u\|_a^{\alpha^+}.$$

Since $p^- > \alpha^+$ so $J_\lambda(u) \to +\infty$ as $\|u\|_a \to +\infty$.
In other words, the functional J_λ is coercive and bounded below on $\mathcal{N}_\lambda$. □

4 Existence of Minimizers on $\mathcal{N}_\lambda^+$ and $\mathcal{N}_\lambda^-$:

By Lemma 9, for $0 < \lambda < \lambda_1$ we write $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ and define

$$m_\lambda = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u), m_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \text{ and } m_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

With the help of Lemma 11, we have the following results.

Theorem 4.1. *If* $0 < \lambda < \lambda_1$, *then we have the assumptions:*

(i). $m_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0$.

(ii). There exists a minimizer of J_λ *on* $\mathcal{N}_\lambda^+$.

Proof. (i). If $u \in \mathcal{N}_\lambda$, then we have,

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)}\left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx - \lambda\int_\Omega \frac{f(x)}{\alpha(x)}|u|^{\alpha(x)}\,dx - \int_\Omega \frac{g(x)}{\beta(x)}|u|^{\beta(x)}\,dx.$$

And so that

$$\langle\nabla\Phi_\lambda(u), u\rangle = \int_\Omega p(x)\left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$
$$-\lambda\int_\Omega \alpha(x)f(x)|u|^{\alpha(x)}\,dx - \int_\Omega \beta(x)g(x)|u|^{\beta(x)}\,dx > 0.$$

We obtain,

$$J_\lambda(u) \leq \frac{1}{p^-}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$
$$-\frac{\lambda}{\alpha^+}\int_\Omega f(x)|u|^{\alpha(x)}\,dx - \frac{1}{\beta^+}\int_\Omega g(x)|u|^{\beta(x)}\,dx,$$

and

$$\frac{p^+}{\beta^-}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx - \frac{\lambda\alpha^-}{\beta^-}\int_\Omega |u|^{\alpha(x)}\,dx > \int_\Omega g(x)|u|^{\beta(x)}\,dx.$$

Then,

$$J_\lambda(u) \leq \frac{1}{p^-}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx - \frac{\lambda}{\alpha^+}\int_\Omega f(x)|u|^{\alpha(x)}dx$$
$$-\frac{1}{\beta^+}\left(\frac{p^+}{\beta^-}\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx - \frac{\lambda\alpha^-}{\beta^-}\int_\Omega |u|^{\alpha(x)}dx\right).$$

Thus,

$$J_\lambda(u) < \left(\frac{1}{p^-} - \frac{p^+}{\beta^+\beta^-}\right)\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$
$$-\lambda\left(\frac{1}{\alpha^+} - \frac{\alpha^-}{\beta^+\beta^-}\right)\int_\Omega f(x)|u|^{\alpha(x)}dx.$$

Then,

$$J_\lambda(u) < \left(\frac{\beta^+\beta^- - p^-p^+}{p^-\beta^+\beta^-}\right)\int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)}\right)dx$$
$$-\lambda\left(\frac{\beta^+\beta^- - \alpha^+\alpha^-}{\alpha^+\beta^+\beta^-}\right)\int_\Omega f(x)|u|^{\alpha(x)}dx.$$

We consider the following in two cases.
Case (i), if $\|u\|_a > 1$, we have

$$J_\lambda(u) < \left(\frac{\beta^+\beta^- - p^+p^-}{p^-\beta^+\beta^-}\right)\|u\|_a^{p^+} + \lambda\left(\frac{\alpha^+\alpha^- - \beta^+\beta^-}{\alpha^+\beta^+\beta^-}\right)c_4\|u\|_a^{\alpha^+}.$$

Since $\alpha^+ < p^+$ and λ is sufficiently small, we get

$$J_\lambda(u) < 0.$$

Case (ii), if $\|u\|_a < 1$, then

$$J_\lambda(u) < \left(\frac{\beta^+\beta^- - p^+p^-}{p^-\beta^+\beta^-}\right)\|u\|_a^{p^+} + \lambda\left(\frac{\alpha^+\alpha^- - \beta^+\beta^-}{\alpha^+\beta^+\beta^-}\right)c_3\|u\|_a^{\alpha^-}.$$

Since $\alpha^+ < p^+$ and λ is sufficiently small, we get

$$J_\lambda(u) < 0.$$

i.e.,

$$\inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) < 0.$$

(ii). Since the functional J_λ is bounded below on $\mathcal{N}_\lambda$, and so on $\mathcal{N}_\lambda^+$. Then, there exists a minimizing sequence noted $(u_m^+)_m \subset \mathcal{N}_\lambda^+$, such that

$$\lim_{m\to+\infty} J_\lambda(u_m^+) = \inf_{v\in\mathcal{N}_\lambda^+} J_\lambda(v) := m_\lambda^+ < 0.$$

Here, we will prove that $(u_m^+)_m$ has a convergent subsequence in X, And since J_λ is coercive, then $(u_m^+)_m$ is bounded in X. Using compact embedding and going if necessary to a subsequence, there exists $u_2 \in \mathcal{N}_\lambda^+$ such that
(1). $u_m^+ \to u_2$ in X.
(2). $u_m^+ \to u_2$ in $L^{s(x)}(\Omega)$ such that $1 \leq s(x) < p^*$.
(3). $u_m^+ \to u_2$ in $L^{\alpha(x)}_{f(x)}(\Omega)$.
(4). $u_m^+ \to u_2$ in $L^{\beta(x)}_{g(x)}(\Omega)$.
(5). $u_m^+ \to u_2$ a.e in Ω.
Now, if $u_2 \in \mathcal{N}_\lambda^+$, then there exists $t^+ > 0$ such that $t^+u_2 \in \mathcal{N}_\lambda^+$ with

$$J_\lambda(u_2) \geq J_\lambda(t^+u_2).$$

Indeed

$$\langle \nabla\Phi_\lambda(u_2), u_2\rangle = \int_\Omega p(x)\left(|\Delta u_2|^{p(x)} + a(x)|u_2|^{p(x)}\right)dx$$

$$-\lambda\int_\Omega \alpha(x)|u_2|^{\alpha(x)}dx - \int_\Omega \beta(x)g(x)|u_2|^{\beta(x)}dx,$$

and

$$\left\langle \nabla\Phi_\lambda(t^+u_2), t^+u_2\right\rangle = \int_\Omega p(x)\left(\left|\Delta(t^+u_2)\right|^{p(x)} + a(x)\left|t^+u_2\right|^{p(x)}\right)dx$$

$$-\lambda\int_\Omega \alpha(x)\left|t^+u_2\right|^{\alpha(x)}dx - \int_\Omega \beta(x)g(x)\left|t^+u_2\right|^{\beta(x)}dx.$$

Then,

$$\left\langle \nabla\Phi_\lambda(t^+u_2), t^+u_2\right\rangle \le p^+\left(t^+\right)^{p^+}\int_\Omega\left(|\Delta u_2|^{p(x)} + a(x)|u_2|^{p(x)}\right)dx$$

$$-\lambda\left(t^+\right)^{\beta^-}\alpha^-\int_\Omega\left|t^+u_2\right|^{\alpha(x)}dx - \beta^-\left(t^+\right)^{\beta^-}\int_\Omega g(x)\left|t^+u_2\right|^{\beta(x)}dx.$$

Next, we show that $u_m^+ \to u_2$ in X.
Using contradiction, suppose that $u_m^+ \nrightarrow u_2$ in X and by applying Fatto's Lemmas, we have

$$\lim\inf_{m\to+\infty}\int_\Omega\left|\Delta\left(u_m^+\right)\right|^{p(x)}dx > \int_\Omega|\Delta(u_2)|^{p(x)}dx.$$

Then,

$$J_\lambda(t^+u_2) = \int_\Omega\frac{1}{p(x)}\left(\left|\Delta\left(t^+u_2\right)\right|^{p(x)} + a(x)\left|t^+u_2\right|^{p(x)}\right)dx$$

$$-\lambda\int_\Omega\frac{f(x)}{\alpha(x)}\left|t^+u_2\right|^{\alpha(x)}dx - \int_\Omega\frac{g(x)}{\beta(x)}\left|t^+u_2\right|^{\beta(x)}dx.$$

Thus,

$$J_\lambda(t^+u_2) \le \frac{(t^+)^{p^+}}{p^-}\int_\Omega\left(|\Delta u_2|^{p(x)} + a(x)|u_2|^{p(x)}\right)dx$$

$$-\frac{\lambda(t^+)^{\alpha^-}}{\alpha^+}\int_\Omega f(x)|u_2|^{\alpha(x)}dx - \frac{(t^+)^{\beta^+}}{\beta^+}\int_\Omega g(x)|u_2|^{\beta(x)}dx.$$

Hence,

$$J_\lambda(t^+u_2) \le \lim_{m\to+\infty}\left[\frac{(t^+)^{p^+}}{p^-}\int_\Omega\left(\left|\Delta u_m^+\right|^{p(x)} + a(x)\left|u_m^+\right|^{p(x)}\right)dx\right.$$

$$\left.-\frac{\lambda(t^+)^{\alpha^-}}{\alpha^+}\int_\Omega f(x)\left|u_m^+\right|^{\alpha(x)}dx - \frac{(t^+)^{\beta^+}}{\beta^+}\int_\Omega g(x)\left|u_m^+\right|^{\beta(x)}dx\right].$$

$$\le \lim_{m\to+\infty}J_\lambda\left(t^+u_m^+\right) < \lim_{m\to+\infty}J_\lambda\left(u_m^+\right) = \inf_{v\in\mathcal{N}_\lambda^-}J_\lambda(v).$$

Finally we have that u_2 is a minimizer for J_λ on $\mathcal{N}_\lambda^+$.

□

Theorem 4.2. *If* $0 < \lambda < \lambda^*$, *then we have the assumptions,*

(i). $m_\lambda^- = \inf_{u\in\mathcal{N}_\lambda^-}J_\lambda(u) > 0.$

(ii). There exists a minimizer of J_λ *on* $\mathcal{N}_\lambda^-$.

Proof. (i). If $u \in \mathcal{N}_\lambda^-$, then we have:

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx$$
$$-\lambda \int_\Omega \frac{f(x)}{\alpha(x)} |u|^{\alpha(x)} dx - \int_\Omega \frac{g(x)}{\beta(x)} |u|^{\beta(x)} dx.$$

And so that

$$\langle \nabla \Phi_\lambda(u), u \rangle = \int_\Omega p(x) \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx$$
$$-\lambda \int_\Omega \alpha(x) f(x) |u|^{\alpha(x)} dx - \int_\Omega \beta(x) g(x) |u|^{\beta(x)} dx < 0$$

We obtain,

$$J_\lambda(u) \geq \frac{1}{p^+} \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx$$
$$-\frac{\lambda}{\alpha^-} \int_\Omega f(x)|u|^{\alpha(x)} dx - \frac{1}{\beta^-} \int_\Omega g(x)|u|^{\beta(x)} dx,$$

and

$$\frac{p^+}{\beta^-} \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx - \frac{\lambda\alpha^-}{\beta^-} \int_\Omega |u|^{\alpha(x)} dx < \int_\Omega g(x)|u|^{\beta(x)} dx$$

Then,

$$J_\lambda(u) \geq \frac{1}{p^+} \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx - \frac{\lambda}{\alpha^-} \int_\Omega f(x)|u|^{\alpha(x)} dx$$
$$-\frac{1}{\beta^-} \left(\frac{p^+}{\beta^-} \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx - \frac{\lambda\alpha^-}{\beta^-} \int_\Omega |f(x)u|^{\alpha(x)} dx \right).$$

Thus,

$$J_\lambda(u) \geq \left(\frac{1}{p^+} - \frac{p^+}{(\beta^-)^2} \right) \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx$$
$$-\lambda \left(\frac{1}{\alpha^-} - \frac{\alpha^-}{(\beta^-)^2} \right) \int_\Omega f(x)|u|^{\alpha(x)} dx.$$

Hence,

$$J_\lambda(u) \geq \left(\frac{(\beta^-)^2 - (p^+)^2}{\beta^- p^+ \beta^-} \right) \int_\Omega \left(|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right) dx$$
$$+\lambda \left(\frac{(\alpha^-)^2 - (\beta^-)^2}{\beta^- \alpha^- \beta^-} \right) \int_\Omega f(x)|u|^{\alpha(x)} dx.$$

We consider the following in two cases.
Case (i), if $\|u\|_a < 1$, we have

$$J_\lambda(u) \geq \left(\frac{(\beta^-)^2 - (p^+)^2}{\beta^- p^+ \beta^-} \right) \|u\|_a^{p^-} + \lambda \left(\frac{(\alpha^-)^2 - (\beta^-)^2}{\beta^- \alpha^- \beta^-} \right) c_3 \|u\|_a^{\alpha^-}.$$

Since $\alpha^- < p^-$ and λ is sufficiently small, we get

$$J_\lambda(u) > 0.$$

Case (ii), if $\|u\|_a > 1$, then

$$J_\lambda(u) \geq \left(\frac{(\beta^-)^2 - (p^+)^2}{\beta^- p^+ \beta^-} \right) \|u\|_a^{p^+} + \lambda \left(\frac{(\alpha^-)^2 - (\beta^-)^2}{\beta^- \alpha^- \beta^-} \right) c_4 \|u\|_a^{\alpha^+}.$$

Since $\alpha^+ < p^+$ and λ is sufficiently small, we get

$$J_\lambda(u) > 0.$$

i.e.,

$$\inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u) > 0.$$

(i) The same proof in the above theorem, therefore we omit it.

□

4.1 Sketch of Generic Theorem's Proof

Proof. The proof of the generic theorem is regarded as a result of the above Theorems 12 and 13. □

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