Palestine Journal of Mathematics

Vol 14(4)(2025) , 380–394

ROTHE TIME-DISCRETIZATION APPROACH FOR A NONLINEAR PARABOLIC PROBLRM INVOLVING P(.)-LAPLACE OPERATOR IN WEIGHTED SOBOLEV SPACES

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Communicated by: Salim Messaoudi

MSC 2010 Classifications: Primary 35A15, 34D30, 35J60; Secondary 35J65, 74G30.

Keywords and phrases: Degenerate parabolic problem, entropy solution, existence, uniqueness, entropy solution, weighted variable Sobolev space.

Abstract In this paper, we prove the existence and uniqueness of entropy solutions for the following problem in weighted Sobolev variable spaces:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(\omega|\nabla u - \Theta(u)|^{p(x)-2}\big(\nabla u - \Theta(u)\big)\Big) + \alpha(u) = f(x,t) \;\; in \;\; Q_T := \Omega \times (0,\, T), \\ u = 0 \;\; on \;\; \Sigma_T := \partial\Omega \times (0,\, T), \\ u(.,0) = u_0 \;\; in \;\; \Omega. \end{cases}$$

The main method used here is the Rothe's time-discretization approach combined with the theory of weighted variable Sobolev spaces.

1 Introduction

Let $\Omega \subset \mathbb{R}^d, (d \geq 2)$ be an open bounded domain with a connected Lipschitz boundary $\partial\Omega$, $p \in (1,\infty)$, and let T be a fixed positive real number. Our goal of this paper is to prove the existence and uniqueness results of entropy solutions for the nonlinear degenerate parabolic problem

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\Big(\omega|\nabla u - \Theta(u)|^{p(x)-2}\big(\nabla u - \Theta(u)\big)\Big) + \alpha(u) = f(x,t) \;\; in \;\; Q_T := \Omega \times (0,\, T), \\ u = 0 \;\; on \;\; \Sigma_T := \partial\Omega \times (0,\, T), \\ u(.,0) = u_0 \;\; in \;\; \Omega. \end{cases} \tag{1.1}$$

where $p(.)$ is a continuous function defined on $\overline{\Omega}$ with $p(x) > 1$ for all $x \in \overline{\Omega}$, ω is a measurable positive and a.e finite function defined on $\mathbb{R}^d$, α is a non decreasing continuous real function defined on $\mathbb{R}$ and Θ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}^N$, the datum f is in $L^1(\Omega)$.

In recent years, the study of partial differential equations and variational problems has received considerable attention in many models coming from various branches of mathematical physics, such as elastic mechanics, electrorheological fluid dynamics and image processing, etc. Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating and electrochemical problems (see for example [3, 7, 11]).

In general, the Sobolev spaces $W^{k,p(.)}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted variable Sobolev spaces. For more information about these spaces and their properties in the constant or variable exponent, we refer to articles [2, 4, 5, 14, 15]) and references therein.

the problem (1.1), or some of its special cases, is a model of several physical phenomena,

for example, we state here the following two parabolic models:
• ***Model 1. Filtration in a porous medium*** . The filtration phenomena of fluids in porous media are modeled by following equation,

$$\frac{\partial c(p)}{\partial t} = \nabla a[k(c(p))(\nabla p + e)], \tag{1.2}$$

where p is the unknown pressure, c volumetric moisture content, k the hydraulic conductivity of the porous medium, a the heterogeneity matrix and $-e$ is the direction of gravity.
• ***Model 2. Fluid flow through porous media.*** This model is governed by following equation,

$$\frac{\partial \theta}{\partial t} - div\left(|\nabla\varphi(\theta) - K(\theta)e|^{p-2}(\nabla\varphi(\theta) - K(\theta)e)\right) = 0, \tag{1.3}$$

where θ is the volumetric content of moisture, $K(\theta)$ the hydraulic conductivity, $\varphi(\theta)$ the hydrostatic potential and e is the unit vector in the vertical direction.
In this paper, we study the existence and uniqueness question of entropy solutions to the problem (1.1), we apply here a time discretization of the problem (1.1) by Euler forward scheme and we show existence, uniqueness and stability of entropy solutions to the discretized problem. After, we will construct from the entropy solution of the discretized problem a sequence that we show converging to an entropy solution of the nonlinear parabolic problem (1.1). We recall that the Rothe's method was introduced by E. Rothe in 1930 and it has been used and developed by many authors, e.g P.P. Mosolov, K. Rektorys in linear and quasilinear parabolic problems. This method has been used by several authors while studying time discretization of nonlinear parabolic problems, we refer to the works [8, 12, 13] for some details. The advantage of our method is that we cannot only obtain the existence and uniqueness of weak solutions to the problem (1.1), but also compute the numerical approximations.
The plan of our paper is divided into four sections organized as follows, in section 2, we present some preliminaries on weighted variable Sobolev spaces and some basic tools to prove our main result of this paper, in section 3, we introduce some assumptions, and we give the definition of entropy solutions of problem (1.1), we finish this paper by proving the main result of this paper.

2 Related properties of variable exponent problems

In this section, we give some notations and definitions and state some results which will be used in this work.
Let Ω be a bounded open domain in $\mathbb{R}^d$ $(d \geq 2)$, we consider the following set

$$C^+(\overline{\Omega}) = \{p : \overline{\Omega} \to \mathbb{R}^+ : \text{p is continuous and such that } 1 < p_- < p_+ < \infty\},$$

where

$$p_- = \min_{x\in\overline{\Omega}} p(x) \text{ and } p_+ = \max_{x\in\overline{\Omega}} p(x).$$

Let ω be a measurable positive and a.e finite function defined on $\mathbb{R}^d$ and satisfied the following integrability conditions :

(H_1) : $\omega \in L^1_{loc}(\Omega)$ and $\omega^{\frac{-1}{p(x)-1}} \in L^1_{loc}(\Omega)$,

(H_2) : $\omega^{-s(x)} \in L^1_{loc}(\Omega)$ where $s(x) \in \left(\frac{d}{p(x)}, \infty\right) \cap \left[\frac{1}{p(x)-1}, \infty\right)$.

For $p(.) \in C^+(\overline{\Omega})$, we define the weighted Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega,\omega)$ by

$$L^{p(\cdot)}(\Omega,\omega) = \{u : \Omega \to \mathbb{R} : \text{u is measurable and } \int_\Omega |u|^{p(x)}\omega(x)dx < \infty\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(.),\omega} = \|u\|_{L^{p(\cdot)}(\Omega,\omega)} = \inf\left\{\lambda > 0, \ \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} \omega(x)dx \leq 1\right\}.$$

We denote by $L^{p'(.)}(\Omega,\omega^*)$ the conjugate space of $L^{p(.)}(\Omega,\omega)$, where $\frac{1}{p(x)}+\frac{1}{p'(x)}=1$ and where $\omega^*(x)=\omega(x)^{1-p'(x)}$ for all $x\in\Omega$.
On the space $L^{p(.)}(\Omega,\omega)$, we consider the modular function $\rho_{p(.),\omega}: L^{p(.)}(\Omega,\omega)\to\mathbb{R}$ defined by

$$\rho_{p(.),\omega}(u)=\rho_{L^{p(.)}(\Omega,\omega)}(u)=\int_\Omega |u(x)|^{p(x)}\omega(x)dx.$$

The relation between $\rho_{p(.),\omega}$ and $\|.\|_{p(.),\omega}$ is given by the next Proposition

Proposition 2.1 ([6]). *Let u be an element of $L^{p(.)}(\Omega,\omega)$, the following assertions hold:*

i) $\|u\|_{p(.),\omega}<1$ *(respectively* $>,=1$*)* $\Leftrightarrow \rho_{p(.),\omega}(u)<1$ *(respectively* $>,=1$*).*

ii) *If* $\|u\|_{p(.),\omega}<1$ *then* $\|u\|^{p_+}_{p(.),\omega}\le\rho_{p(.),\omega}(u)\le\|u\|^{p_-}_{p(.),\omega}$.

iii) *If* $\|u\|_{p(.),\omega}>1$ *then* $\|u\|^{p_-}_{p(.),\omega}\le\rho_{p(.),\omega}(u)\le\|u\|^{p_+}_{p(.),\omega}$.

iv) $\|u_n\|_{p(.),\omega}\to 0\Leftrightarrow\rho_{p(.),\omega}(u_n)\to 0$ *and* $\|u_n\|_{p(.),\omega}\to\infty\Leftrightarrow\rho_{p(.),\omega}(u_n)\to\infty$.

Lemma 2.2 ([6]). ***(Hölder-type inequality).*** *If $u\in L^{p(.)}(\Omega)$ and $v\in L^{p'(.)}(\Omega)$, then*

$$\left|\int_\Omega uvdx\right|\leqslant\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{L^{p(.)}(\Omega)}\|v\|_{L^{p'(.)}(\Omega)}\leqslant 2\|u\|_{L^{p(.)}(\Omega)}\|v\|_{L^{p'(.)}(\Omega)}.$$

The weighted Sobolev space with variable exponent is defined by

$$W^{1,p(.)}(\Omega,\omega)=\left\{u\in L^{p(.)}\text{ and }|\nabla u|\in L^{p(.)}(\Omega,\omega)\right\},$$

with the norm

$$\|u\|_{1,p(.),\omega}=\|u\|_{p(.)}+\|u\|_{p(.),\omega},\ \forall\, u\in W^{1,p(.)}(\Omega,\omega).$$

In the following, the space $W_0^{1,p(.)}(\Omega,\omega)$ denotes the closure of C_0^∞ in $W^{1,p(.)}(\Omega,\omega)$ endowed with the norm

$$\|u\|_{W_0^{1,p(.)}(\Omega,\omega)}=\|\nabla u\|_{L^{p(.)}(\Omega,\omega)}.$$

Let $p(.),s(.)$ are two elements of the set $C^+(\overline{\Omega})$ where the function s(.) satisfies the hypothesis (H_2), we define the following functions

$$p^*(x)=\frac{dp(x)}{d-p(x)}\ \ for\ \ p(x)<d,$$

$$p_s(x)=\frac{p(x)s(x)}{1+s(x)}<p(x),$$

$$p_s^*(x)=\begin{cases}\dfrac{p(x)s(x)}{(1+s(x))d-p(x)s(x)} & \text{if } d>p_s(x),\\ +\infty & \text{if } d\le p_s(x).\end{cases}$$

for almost all $x\in\Omega$.

Proposition 2.3 ([10]). *Let Ω be an open set of $\mathbb{R}^d$, p(.) $\in C^+(\overline{\Omega})$ and let hypothesis (H_1) be satisfied, we have*

$$L^{p(.)}(\Omega,\omega)\hookrightarrow L^1_{Loc}(\Omega).$$

Proposition 2.4 ([10]). *Let hypothesis (H_1) be satisfied, the space $\left(W^{1,p(.)}(\Omega,\omega),\|u\|_{1,p(.),\omega}\right)$ is a separable and reflexive Banach space.*

Proposition 2.5 ([10]). *Assume that hypotheses (H_1) and (H_2) hold and $p(.),s(.)\in C^+(\overline{\Omega})$, then we have the continuous embedding*

$$W^{1,p(x)}(\Omega,\omega)\hookrightarrow W^{1,p_s(x)}(\Omega,\omega).$$

Moreover, we have the compact embedding

$$W^{1,p(x)}(\Omega,\omega)\hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$$

provided that $r\in C^+(\overline{\Omega})$ and $1\le r(x)<p_s^*(x)$ for all $x\in\Omega$.

Proposition 2.6 ([14]). *Let Ω be an open set of $\mathbb{R}^d$, $p(.) \in C^+(\overline{\Omega})$ and let $u \in W_0^{1,p(.)}(\Omega,\omega)$. Then there exists a positive constant C_Ω depending only on Ω such that*

$$\int_\Omega \omega(x)|u|^{p(x)} \le C_\Omega \int_\Omega \omega(x)|\nabla u|^{p(x)}.$$

Let X be a Banach space and let $T > 0$. For $1 \le p \le \infty$, the space $L^p(0,T;X)$ consists of all measurable functions $u : (0,T) \to X$ such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}} < \infty \quad \text{if } 1 \le p < \infty$$

and

$$\|u\|_{L^\infty(0,T;X)} = \text{esssup}_{t\in(0,T)} \|u(t)\|_X < \infty.$$

The space $C(0,T;X)$ is a space of all continuous functions $u : (0,T) \to X$ such that

$$\|u\|_{C(0,T;X)} = \max_{t\in(0,T)} \|u(t)\|_X < \infty.$$

The spaces $L^p(0,T;X)$ and $C(0,T;X)$ equipped with the norms from the above definitions are the Banach spaces.
Given a constant $k > 0$, we define the cut function $T_k : \mathbb{R} \to \mathbb{R}$ as

$$T_k(s) = \min(k, \max(s,-k)) = \begin{cases} s & \text{if } |s| \le k, \\ k & \text{if } s > k, \\ -k & \text{if } s < -k. \end{cases}$$

For a function $u = u(x)$ defined on Ω, we define the truncated function $T_k u$ as follows, for every $x \in \Omega$, the value of $(T_k u)$ at x is just $T_k(u(x))$.
Let the function $J_k : \mathbb{R} \to \mathbb{R}^+$ (is the primitive function of T_k) defined by

$$J_k(s) = \int_0^s T_k(t)dt = \begin{cases} \frac{s^2}{2} & \text{if } |s| \le k \\ k|s| - \frac{k^2}{2} & \text{if } |s| > k. \end{cases}$$

We have as in the paper [9]

$$\langle \frac{\partial s}{\partial t}, T_k(s)\rangle = \frac{d}{dt}\int_\Omega J_k(s)dx \ \text{ in } \ L^1(]0,T[),$$

which implies

$$\int_0^t \langle \frac{\partial s}{\partial t}, T_k(s)\rangle dt = \int_\Omega J_k(s(t))dx - \int_\Omega J_k(s(0))dx.$$

We define also the space

$$\mathcal{T}_0^{1,p(.)}(\Omega,\omega) = \left\{u : \Omega \to \mathbb{R}, \ \ u \text{ is measurable and } \ T_k(u) \in W_0^{1,p(x)}(\Omega,\omega) \ \ for\ all \ \ k > 0\right\}.$$

By Proposition 1.1 in [16], we have the following result

Proposition 2.7. *For every function $u \in \mathcal{T}_0^{1,p(.)}(\Omega,\omega)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}^d$, which we call the very weak gradient of u and denote $v = \nabla u$ such that*

$$\nabla T_k(u) = v\chi_{\{|u|\le k\}} \text{ for a.e } x \in \Omega \text{ and for all } k > 0,$$

where χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^d$. Moreover, if u belongs to $W_0^{1,p(.)}(\Omega,\omega)$, then the function v coincides with the weak gradient of u.

Lemma 2.8 ([1]). *For $\xi,\ \eta \in \mathbb{R}^d$ and $1 < p < \infty$, we have*

$$\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \le |\xi|^{p-2}\xi(\xi - \eta).$$

Lemma 2.9 ([1]). *For $a \geq 0$, $b \geq 0$ and $1 \leq p < +\infty$, we have*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

Remark 2.10. Hereinafter, $C_i, i \in \{1;2;...\}$ is a positive constant, $meas\{A\}$ denotes the measure of the measurable set $A \subset \mathbb{R}^d$ and χ_B is the characteristic function of the measurable set $B \subset \mathbb{R}^d$.

3 Main result

In this section, in order to get our main results, we do the following assumptions in addition to the conditions $(H1)$ and $(H2)$ listed earlier:

(H_3) : α is a non decreasing and continuous function on $\mathbb{R}$ such that $\alpha(0) = 0$.

(H_4) : Θ is a continuous function from $\mathbb{R}$ to $\mathbb{R}^d$ such that $\Theta(0) = 0$ and for all real numbers x, y, we have $|\Theta(x) - \Theta(y)| \leq \lambda|x-y|$, where λ is a real constant such that $0 < \lambda < \dfrac{1}{2C_\Omega^{\frac{1}{p_+}}}$ and C_Ω is the constant given in Proposition 2.6.

(H_5) : $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$.

Definition 3.1. A measurable function $u : Q_T \to \mathbb{R}$ is an entropy solution of the parabolic problem (1.1) in Q_T if $u(.,0) = u_0$ in Ω, $u \in C(0,T;L^1(\Omega))$, $T_k(u) \in L^{p_-}(0,T;W_0^{1,p(.)}(\Omega,\omega))$ for all $k > 0$ and

$$\int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u-\varphi) \right\rangle ds + \int_0^t \int_\Omega \omega(x)\Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)dxds + \int_0^t \int_\Omega \alpha(u)T_k(u-\varphi)dxds$$

$$\leq \int_\Omega J_k\big(u(0)-\varphi(0)\big)dx - \int_\Omega J_k\big(u(t)-\varphi(t)\big)dx + \int_0^t \int_\Omega fT_k(u-\varphi)dxds \tag{3.1}$$

for all $\varphi \in L^\infty(Q_T) \cap L^{p_-}(0,T;W^{1,p(.)}(\Omega,\omega)) \cap W^{1,1}(0,T;L^1(\Omega))$ and $t \in (0,\ T)$, where

$$\Phi(\xi) = |\xi|^{p(x)-2}\xi, \quad \forall\, \xi \in \mathbb{R}^d.$$

The main result of this work is the following Theorem:

Theorem 3.2. *Let hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) be satisfied. If $1 < p_- < p_+ \leq 2$ or $p_- > \max\left(2; \frac{p_+}{2}+1\right)$, then the problem (1.1) has a unique entropy solution.*

4 Proof of the main result

The proof of Theorem 3.2 is divided into three steps, in the first one, by using Euler forward scheme, we discretize the continuous problem (1.1) and we study the existence and uniqueness of entropy solutions to the discretized problems. In the second step, some stability results for the discrete entropy solutions will be given. Finally and by Rothe's function, we construct a sequence of functions that we show that this sequence converges to an entropy solution of the problem (1.1). We finish this step by proving the uniqueness result of entropy solutions.

4.1 The semi-discrete problem

Thanks to Euler forward scheme, we discretize the problem (1.1) and obtain the following problems

$$\begin{cases} U_n - \tau \text{div}\Big(\omega\Phi(\nabla U_n - \Theta(U_n))\Big) + \tau\alpha(U_n) = \tau f_n + U_{n-1} \ in\ \Omega, \\ U_n = 0 \ \ on\ \partial\Omega, \\ U_0 = u_0 \ \ in\ \Omega, \end{cases} \tag{4.1}$$

where $N\tau = T,\ 0 < \tau < 1,\ 1 \leq n \leq N, t_n = n\tau$ and

$$f_n(.) = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(s,.)ds \text{ in } \Omega.$$

Definition 4.1. An entropy solution to the discretized problem (4.1) is a sequence $(U_n)_{0\leq n\leq N}$ such that $U_0 = u_0$ and for $n = 1, 2, ..., N$, U_n is defined by induction as an entropy solution to the problem

$$\begin{cases} u - \tau \text{div}\Big(\omega\Phi(\nabla U_n - \Theta(U_n))\Big) + \tau\alpha(u) = \tau f_n + U_{n-1} \ in\ \Omega, \\ u = 0 \ \ on\ \partial\Omega. \end{cases}$$

i.e. for all $n \in \{1, 2, ..., N\}$, $U_n \in \mathcal{T}_0^{1,p(.)}(\Omega, \omega)$ and for all $\varphi \in W_0^{1,p(.)}(\Omega, \omega) \cap L^\infty(\Omega)$, $k > 0, \tau > 0$, we have

$$\int_\Omega U_n T_k(U_n - \varphi)dx + \tau\int_\Omega \omega(x)\Phi(\nabla U_n - \Theta(U_n))\nabla T_k(U_n - \varphi)dx + \tau\int_\Omega \alpha(U_n)T_k(U_n - \varphi)dx$$
$$\leq \int_\Omega (\tau f_n + U_{n-1})T_k(U_n - \varphi)dx. \tag{4.2}$$

Lemma 4.2. *Let hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) be satisfied. If $(U_n)_{0\leq n\leq N}$ is an entropy solution of the discretized problem (4.1), then we have $U_n \in L^1(\Omega)$ for all $n = 1, ..., N$.*

Proof. For $n = 1$, if we take $\varphi = 0$ in (4.2), we get

$$\int_\Omega U_1 T_k(U_1)dx + \tau\int_\Omega \omega(x)\Phi(\nabla U_1 - \Theta(U_1))\nabla T_k(U_1)dx + \tau\int_\Omega \alpha(U_1)T_k(U_1)dx$$
$$\leq \int_\Omega \tau f_1 T_k(U_1)dx + \int_\Omega u_0 T_k(U_1)dx. \tag{4.3}$$

On the one hand, using the Lemma 2.8, we obtain

$$\int_\Omega \omega(x)\Phi(\nabla U_1 - \Theta(U_1))\nabla T_k(U_1)dx + \frac{1}{p(x)}\int_\Omega \omega(x) \mid \Theta(T_k(U_1)) \mid^{p(x)} dx \geqslant 0.$$

On the other hand, by assumption (H_3), we conclude that

$$\tau\int_\Omega \alpha(U_1)T_k(U_1)dx \geqslant 0.$$

Therefore, the inequality (4.3) becomes

$$\int_\Omega U_1 T_k(U_1)dx \leq \int_\Omega \tau f_1 T_k(U_1)dx \int_\Omega u_0 T_k(U_1)dx + \frac{\tau}{p_-}\int_\Omega \omega(x) \mid \Theta(T_k(U_1)) \mid^{p(x)} dx.$$

Using hypothesis (H_4) we get

$$\int_\Omega U_1 T_k(U_1)dx \leq \int_\Omega \tau f_1 T_k(U_1)dx + \int_\Omega u_0 T_k(U_1)dx + \frac{\tau}{p_-}(k\lambda)^{p_-}\int_\Omega \omega(x)dx.$$

This implies

$$0 \leq \int_\Omega U_1 \frac{T_k(U_1)}{k}dx \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} + \frac{k^{p_- - 1}\lambda^{p_-}}{p_-}\|w\|_{L^1(\Omega)}.$$

For each $x \in \Omega$, we have

$$\lim_{k\to 0} U_1(x)\frac{T_k(U_1(x))}{k} = |U_1(x)|.$$

By Fatou's Lemma, we deduce that $U_1 \in L^1(\Omega)$ and

$$\|U_1\|_{L^1(\Omega)} \leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}.$$

By induction, we deduce in the same manner that $U_n \in L^1(\Omega)$ for all $n = 1, ..., N$. □

Theorem 4.3. *Assume that hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) hold. Then the discretized problem (4.1) has a unique entropy solution $(U_n)_{0\leq n\leq N}$ and $U_n \in L^1(\Omega)\cap\mathcal{T}_0^{1,p(.)}(\Omega,\omega)$ for all $n = 1, ..., N$.*

Proof. For $n = 1$, we rewrite the discretized problem (4.1) as

$$\begin{cases} -\tau \text{div}\Big(\omega\Phi(\nabla u - \Theta(u))\Big) + \bar{\alpha}(u) = F \ in \ \Omega, \\ u = 0 \ \ on \ \partial\Omega, \end{cases} \tag{4.4}$$

where $u = U_1$ and $F = \tau f_1 + u_0$. According to the hypothesis (H_5), we have $F \in L^1(\Omega)$ and, by hypothesis (H_3), the function defined by $\bar{\alpha}(s) := \tau\alpha(s) + s$ is non-decreasing, continuous and satisfies $\bar{\alpha}(0) = 0$. Then, the problem (4.1) has a unique entropy solution U_1 in $L^1(\Omega) \cap \mathcal{T}_0^{1,p(.)}(\Omega,\omega)$ (see [17, Theorems 13 and 14]). By induction, using the same argument above, we prove that the problem (4.1) has a unique entropy solution $(U_n)_{0\leq n\leq N}$ and $U_n \in L^1(\Omega) \cap \mathcal{T}_0^{1,p(.)}(\Omega,\omega)$ for all $n = 1, ..., N$. □

4.2 Stability results

Theorem 4.4. *Assume that hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) hold. If $(U_n)_{1\leq n\leq N}$ is an entropy solution of the discretized problem (4.1), then for all $n = 1, ..., N$, we have*

$$\textbf{(a)} \quad \|U_n\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(b)} \quad \tau\sum_{i=1}^{n}\|\alpha(U_i)\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(c)} \quad \sum_{i=1}^{n}\|U_i - U_{i-1}\|_{L^1(\Omega)} \leq C(u_0, f),$$

$$\textbf{(d)} \quad \tau\sum_{i=1}^{n}\|T_k(U_i)\|^{p_-}_{W_0^{1,p(.)}(\Omega,\omega)} \leq C(u_0, f, T, k),$$

where $C(u_0, f)$ and $C(u_0, f, T, k)$ are positive constants independents of N.

Proof. **For (a) and (b).** Let $i \in \{1, 2, ..., N\}$, we take $\varphi = 0$ as a test function in entropy formulation of the discretized problem (4.1), we obtain

$$\int_\Omega U_i T_k(U_i)dx + \tau\int_\Omega \omega(x)\Phi(\nabla U_i - \Theta(U_i))\nabla T_k(U_i)dx + \frac{\tau}{p(x)}\int_\Omega \omega(x) \mid \Theta(T_k(U_i)) \mid^{p}(x)dx$$
$$+\tau\int_\Omega \alpha(U_i)T_k(U_i)dx \leq \tau\int_\Omega f_i T_k(U_i)dx + \int_\Omega U_{i-1}T_k(U_i)dx.$$

This inequality implies that

$$\int_\Omega U_i\frac{T_k(U_i)}{k}dx + \int_\Omega \alpha(U_i)\frac{T_k(U_i)}{k}dx \leq \tau\|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)} + \frac{1}{kp_-}\|\Theta(T_k(U_i))\|^{p_-}_{L^{p(.)}(\Omega,\omega)}. \tag{4.5}$$

Note that

$$\lim_{k\to 0}\frac{T_k(s)}{k} = sign(s), \tag{4.6}$$

where

$$\text{sign}(s) := \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0 \\ -1 & \text{if } s < 0 \end{cases}.$$

Then, passing to limit in (4.5), using Fatou's lemma and hypothesis (H_4), we deduce that

$$\|U_i\|_{L^1(\Omega)} + \tau\|\alpha(U_i)\|_{L^1(\Omega)} \leq \tau\|f_i\|_{L^1(\Omega)} + \|U_{i-1}\|_{L^1(\Omega)}.$$

Summing the above inequality from $i=1$ to n, we conclude that

$$\begin{aligned} \|U_n\|_{L^1(\Omega)} + \tau \sum_{i=1}^{n} \|\alpha(U_i)\|_{L^1(\Omega)} &\leq \tau \frac{1}{\tau} \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \int_\Omega |f| dx dt + \|u_0\|_{L^1(\Omega)} \\ &\leq \int_0^T \int_\Omega |f| dx dt + \|u_0\|_{L^1(\Omega)} \\ &\leq \|f\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} \end{aligned}$$

Hence, the stability results (a) and (b) are then proved.
For (c). Let $i \in \{1, 2, ..., N\}$. Taking $\varphi = T_h\big(U_i - sign(U_i - U_{i-1})\big)$ as a test function in entropy formulation of the discretized problem (4.1), and letting $h \to \infty$, we get, for $k \geqslant 1$

$$\tau \lim_{h\to\infty} \mathcal{I}(k,h) + \|U_i - U_{i-1}\|_{L^1(\Omega)} \leq \tau \left[\|f_i\|_{L^1(\Omega)} + \|\alpha(U_i)\|_{L^1(\Omega)} \right], \tag{4.7}$$

where

$$\begin{aligned} \mathcal{I}(k,h) &:= \int_\Omega \omega(x) \Phi\left(\nabla U_i - \Theta(U_i)\right) \nabla T_k\left(U_i - T_h\left(U_i - \text{sign}\left(U_i - U_{i-1}\right)\right)\right) dx \\ &= \int_{\Omega(k,h)\cap\overline{\Omega}(h)} \omega(x) \Phi\left(\nabla U_i - \Theta(U_i)\right) \nabla U_i dx \end{aligned}$$

and

$$\begin{aligned} \Omega(k,h) &:= \{|U_i - T_h\left(U_i - \text{sign}\left(U_i - U_{i-1}\right)\right)| \leq k\} \\ \overline{\Omega}(h) &:= \{|U_i - \text{sign}\left(U_i - U_{i-1}\right)| > h\}. \end{aligned}$$

As

$$\Omega(k,h) \cap \overline{\Omega}(h) \subset \{k-1 \leq |U_i| \leq k+h\},$$

we conclude by using [17, Lemma 20] that

$$\lim_{h\to\infty} \mathcal{I}(k,h) = 0$$

This follows by (4.7) that

$$\|U_i - U_{i-1}\|_{L^1(\Omega)} \leq \tau \left[\|f_i\|_{L^1(\Omega)} + \|\alpha(U_i)\|_{L^1(\Omega)} \right].$$

Summing up the above inequality from $i=1$ to n and using the stability result (b), we get the stability result (c).
For (d). Let $i \in \{1, 2, ..., N\}$. Taking $\varphi = 0$ as a test function in (4.2), and using hypothesis (H_3), we get

$$\tau \int_\Omega \omega(x) \Phi(\nabla U_i - \Theta(U_i)) \nabla T_k(U_i) dx \leq \tau k \|f_i\|_{L^1(\Omega)} + k\|U_i - U_{i-1}\|_{L^1(\Omega)}.$$

Thanks to Lemmas 2.8 and 2.9 and hypothesis (H_4) , we deduce that

$$\tau \|\nabla T_k(U_i)\|^{p(x)}_{L^{p(\cdot)}(\Omega,\omega)} \leq \tau k \|f_i\|_{L^1(\Omega)} + k\|U_i - U_{i-1}\|_{L^1(\Omega)}. \tag{4.8}$$

This inequality implies that

$$\tau \sum_{i=1}^{n} \rho_{p(.),\omega}(\nabla T_k(U_i)) dx \leq \tau k \sum_{i=1}^{n} \|f_i\|_{L^1(\Omega)} + k \sum_{i=1}^{n} \|U_i - U_{i-1}\|_{L^1(\Omega)} dx.$$

Then by hypothesis (H_5) and the stability result (c), we deduce that

$$\tau \sum_{i=1}^{n} \rho_{p(.),\omega}(\nabla T_k(U_i)) dx \leq C_1(f, u_0, k). \tag{4.9}$$

On the other hand, let $s_0 = \left\{ i \in 1; 2; ...; n : \|\nabla T_k(U_i)\|_{L^{p(.)}(\Omega,\omega)} \leq 1 \right\}$, we have

$$\begin{aligned} \tau \sum_{i=1}^{n} \|\nabla T_k(U_i)\|^{p_-}_{L^{p(.)}(\Omega,\omega)} &= \tau \sum_{i \in s_0} \|\nabla T_k(U_i)\|^{p_-}_{L^{p(.)}(\Omega,\omega)} + \tau \sum_{i \notin s_0} \|\nabla T_k(U_i)\|^{p_-}_{L^{p(.)}(\Omega,\omega)} \\ &\leq T + \tau \sum_{i \notin s_0} \rho_{p(.),\omega}(\nabla T_k(U_i)). \end{aligned}$$

We conclude by inequality (4.9) that

$$\tau \sum_{i=1}^{n} \|\nabla T_k(U_i)\|^{p_-}_{p(x),\omega(x)} \leq C_2(f, u_0, T, k).$$

Hence the stability result (d) is established. □

4.3 Entropy solution of the continuous problem

Let us introduce the following piecewise linear extension (called Rothe function)

$$\begin{cases} u_N(0) := u_0, \\ u_N(t) := U_{n-1} + (U_n - U_{n-1})\frac{(t-t_{n-1})}{\tau}, \ \forall\, t \in]t_{n-1},\ t_n], \ \ n = 1, ..., N \ \ in\ \Omega. \end{cases} \tag{4.10}$$

And the following piecewise constant function

$$\begin{cases} \overline{u}_N(0) := u_0, \\ \overline{u}_N(t) := U_n \ \forall\, t \in]t_{n-1},\ t_n], \ \ n = 1, ..., N \ \ in\ \Omega. \end{cases} \tag{4.11}$$

We have by Theorem 4.3 that for any $N \in \mathbb{N}$, the entropy solution $(U_n)_{1 \leq n \leq N}$ of problems (4.1) is unique, thus, the two sequences $(u_N)_{N \in \mathbb{N}}$ and $(\overline{u}_N)_{N \in \mathbb{N}}$ are uniquely defined.

Lemma 4.5. *Let hypotheses (H_1), (H_2), (H_3), (H_4) and (H_5) be satisfied, then for all $N \in \mathbb{N}$, we have*

(1) $\|\overline{u}_N - u_N\|_{L^1(Q_T)} \leq \dfrac{1}{N} C(T, u_0, f)$,

(2) $\|\dfrac{\partial u_N}{\partial t}\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

(3) $\|u_N\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

(4) $\|\overline{u}_N\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

(5) $\|\alpha(\overline{u}_N)\|_{L^1(Q_T)} \leq C(T, u_0, f)$,

(6) $\|T_k(\overline{u}_N)\|_{L^{p_-}(0,T,W_0^{1,p(.)}(\Omega,\omega))} \leq C(T, u_0, f, k)$,

where $C(T, u_0, f)$ and $C(T, u_0, f, k)$ are positive constants independents of N.

Proof. **For (1).** For $N \in \mathbb{N}$, we have

$$\begin{aligned} \|\overline{u}_N - u_N\|_{L^1(Q_T)} &= \int_0^T \int_\Omega |\overline{u}_N - u_N| dx dt \\ &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|U_n - U_{n-1}\|_{L^1(\Omega)} \frac{(t_n - t)}{\tau} dt \\ &= \frac{\tau}{2} \sum_{n=1}^{N} \|U_n - U_{n-1}\|_{L^1(\Omega)} \\ &\leq \frac{T}{2N} \sum_{n=1}^{N} \|U_n - U_{n-1}\|_{L^1(\Omega)}. \end{aligned}$$

Thanks to Theorem 4.4, we conclude the result (1).
For (2). We have

$$\frac{\partial u_N(t)}{\partial t} = \frac{(U_n - U_{n-1})}{\tau}.$$

for $n = 1, ..., N$ and $t \in]t_{n-1},\ t_n]$. This implies that

$$\begin{aligned} \|\frac{\partial u_N}{\partial t}\|_{L^1(Q_T)} &= \int_0^T \int_\Omega |\frac{\partial u_N}{\partial t}| dxdt \\ &= \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \frac{1}{\tau}\|U_n - U_{n-1}\|_{L^1(\Omega)} dt \\ &= \sum_{n=1}^N \|U_n - U_{n-1}\|_{L^1(\Omega)}. \end{aligned}$$

Using Theorem 4.4, we deduce the result (2). We follow the same techniques used above to show the estimates (3), (4), (5) and (6). □

Lemma 4.6. *Let hypotheses* $(H_1), (H_2), (H_3), (H_4)$ *and* (H_5) *be satisfied. The sequence* $(\overline{u}_N)_{N\in\mathbb{N}}$ *converges in measure and a.e. in* Q_T.

Proof. Let ε, r, k be positive real numbers. The $N, M \in \mathbb{N}$, we have the following inclusion

$$\{|\overline{u}_N - \overline{u}_M| > r\} \subset \{\{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\} \cup \{|\overline{u}_N| \le k, |\overline{u}_M| \le k, |\overline{u}_N - \overline{u}_M| > r\}\}.$$

By Markov inequality and Lemma 4.5, we deduce

$$\begin{aligned} meas\{|\overline{u}_N| > k\} &\le \frac{1}{k}\|\overline{u}_N\|_{L^1(Q_T)} \\ &\le \frac{1}{k} C(T, u_0, f), \end{aligned}$$

and

$$meas\{|\overline{u}_M| > k\} \le \frac{1}{k} C(T, u_0, f).$$

This implies for k sufficiently large that

$$meas\left(\{|\overline{u}_N| > k\} \cup \{|\overline{u}_M| > k\}\right) \le \frac{\varepsilon}{2}. \tag{4.12}$$

On the other hand, by Lemma 4.5, the sequence $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ is bounded in the space $L^{p(.)}(Q_T, \omega)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that $(T_k(\overline{u}_N))_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^p(Q_T, \omega)$ and in measure. Therefore, there exists an $N_0 \in \mathbb{N}$ such that for all $N, M \ge N_0$, we have

$$meas\left(\{|\overline{u}_N| \le k, |\overline{u}_M| \le k, |\overline{u}_N - \overline{u}_M| > r\}\right) < \frac{\varepsilon}{2}. \tag{4.13}$$

Hence, by (4.12) and (4.13), $(\overline{u}_N)_{N\in\mathbb{N}}$ converges in measure and there exists a measurable function on Q_T, u such that

$$\overline{u}_N \to u \text{ a.e. in } Q_T.$$

□

Lemma 4.7. *There exists a function* u *in* $L^1(Q_T)$ *such that* $T_k(u) \in L^{p_-}(0, T; W_0^{1,p(.)}(\Omega, \omega))$ *for all* $k > 0$ *and*

(i) u_N *converges to* u *in* $L^1(Q_T)$,

(ii) $\overline{u}_N$ *converges to* u *in* $L^1(Q_T)$,

(iii) $\alpha(\overline{u}_N)$ *converges to* $\alpha(u)$ *in* $L^1(Q_T)$,

(iv) $\nabla T_k(\overline{u}_N)$ *converges to* $\nabla T_k(u)$ *weakly in* $L^{p(.)}(Q_T, \omega)$,

(v) $T_k(\overline{u}_N)$ *converges to* $T_k(u)$ *weakly in* $L^{p_-}(0, T; W_0^{1,p(.)}(\Omega, \omega))$.

Proof. **For (iv) and (v).** By (6) of Lemma 4.5, we have

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ is bounded in } L^{p(.)}(Q_T,\omega).$$

Then, there exists a subsequence, still denoted $(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}}$ such that

$$(\nabla T_k(\overline{u}_N))_{N\in\mathbb{N}} \text{ converges weakly to } v\in\ L^{p(.)}(Q_T,\omega).$$

However

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ in } L^{p(.)}(Q_T,\omega).$$

Hence, it follows that

$$\nabla T_k(\overline{u}_N) \text{ converges to } \nabla T_k(u) \text{ weakly in } L^{p(.)}(Q_T,\omega),$$

and by (6) of Lemma 4.5, we deduce that

$$T_k(\overline{u}_N) \text{ converges to } T_k(u) \text{ weakly in } L^{p_-}(0,T;W_0^{1,p(.)}(\Omega,\omega)).$$

□

In order to prove that the limit function u is an entropy solution of the problem (1.1), we need the following result.

Lemma 4.8. *The sequence $(u_N)_{N\in\mathbb{N}}$ converges to u in $C\left(0,\ T;\ L^1(\Omega)\right)$.*

Proof. For $\varphi\in L^\infty(Q_T)\cap L^{p_-}(0,T;W_0^{1,p(.)}(\Omega,\omega))\cap W^{1,1}(0,T;L^1(\Omega))$, the inequality (4.2) implies that

$$\int_0^t\left\langle\frac{\partial u_N}{\partial s},T_k(\overline{u}_N-\varphi)\right\rangle ds+\int_0^t\int_\Omega\omega(x)\Phi(\nabla\overline{u}_N-\Theta(\overline{u}_N))\nabla T_k(\overline{u}_N-\varphi)dxds$$
$$+\int_0^t\int_\Omega\alpha(\overline{u}_N)T_k(\overline{u}_N-\varphi)dxds\leq\int_0^t\int_\Omega f_NT_k(\overline{u}_N-\varphi)dxds, \tag{4.14}$$

where $f_N(t,x)=f_n(x)$ for $t\in]t_{n-1},\ t_n]$, $n=1,...,N$.
We consider the two partitions $(t_n=n\tau_N)_{n=1}^N$ and $(t_m=m\tau_M)_{m=1}^M$ of interval $[0,\ T]$ and the corresponding semi-discrete solutions $(u_N(t),\overline{u}_N(t))$, $(u_M(t),\overline{u}_M(t))$ defined by (4.10) and (4.11).
Let $h>0$, for the semi-discrete solution $(u_N(t),\overline{u}_N(t))$ we take $\varphi=T_h(\overline{u}_M)$ and for the semi-discrete solution $(u_M(t),\overline{u}_M(t))$ we take $\varphi=T_h(\overline{u}_N)$. Summing the two inequalities and letting h go to infinity, we have for $k=1$ that

$$\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)\right\rangle ds+\lim_{h\to\infty}\mathcal{I}_{N,M}(h)\leq\|f_N-f_M\|_{L^1(Q_T)}+\|\alpha(\overline{u}_N)-\alpha(\overline{u}_M)\|_{L^1(Q_T)}, \tag{4.15}$$

where

$$\mathcal{I}_{N,M}(h)=$$
$$\int_0^t\int_\Omega\omega(x)\Big(\Phi(\nabla\overline{u}_N-\Theta(\overline{u}_N))\nabla T_1(\overline{u}_N-T_h(\overline{u}_M))+\Phi(\nabla\overline{u}_M-\Theta(\overline{u}_M))\nabla T_1(\overline{u}_M-T_h(\overline{u}_N))\Big)dxds.$$

The inequality (4.15) becomes

$$\int_\Omega J_1\left(u_N(t)-u_M(t)\right)dx+\lim_{h\to\infty}\mathcal{I}_{N,M}(h)\leq$$
$$\|f_N-f_M\|_{L^1(Q_T)}+\|\alpha(\overline{u}_N)-\alpha(\overline{u}_M)\|_{L^1(Q_T)}+\left|\int_0^t\left\langle\frac{\partial(u_N-u_M)}{\partial s},\ T_1(\overline{u}_N-\overline{u}_M)-T_1(u_N-u_M)\right\rangle ds\right|. \tag{4.16}$$

Thanks to Hölder's type inequality

$$\left|\int_0^t \left\langle \frac{\partial(u_N - u_M)}{\partial s},\ T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\right\rangle ds\right|$$
$$\leq \left\|\frac{\partial(u_N - u_M)}{\partial s}\right\|_{L^1(Q_T)} \|T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\|_{L^\infty(Q_T)}.$$

This implies by Lemma 4.5 that

$$\left|\int_0^t \left\langle \frac{\partial(u_N - u_M)}{\partial s}, T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\right\rangle ds\right|$$
$$\leq C(T, u_0, f)\|T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\|_{L^\infty(Q_T)}.$$

We have

$$\lim_{N,M\to\infty} \|T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\|_{L^\infty(Q_T)} = 0.$$

Then

$$\lim_{N,M\to\infty}\left|\int_0^t \left\langle \frac{\partial(u_N - u_M)}{\partial s},\ T_1(\overline{u}_N - \overline{u}_M) - T_1(u_N - u_M)\right\rangle ds\right| = 0. \tag{4.17}$$

We also have

$$\lim_{N,M\to\infty}\Big(\|f_N - f_M\|_{L^1(Q_T)} + \|\alpha(\overline{u}_N) - \alpha(\overline{u}_M)\|_{L^1(Q_T)}\Big) = 0.$$

Then, the inequality (4.16) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u_N(t) - u_M(t)\right) dx + \lim_{N,M\to\infty}\lim_{h\to\infty} \mathcal{I}_{N,M}(h) \leq 0. \tag{4.18}$$

Using the same technique used in the proof of [17, Theorem 14], we prove that

$$\lim_{N,M\to\infty}\lim_{h\to\infty} \mathcal{I}_{N,M}(h) \geq 0. \tag{4.19}$$

As $J_1 : \mathbb{R} \to \mathbb{R}^+$, Then, by inequality (4.18), we get

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u_N(t) - u_M(t)\right) dx = 0. \tag{4.20}$$

By definition of J_1, we have

$$\int_{\{|u_N-u_M|<1\}} |u_N(t)-u_M(t)|^2 dx + \frac{1}{2}\int_{\{|u_N-u_M|\geq 1\}} |u_N(t)-u_M(t)| dx \leq \int_\Omega J_1\left(u_N(t) - u_M(t)\right) dx.$$

This implies that

$$\begin{aligned}
\int_\Omega |u_N(t) - u_M(t)| dx &= \int_{\{|u_N-u_M|<1\}} |u_N(t) - u_M(t)| dx + \int_{\{|u_N-u_M|\geq 1\}} |u_N(t) - u_M(t)| dx \\
&\leq C(\Omega)\left(\int_{\{|u_N-u_M|<1\}} |u_N(t) - u_M(t)|^2 dx\right)^{\frac{1}{2}} \\
&+ \int_{\{|u_N-u_M|\geq 1\}} |u_N(t) - u_M(t)| dx \\
&\leq C(\Omega)\left(\int_\Omega J_1\big(u_N(t) - u_M(t)\big) dx\right)^{\frac{1}{2}} + 2\int_\Omega J_1\big(u_N(t) - u_M(t)\big) dx.
\end{aligned}$$

Therefore, by the result (4.20), we conclude that $(u_N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,\ T; L^1(\Omega))$ and

$$(u_N)_{N\in\mathbb{N}} \textit{ converges to } u \textit{ in } C(0,\ T;\ L^1(\Omega)).$$

□

It remains to prove that the limit function u is an entropy solution of the problem (1.1). Since $u_N(0) = U_0 = u_0$ for all $N \in \mathbb{N}$, then $u(.,0) = u_0$, and by (4.14) we get

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N - \varphi) - T_k(u_N - \varphi) \right\rangle ds + \int_0^t \int_\Omega \omega(x)\Phi(\nabla \overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N - \varphi)dxds +$$

$$\int_0^t \int_\Omega \alpha(\overline{u}_N) T_k(\overline{u}_N - \varphi)dxds \leqslant - \int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u_N - \varphi) \right\rangle ds$$

$$+ \int_\Omega J_k(u_N(0) - \varphi(0))dx - \int_\Omega J_k(u_N(t) - \varphi(t))dx + \int_0^t \int_\Omega f_N T_k(\overline{u}_N - \varphi)dxds. \quad (4.21)$$

In the same argument, as used for the proof of equality (4.17), we deduce that

$$\lim_{N\to\infty} \int_0^t \left\langle \frac{\partial u_N}{\partial s}, T_k(\overline{u}_N - \varphi) - T_k(u_N - \varphi) \right\rangle ds = 0. \quad (4.22)$$

following the same technique used in the proof of [17, Theorem 13], we show that

$$\lim_{N\to\infty} \int_0^t \int_\Omega \omega(x)\Phi(\nabla \overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N - \varphi)dxds = \int_0^t \int_\Omega \omega(x)\Phi(\nabla u - \Theta(u))\nabla T_k(u - \varphi)dxds. \quad (4.23)$$

And by Lemma 4.8, we deduce that $u_N(t) \to u(t)$ in $L^1(\Omega)$ for all $t \in [0,\ T]$, which implies that

$$\int_\Omega J_k(u_N(t) - \varphi(t))dx \to \int_\Omega J_k(u(t) - \varphi(t))dx \quad (4.24)$$

Finally, taking limits as N goes to infinity, and using the above results, the continuity of α and Θ, the facts that $f_N \to f$ in $L^1(Q_T)$, and $T_k(\overline{u}_N - \varphi) \to T_k(u - \varphi)$ in $L^\infty(Q_T)$, we deduce that u is an entropy solution of the nonlinear parabolic problem (1.1).

Uniqueness. Let v another entropy solution of the nonlinear parabolic problem (1.1). Taking $\varphi = T_h(u_N)$ as a test function in (3.1) and letting h goes to infinity, we get

$$\int_\Omega J_k(v(t) - u_N(t))dx + \int_0^t \left\langle \frac{\partial u_N}{\partial s},\ T_k(v - u_N) \right\rangle ds + \lim_{h\to\infty} \mathcal{J}_1^N(k,h)$$

$$+ \int_0^t \int_\Omega \alpha(v) T_k(v - u_N)dxds \leq \int_0^t \int_\Omega f T_k(v - u_N)dxds, \quad (4.25)$$

where

$$\mathcal{J}_1^N(k,h) = \int_0^t \int_\Omega \omega(x)\Phi(\nabla v - \Theta(v))\nabla T_k(v - T_h(u_N))dxds.$$

On the other hand, taking $\varphi = T_h(v)$ as a test function in the inequality (4.14) and taking h goes to infinity, we obtain

$$\int_0^t \left\langle \frac{\partial u_N}{\partial s},\ T_k(\overline{u}_N - v) \right\rangle ds + \lim_{h\to\infty} \mathcal{J}_2^N(k,h) + \int_0^t \int_\Omega \alpha(\overline{u}_N) T_k(\overline{u}_N - v)dxds$$

$$\leq \int_0^t \int_\Omega f_N T_k(\overline{u}_N - v)dxds \quad (4.26)$$

where

$$\mathcal{J}_2^N(k,h) = \int_0^t \int_\Omega \omega(x)\Phi(\nabla \overline{u}_N - \Theta(\overline{u}_N))\nabla T_k(\overline{u}_N - T_h(v))dxds.$$

Adding (4.25) and (4.26), we get

$$\int_\Omega J_k(v(t)-u_N(t))dx + \int_0^t \left\langle \frac{\partial u_N}{\partial s},\, T_k(v-u_N)+T_k(\overline{u}_N-v)\right\rangle ds + \lim_{h\to\infty}\mathcal{J}^N(k,h)$$

$$+\int_0^t\int_\Omega [\alpha(v)T_k(v-u_N)+\alpha(\overline{u}_N)T_k(\overline{u}_N-v)]\,dxds \le \int_0^t\int_\Omega [fT_k(v-u_N)+f_NT_k(\overline{u}_N-v)]\,dxds,$$

where

$$\mathcal{J}^N(k,h)=\mathcal{J}_1^N(k,h)+\mathcal{J}_2^N(k,h).$$

Taking N goes to infinity, using the above convergence results, and the hypothesis (H_3), we get

$$\int_\Omega J_k(v(t)-u(t))dx + \lim_{N\to\infty}\lim_{h\to\infty}\mathcal{J}^N(k,h)\le 0. \tag{4.27}$$

Applying the technique used in (4.19), we deduce that

$$\lim_{N\to\infty}\lim_{h\to\infty}\mathcal{J}^N(k,h)\ge 0. \tag{4.28}$$

Therefore, the inequality (4.27) implies that

$$\int_\Omega J_k(v(t)-u(t))dx\le 0,$$

i.e.

$$\int_\Omega \frac{J_k(v(t)-u(t))}{k}dx\le 0.$$

However

$$\lim_{k\to 0}\frac{J_k(x)}{k}=|x|.$$

Then, by Dominated Convergence Theorem, we get

$$\|v(t)-u(t)\|_{L^1(\Omega)}\le 0,\quad \text{for } t\in[0,\ T].$$

Compliance with ethical standards
Conflict of interest The authors declare that there is no conflict of interest.

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Received: 2025-01-02
Accepted: 2025-04-12