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A CONFORMABLE-TYPE FRACTIONAL DERIVATIVE VIA A TRIGONOMETRIC FUNCTION

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Abstract *In this paper, we introduce a new conformable-type fractional derivative that employs a trigonometric cosine function. For a function f, the operator is defined as*

$$(D^{\alpha}f)(t) = \lim_{h\to 0} \frac{f\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - f(t)}{h},$$

for $0 < \alpha \leq 1$ and $t > 0$. Unlike many classical fractional derivatives, this definition preserves important properties such as linearity and the familiar product, quotient, and chain rules. This approach further enables the derivation of fractional analogues of Rolle's theorem and the mean value theorem. In addition, we develop a corresponding fractional integral operator and illustrate its application by solving several fractional differential equations. Overall, this work provides an accessible and robust framework for further applications in fractional calculus.

1 Introduction

Fractional calculus (FC) originated in 1695 as a generalization of classical calculus and has since evolved into an important branch of mathematics. In recent decades, FC has experienced significant theoretical and applied advancements and now serves as a powerful tool for modeling complex systems in various disciplines [1, 2, 3, 4, 5, 6]. For further studies on FC and its applications, we refer the reader to [7, 8, 9, 10].

Many prominent mathematicians—including Riemann, Liouville, Grünwald, Letnikov, Sonine, Marchaud, Weyl, Riesz, Hadamard, Kober, Erdélyi, and Caputo—have contributed to various fractional integral and differential operators. However, a noted drawback of the classical fractional derivatives is the loss of the standard product and chain rules.

To address these limitations, a novel definition of the fractional derivative, closely aligned with the properties of standard integer-order derivatives, was introduced about a decade ago. This approach, known as the conformable fractional derivative, preserves consistency with familiar derivative properties such as sum, product, and chain rules. Furthermore, it has facilitated the development of fractional analogues of Rolle's theorem and the mean value theorem.

Building on this foundation, several conformable-like fractional derivatives have been introduced and studied, extending the scope of this framework. The concept originated with R. Khalil et al., who defined the conformable fractional derivative as follows [11]: Let $f : [0,\infty) \to \mathbb{R}$, for all $0 < \alpha \leq 1$, $t > 0$, the conformable fractional derivative is given by

$$(D^{\alpha}f)(t) = \lim_{h\to 0} \frac{f(t + ht^{1-\alpha}) - f(t)}{h}$$

Subsequently, Katugampola proposed a generalization of this derivative [12]: Let $f : [0, \infty) \to \mathbb{R}$*, for all* $0 < \alpha \leq 1$ *,* $t > 0$*, the fractional derivative of order* α *is given by*

$$(D_k^\alpha f)(t) = \lim_{h \to 0} \frac{f(te_k^{ht^{-\alpha}}) - f(t)}{h}$$

where

$$e_k^{ht^{-\alpha}} = \sum_{i=0}^{k} \frac{\left(ht^{-\alpha}\right)^i}{i!}$$

Clearly, $D_1^\alpha f$ *coincides with the conformable fractional derivative defined in [11].*

Despite the appealing properties of the fractional derivatives introduced in [11] and [12], issues such as boundedness emerge when dealing with higher-order integer derivatives of these fractional derivatives [13]. To address these challenges, A. Kajouni et al. introduced a conformable-like fractional derivative designed to resolve the boundedness issues [13]: Let $f : [0, \infty) \to \mathbb{R}$*, for all* $0 < \alpha \leq 1$ *,* $t > 0$*, the conformable fractional derivative of order* α *is given by*

$$(D^\alpha f)(t) = \lim_{h \to 0} \frac{f(t + he^{(\alpha-1)t}) - f(t)}{h}$$

More recently, a conformable-like fractional derivative utilizing a hyperbolic function was introduced [14]: Let $f : [0, \infty) \to \mathbb{R}$*, for all* $0 < \alpha \leq 1$ *,* $t > 0$*, the hyperbolic fractional derivative of order* α *is given by*

$$(D^\alpha f)(t) = \lim_{h \to 0} \frac{f(t + h \cosh{(1-\alpha)t}) - f(t)}{h}$$

In this paper, we extend these ideas by proposing a new fractional derivative based on the trigonometric cosine function. We establish its basic properties—including the sum, product, quotient, and chain rules—and derive fractional versions of Rolle's theorem and the mean value theorem. Finally, we present a corresponding fractional integral and demonstrate its utility in solving fractional differential equations.

2 Trigonometric Fractional Derivative

In this section, we present a conformable-type fractional derivative using a trigonometric cosine function.

Definition 2.1. Given a function $f : [0, \infty) \to \mathbb{R}$, for all $0 < \alpha \leq 1$ and $t > 0$, the α-trigonometric fractional derivative is defined by

$$(D^\alpha f)(t) = \lim_{h \to 0} \frac{f\Big(t + h \, \cos\big((1-\alpha)t\big)\Big) - f(t)}{h}.$$

It is clear that the above definition coincides with the usual derivative when $\alpha = 1$*. If the* α*-trigonometric fractional derivative of* f *exists, we say that* f *is* α*-trigonometric differentiable. If* f *is* α*-trigonometric differentiable in* $(0, a)$ *for some* $a > 0$ *and* $\alpha \in (0, 1]$*, and if* $\lim_{t \to 0^+} (D^\alpha f)(t)$ *exists, then we define*

$$(D^\alpha f)(0) = \lim_{t \to 0^+} (D^\alpha f)(t).$$

Remark 2.2. Definition 2.1 is equivalent to

$$(D^\alpha f)(t_0) = \lim_{t \to t_0} \frac{f\Big(t_0 + (t - t_0) \, \cos\big((1-\alpha)t_0\big)\Big) - f(t_0)}{t - t_0}.$$

We now discuss key properties of the new fractional derivative. Much like in classical calculus where differentiability implies continuity, here α*-trigonometric differentiability implies continuity.*

Theorem 2.3. *If a function $f : [0, \infty) \to \mathbb{R}$ is α-trigonometric differentiable at $t_0 > 0$, then f is continuous at t_0.*

Proof. Since

$$f\Big(t_0 + h\,\cos\big((1-\alpha)t_0\big)\Big) - f(t_0) = \frac{f\Big(t_0 + h\,\cos\big((1-\alpha)t_0\big)\Big) - f(t_0)}{h}\,h,$$

taking the limit as $h \to 0$ yields

$$\lim_{h\to 0}\Big[f\Big(t_0 + h\,\cos\big((1-\alpha)t_0\big)\Big) - f(t_0)\Big] = (D^\alpha f)(t_0)\cdot 0 = 0.$$

Letting $\epsilon = h\,\cos\big((1-\alpha)t_0\big)$ shows that

$$\lim_{\epsilon\to 0}\Big[f(t_0+\epsilon) - f(t_0)\Big] = 0,$$

and so f is continuous at t_0. □

Next, we present the basic differentiation rules.

Theorem 2.4. *Let f and g be α-trigonometric differentiable in $(0, a)$ for $a > 0$ and $\alpha \in (0, 1]$. Then:*

(i) $D^\alpha(c_1 f + c_2 g)(t) = c_1\,(D^\alpha f)(t) + c_2\,(D^\alpha g)(t)$ *for all* $c_1, c_2 \in \mathbb{R}$.

(ii) $D^\alpha(fg)(t) = f(t)\,(D^\alpha g)(t) + g(t)\,(D^\alpha f)(t)$.

(iii) $D^\alpha\Big(\frac{f}{g}\Big)(t) = \frac{g(t)\,(D^\alpha f)(t) - f(t)\,(D^\alpha g)(t)}{g^2(t)}$, *provided* $g(t) \neq 0$.

Proof. (i) This follows directly from the linearity of limits.

(ii) We have

$$\begin{aligned} D^\alpha(fg)(t) &= \lim_{h\to 0}\frac{f\Big(t + h\,\cos\big((1-\alpha)t\big)\Big)g\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - f(t)g(t)}{h} \\ &= \lim_{h\to 0}\Bigg\{\frac{f\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - f(t)}{h}\,g\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) \\ &\qquad + f(t)\,\frac{g\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - g(t)}{h}\Bigg\} \\ &= (D^\alpha f)(t)\,g(t) + f(t)\,(D^\alpha g)(t), \end{aligned}$$

where we have used the continuity of g at t.

(iii) Similarly,

$$\begin{aligned} D^\alpha\Big(\frac{f}{g}\Big)(t) &= \lim_{h\to 0}\frac{\dfrac{f\Big(t+h\,\cos\big((1-\alpha)t\big)\Big)}{g\Big(t+h\,\cos\big((1-\alpha)t\big)\Big)} - \dfrac{f(t)}{g(t)}}{h} \\ &= \frac{1}{g^2(t)}\Bigg[g(t)\,\lim_{h\to 0}\frac{f\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - f(t)}{h} \\ &\qquad - f(t)\,\lim_{h\to 0}\frac{g\Big(t + h\,\cos\big((1-\alpha)t\big)\Big) - g(t)}{h}\Bigg] \\ &= \frac{g(t)\,(D^\alpha f)(t) - f(t)\,(D^\alpha g)(t)}{g^2(t)}. \end{aligned}$$

□

Theorem 2.5. *If f is differentiable at $t > 0$, then*

$$(D^\alpha f)(t) = \cos\big((1-\alpha)t\big)\, f'(t).$$

Proof. Starting from the definition,

$$(D^\alpha f)(t) = \lim_{h\to 0} \frac{f\Big(t + h\cos\big((1-\alpha)t\big)\Big) - f(t)}{h},$$

and letting $\epsilon = h\cos\big((1-\alpha)t\big)$, it follows that

$$(D^\alpha f)(t) = \cos\big((1-\alpha)t\big) \lim_{\epsilon\to 0} \frac{f(t+\epsilon) - f(t)}{\epsilon} = \cos\big((1-\alpha)t\big)\, f'(t).$$

□

Corollary 2.6. *Let $p, \lambda \in \mathbb{R}$ and $0 < \alpha \le 1$. Then:*

(i) $D^\alpha(\lambda) = 0$.

(ii) $D^\alpha(t^p) = p\, t^{p-1} \cos\big((1-\alpha)t\big)$.

(iii) $D^\alpha\big(\sin\big((1-\alpha)t\big)\big) = (1-\alpha)\cos^2\big((1-\alpha)t\big)$.

(iv) $D^\alpha\big(\cos\big((1-\alpha)t\big)\big) = -(1-\alpha)\sin\big((1-\alpha)t\big)\cos\big((1-\alpha)t\big)$.

(v) $D^\alpha\big(\tan\big((1-\alpha)t\big)\big) = (1-\alpha)\sec\big((1-\alpha)t\big)$.

(vi) $D^\alpha\big(\sec\big((1-\alpha)t\big)\big) = (1-\alpha)\tan\big((1-\alpha)t\big)$.

(vii) $D^\alpha\big(\csc\big((1-\alpha)t\big)\big) = -(1-\alpha)\cot^2\big((1-\alpha)t\big)$.

(viii) $D^\alpha\big(\cot\big((1-\alpha)t\big)\big) = -(1-\alpha)\csc\big((1-\alpha)t\big)\cot\big((1-\alpha)t\big)$.

(ix) $D^\alpha\big(e^{\lambda t}\big) = \lambda\, e^{\lambda t}\cos\big((1-\alpha)t\big)$.

(x) $D^\alpha\Big(\frac{1}{1-\alpha}\ln\Big|\sec\big((1-\alpha)t\big) + \tan\big((1-\alpha)t\big)\Big|\Big) = 1$.

Next, we present a chain rule for α-trigonometric differentiable functions. The proof relies on Carathéodory's Theorem, which characterizes differentiability.

Theorem 2.7. ***(Carathéodory's Theorem)*** *Let $f : I \to \mathbb{R}$ with I an interval containing t_0. Then f is differentiable at t_0 if and only if there exists a function $\varphi : I \to \mathbb{R}$ that is continuous at t_0 and satisfies*

$$f(t) - f(t_0) = \varphi(t)(t - t_0) \quad \text{for } t \in I. \tag{2.1}$$

In this case, $\varphi(t_0) = f'(t_0)$.

Similarly, we have a version of Carathéodory's Theorem for the fractional derivative.

Theorem 2.8. ***(Carathéodory's Theorem for the Trigonometric Fractional Operator)*** *Let $0 < \alpha < 1$, $t_0 > 0$, and let $f : I \to \mathbb{R}$ (with $I \subseteq (0,\infty)$ containing t_0) be given. Then f is α-trigonometric differentiable at t_0 if and only if there exists a function $\varphi : I \to \mathbb{R}$, continuous at t_0, satisfying*

$$f\Big(t_0 + (t - t_0)\cos\big((1-\alpha)t_0\big)\Big) - f(t_0) = \varphi(t)(t - t_0) \quad \text{for } t \in I. \tag{2.2}$$

In this case, $\varphi(t_0) = (D^\alpha f)(t_0)$.

Proof. For the forward direction, assume $(D^\alpha f)(t_0)$ exists and define

$$\varphi(t) = \begin{cases} \dfrac{f\Big(t_0 + (t-t_0)\cos\big((1-\alpha)t_0\big)\Big) - f(t_0)}{t - t_0}, & t \neq t_0, \\ (D^\alpha f)(t_0), & t = t_0. \end{cases}$$

Then continuity of φ at t_0 follows from the definition of the derivative. Conversely, if such a continuous function φ exists satisfying (2.2), then by dividing by $(t - t_0)$ (for $t \neq t_0$) and taking the limit as $t \to t_0$, we deduce that $(D^\alpha f)(t_0)$ exists and equals $\varphi(t_0)$. □

Theorem 2.9. ***(Chain Rule for α-Trigonometric Differentiable Functions)*** *Let I and J be intervals in $(0,\infty)$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions with $f(J) \subseteq I$, and let $t_0 \in J$. If f is differentiable at t_0 and g is α-trigonometric differentiable at $f(t_0)$, then the composite function $g \circ f$ is α-trigonometric differentiable at t_0 and*

$$D^\alpha(g \circ f)(t_0) = D^\alpha\Big(g\big(f(t_0)\big)\Big)\, f'(t_0).$$

Proof. Since f is differentiable at t_0, by Theorem 2.7 there is a function $\varphi : J \to \mathbb{R}$ that is continuous at t_0 with $\varphi(t_0) = f'(t_0)$ such that

$$f(t) - f(t_0) = \varphi(t)(t - t_0) \quad \text{for } t \in J.$$

Likewise, since g is α-trigonometric differentiable at $f(t_0)$, by Theorem 2.8 there is a function $\psi : I \to \mathbb{R}$ that is continuous at $f(t_0)$ with $\psi(f(t_0)) = D^\alpha\Big(g\big(f(t_0)\big)\Big)$ such that

$$g\Big(f(t_0) + (s - f(t_0))\cos\big((1-\alpha)f(t_0)\big)\Big) - g\big(f(t_0)\big) = \psi(s)(s - f(t_0))$$

for all $s \in I$. Setting $s = f(t)$ gives

$$g\Big(f(t_0) + (f(t) - f(t_0))\cos\big((1-\alpha)f(t_0)\big)\Big) - g\big(f(t_0)\big) = \psi\big(f(t)\big)\,\varphi(t)(t - t_0).$$

Then, by Theorem 2.8, the composite function $g \circ f$ is α-trigonometric differentiable at t_0 with

$$D^\alpha(g \circ f)(t_0) = \psi\big(f(t_0)\big)\,\varphi(t_0) = D^\alpha\Big(g\big(f(t_0)\big)\Big)\, f'(t_0).$$

□

Corollary 2.10. *Let $I \subseteq (0,\infty)$ be an interval and $f : I \to (0,\infty)$ be a strictly monotone continuous function with inverse $f^{-1} : f(I) \to \mathbb{R}$. If f^{-1} is α-trigonometric differentiable at $f(t_0)$ and $f'(t_0) \neq 0$, then*

$$(D^\alpha f^{-1})(f(t_0)) = \frac{\cos\big((1-\alpha)t_0\big)}{f'(t_0)}.$$

Proof. Since $(f^{-1} \circ f)(t) = t$ for all $t \in I$, differentiation yields

$$D^\alpha\Big(f^{-1}\big(f(t_0)\big)\Big)\, f'(t_0) = \cos\big((1-\alpha)t_0\big),$$

so that

$$(D^\alpha f^{-1})(f(t_0)) = \frac{\cos\big((1-\alpha)t_0\big)}{f'(t_0)}.$$

□

The definition extends naturally for $\alpha \in (n, n+1]$ with $n \in \mathbb{N}$.

Definition 2.11. Let $\alpha \in (n, n+1]$ for some $n \in \mathbb{N}$, and assume f is n-differentiable at $t > 0$. Then the α-trigonometric fractional derivative is defined by

$$(D^\alpha f)(t) = \lim_{h \to 0} \frac{f^{(n)}\Big(t + h\cos\big((1-\alpha+n)t\big)\Big) - f^{(n)}(t)}{h},$$

provided the limit exists.

Remark 2.12. It follows immediately that if f is $(n+1)$-differentiable at $t > 0$, then

$$(D^\alpha f)(t) = \cos\big((1-\alpha+n)t\big)\, f^{(n+1)}(t).$$

We now state fractional versions of Rolle's and the mean value theorems.

Theorem 2.13. ***(Rolle's Theorem for α-Trigonometric Differentiable Functions)***
Let $\alpha \in (0,1]$, $a > 0$, and let $f : [a,b] \to \mathbb{R}$ satisfy:

(i) Either

$$[a,b] \subseteq \left[\frac{1}{1-\alpha}\Big(-\frac{\pi}{2}+2n\pi\Big), \frac{1}{1-\alpha}\Big(\frac{\pi}{2}+2n\pi\Big)\right]$$

or

$$[a,b] \subseteq \left[\frac{1}{1-\alpha}\Big(\frac{\pi}{2}+2n\pi\Big), \frac{1}{1-\alpha}\Big(\frac{3\pi}{2}+2n\pi\Big)\right],$$

for some $n \in \mathbb{N} \cup \{0\}$;

(ii) f is continuous on $[a,b]$;

(iii) f is α-trigonometric differentiable on (a,b);

(iv) $f(a) = f(b)$.

Then there exists $c \in (a,b)$ such that $(D^\alpha f)(c) = 0$.

Proof. If f is constant, the result is trivial. Otherwise, because f is continuous on $[a,b]$, it attains its absolute maximum or minimum at some point $c \in (a,b)$. Without loss of generality, assume $f(c)$ is the absolute minimum. Then

$$(D^\alpha f)(c) = \lim_{h\to 0^+} \frac{f\Big(c + h\cos\big((1-\alpha)c\big)\Big) - f(c)}{h}$$

and

$$(D^\alpha f)(c) = \lim_{h\to 0^-} \frac{f\Big(c + h\cos\big((1-\alpha)c\big)\Big) - f(c)}{h}.$$

There are two cases:

- In the first interval, $\cos\big((1-\alpha)t\big)$ is nonnegative, so the first limit is nonnegative and the second nonpositive, implying $(D^\alpha f)(c) = 0$.
- In the second interval, $\cos\big((1-\alpha)t\big)$ is nonpositive, and a similar argument yields $(D^\alpha f)(c) = 0$.

□

Theorem 2.14. ***(Mean Value Theorem for α-Trigonometric Differentiable Functions)***
Let $\alpha \in (0,1]$, $a > 0$, and let $f : [a,b] \to \mathbb{R}$ satisfy:

(i) $[a,b] \subseteq \left(\frac{(n-1)\pi}{2(1-\alpha)}, \frac{n\pi}{2(1-\alpha)}\right)$ for some $n \in \mathbb{N}$;

(ii) f is continuous on $[a,b]$;

(iii) f is α-trigonometric differentiable on (a,b).

Then there exists $c \in (a,b)$ such that

$$(D^\alpha f)(c) = \frac{f(b) - f(a)}{\frac{1}{1-\alpha}\Big(\ln\Big|\sec\big((1-\alpha)b\big) + \tan\big((1-\alpha)b\big)\Big| - \ln\Big|\sec\big((1-\alpha)a\big) + \tan\big((1-\alpha)a\big)\Big|\Big)}.$$

Equivalently,

$$(D^\alpha f)(c) = \frac{(1-\alpha)\big(f(b) - f(a)\big)}{\ln\Big|\sec\big((1-\alpha)b\big) + \tan\big((1-\alpha)b\big)\Big| - \ln\Big|\sec\big((1-\alpha)a\big) + \tan\big((1-\alpha)a\big)\Big|}.$$

Proof. Define the auxiliary function

$$g(t) = f(t) - f(a) - K\,h(t),$$

where

$$K = \frac{f(b) - f(a)}{\frac{1}{1-\alpha}\Big(\ln\Big|\sec\big((1-\alpha)b\big) + \tan\big((1-\alpha)b\big)\Big| - \ln\Big|\sec\big((1-\alpha)a\big) + \tan\big((1-\alpha)a\big)\Big|\Big)}$$

and

$$h(t) = \frac{1}{1-\alpha}\Big(\ln\Big|\sec\big((1-\alpha)t\big) + \tan\big((1-\alpha)t\big)\Big| - \ln\Big|\sec\big((1-\alpha)a\big) + \tan\big((1-\alpha)a\big)\Big|\Big).$$

Notice that $h(a) = 0$ and $h(b) = \frac{f(b)-f(a)}{K}$, so that $g(a) = g(b) = 0$. By Rolle's theorem (Theorem 2.13), there exists some $c \in (a, b)$ such that $(D^\alpha g)(c) = 0$. Since from Corollary 2.6 we have $(D^\alpha h)(c) = 1$, it follows that $(D^\alpha f)(c) = K$. □

3 Trigonometric Fractional Integral

The trigonometric fractional integral is defined as follows:

Definition 3.1. Let $\alpha \in (0, 1)$ and let f be defined on $(0, t]$. The α-trigonometric fractional integral of f is given by

$$I^\alpha f(t) = \int_0^t \sec\big((1-\alpha)s\big)\, f(s)\, ds,$$

with the integral interpreted in the Riemann sense.

Theorem 3.2. *Let* $\alpha \in (0, 1)$ *and let* $f : [0, b] \to \mathbb{R}$ *be an* α*-trigonometric differentiable function such that* $\sec\big((1-\alpha)t\big) f(t)$ *is continuous on* $[0, b]$*. Then*

$$D^\alpha\Big(I^\alpha f(t)\Big) = f(t).$$

Proof. Since $\sec\big((1-\alpha)t\big) f(t)$ is continuous, by differentiating under the integral sign we obtain

$$D^\alpha\Big(I^\alpha f(t)\Big) = \cos\big((1-\alpha)t\big)\, \frac{d}{dt}\int_0^t \sec\big((1-\alpha)s\big) f(s)\, ds.$$

By the Fundamental Theorem of Calculus,

$$\frac{d}{dt}\int_0^t \sec\big((1-\alpha)s\big) f(s)\, ds = \sec\big((1-\alpha)t\big) f(t),$$

so that

$$D^\alpha\Big(I^\alpha f(t)\Big) = \cos\big((1-\alpha)t\big)\, \sec\big((1-\alpha)t\big)\, f(t) = f(t).$$

□

Theorem 3.3. *Let* $\alpha \in (0, 1)$ *and let* $f : [0, b] \to \mathbb{R}$ *be differentiable on* $(0, b)$*. Then*

$$I^\alpha\Big(D^\alpha f(t)\Big) = f(t) - f(0).$$

Proof. Since f is differentiable on $(0, b)$, we have, by the earlier result, $D^\alpha f(t) = \cos\big((1-\alpha)t\big) f'(t)$. Thus,

$$I^\alpha\Big(D^\alpha f(t)\Big) = \int_0^t \sec\big((1-\alpha)s\big)\, \cos\big((1-\alpha)s\big)\, f'(s)\, ds = \int_0^t f'(s)\, ds = f(t) - f(0).$$

□

3.1 Applications

We now illustrate the use of these operators by solving some fractional differential equations.

Example 3.4. The general form of a linear fractional differential equation of order α is given by

$$y^{(\alpha)} + p(t)\, y = f(t),$$

where $p(t)$ and $f(t)$ are α-trigonometric differentiable functions.

Solution. This equation can be rewritten, using the relation $y^{(\alpha)} = \cos\big((1-\alpha)t\big)y'(t)$, as

$$\cos\big((1-\alpha)t\big)\, y'(t) + p(t)y = f(t).$$

Dividing by $\cos\big((1-\alpha)t\big)$, we obtain a standard first-order linear differential equation:

$$y'(t) + p(t)\sec\big((1-\alpha)t\big)\, y = \sec\big((1-\alpha)t\big)\, f(t).$$

Its general solution is given by

$$y(t) = \frac{1}{\mu(t)}\, I^{\alpha}\Big(f(t)\,\mu(t)\Big),$$

where the integrating factor is

$$\mu(t) = \exp\Big(I^{\alpha}\big(p(t)\big)\Big).$$

■

Example 3.5. Solve the differential equation

$$y^{(\alpha)} - (1-\alpha)\sin\big((1-\alpha)t\big)\, y = (1-\alpha)\sec\big((1-\alpha)t\big).$$

Solution. Rewriting the equation in terms of the standard derivative (using $y^{(\alpha)} = \cos\big((1-\alpha)t\big)\, y'$), we obtain

$$\cos\big((1-\alpha)t\big)y'(t) - (1-\alpha)\sin\big((1-\alpha)t\big)y = (1-\alpha)\sec\big((1-\alpha)t\big).$$

Dividing throughout by $\cos\big((1-\alpha)t\big)$ yields

$$y'(t) - (1-\alpha)\tan\big((1-\alpha)t\big)y = (1-\alpha)\sec^2\big((1-\alpha)t\big).$$

The integrating factor is

$$\begin{aligned}\mu(t) &= \exp\Big(-\int (1-\alpha)\tan\big((1-\alpha)s\big)\, ds\Big)\\ &= \exp\Big(\ln\big(\cos\big((1-\alpha)t\big)\big)\Big) = \cos\big((1-\alpha)t\big).\end{aligned}$$

Thus, the solution is given by

$$\begin{aligned}y(t) &= \frac{1}{\mu(t)}\, I^{\alpha}\Big((1-\alpha)\sec^2\big((1-\alpha)t\big)\,\mu(t)\Big)\\ &= \sec\big((1-\alpha)t\big)\int (1-\alpha)\sec^2\big((1-\alpha)t\big)\, dt\\ &= \sec\big((1-\alpha)t\big)\tan\big((1-\alpha)t\big) + C.\end{aligned}$$

■

Example 3.6. Consider the differential equation

$$y^{(\alpha)} + (1-\alpha)\sec\big((1-\alpha)t\big)\, y = (1-\alpha)\frac{\sec\big((1-\alpha)t\big)}{y}.$$

Solution. Using the substitution $z = y^2$, from the chain rule we get $z^{(\alpha)} = 2y\, y^{(\alpha)}$. Then the equation becomes

$$z^{(\alpha)} + 2(1-\alpha)\sec\big((1-\alpha)t\big)\, z = 2(1-\alpha)\sec\big((1-\alpha)t\big).$$

The integrating factor is computed as

$$\begin{aligned}\mu(t) &= \exp\Big(I^{\alpha}\big(2(1-\alpha)\sec\big((1-\alpha)t\big)\big)\Big)\\ &= \exp\Big(\int 2(1-\alpha)\sec^2\big((1-\alpha)t\big)\, dt\Big) = \exp\Big(2\tan\big((1-\alpha)t\big)\Big).\end{aligned}$$

Thus, the general solution for z is

$$z(t) = \frac{1}{\mu(t)}\, I^{\alpha}\Big(2(1-\alpha)\sec\big((1-\alpha)t\big)\,\mu(t)\Big) = \exp\Big(-2\tan\big((1-\alpha)t\big)\Big)\Big(\exp\big(2\tan\big((1-\alpha)t\big)\big)+C\Big).$$

Since $z = y^2$, we obtain

$$y(t) = \pm\sqrt{\exp\Big(-2\tan\big((1-\alpha)t\big)\Big)\Big(\exp\big(2\tan\big((1-\alpha)t\big)\big)+C\Big)}.$$

■

4 Conclusion and Remarks

In this paper, we introduced a new conformable-type fractional derivative defined via the cosine function. The proposed operator preserves key properties—including linearity and the product, quotient, and chain rules—making it a robust tool for further theoretical development. We derived fractional versions of Rolle's theorem and the mean value theorem, and developed a corresponding fractional integral operator. The applicability of these operators was demonstrated through the solutions of several fractional differential equations.

Future research may focus on applying these operators in modeling phenomena in physics, engineering, and biology, as well as exploring numerical approximations for this new derivative.

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