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EXISTENCE OF SOLUTION OF A FRACTIONAL INTEGRAL EUATIONS VIA MEASURE OF NONCOMPACTNESS

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Abstract *In this article, we study the existence of solution of a fractional integral equation involving both Riemann-Liouville and Hadamard fractional integral equations by applying Petryshyn's fixed point theorem. We also justify our main result with the help of a suitable example.*

1 Introduction

Fractional calculus is a branch of mathematical analysis which really examines many possible interpretations for representing the real number powers or complex number powers of the differentiation operator. In applied science and scientific analysis, a fractional derivative is a derivative of any real or complex non-integer order. The methods and interpretations of fractional calculus have evolved dramatically over the 19th and early 20th centuries, and innumerable researchers have supplied conceptions for fractional derivatives and integrals.
The fixed point theory (in short, FPT), which was first suggested by Stephen Banach has had enormous attention among the scientific community. The fixed point theory has remarkable application in the area of phase-transition theory, health care, integral equation, differential equation, etc. Seeking solutions for fractional differentials and integral equations using the fixed point theory had irresistible attention among researchers.

Many researchers solving different types of integral equations to over come the different real life situations support of FPT and MNC (e.g., one can see [1, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 16, 17, 19, 20, 21, 22, 23, 24, 25] and references among them).
In this article, we will use the following notations:
- $\mathbf{E}$ *: Banach space with the norm* $\| \, . \, \|_{\mathbf{E}}$*;*
- $B[\theta, \kappa]$*: closed ball with center* θ *and radius* κ *in* $\mathbf{E}$*;*
- $\bar{\Omega}$ *the closure of* Ω;
- *Conv*Ω *the convex closure of* Ω;
- $\mathcal{M}_{\mathbf{E}}$ *the family of all nonempty and bounded subsets of* $\mathbf{E}$;
- $\mathcal{N}_{\mathbf{E}}$ *the subfamily consisting of all relatively compact sets;*
- $\mathbb{R}$ *the set of real numbers;*
- $\mathbb{R}^+ = [0, \infty)$.

The organization of the paper is as follows
In this paper, we discussed definition and some preliminaries of MNC with Petryshyn's fixed point theorem in section 2. Next, the existence of a solution of of a fractional integral equation involving both Riemann-Liouville and Hadamard fractional integral equations by applying

Petryshyn's fixed point theorem in section 3. In section 4 , we have given an example which prove the efficiency of our result. Finally, we have given the conclusion of this paper in section 5.

Definition 1.1. [10] A function $\mu : \mathcal{M}_{\mathbf{E}} \to \mathbb{R}^+$ is said to be a measure of noncompactness in $\mathbf{E}$ if it satisfies the following conditions:

(i) for $\Omega \in \mathcal{M}_{\mathbf{E}}$ and $\mu(\Omega) = 0$ gives Ω is precompact.

(ii) $\ker \mu = \{\Omega \in \mathcal{M}_{\mathbf{E}} : \mu(\Omega) = 0\}$ is nonempty and $\ker \mu \subset \mathcal{N}_{\mathbf{E}}$.

(iii) $\Omega \subseteq \Omega_1 \implies \mu(\Omega) \leq \mu(\Omega_1)$.

(iv) $\mu(\bar{\Omega}) = \mu(\Omega)$.

(v) $\mu(Conv\Omega) = \mu(\Omega)$.

(vi) $\mu(\varpi\Omega + (1-\varpi)\Omega_1) \leq \varpi\mu(\Omega) + (1-\varpi)\mu(\Omega_1)$ for $\varpi \in [0,1]$.

(vii) if $\Omega_\sigma \in \mathcal{M}_{\mathbf{E}}, \Omega_\sigma = \bar{\Omega}_\sigma, \Omega_{\sigma+1} \subset \Omega_\sigma$ for $\sigma \in \mathbb{N}$ and $\lim_{\sigma\to\infty} \mu(\Omega_\sigma) = 0$ then $\Omega_\infty = \bigcap_{\sigma=1}^{\infty} \Omega_\sigma \neq \phi$.

The family $\ker \mu$ *is said to be the kernel of measure* μ. *Also,* $\Omega_\infty \in ker\mu$ *and* $\mu(\Omega_\infty) \leq \mu(\Omega_\sigma)$ *for any* σ, *so* $\mu(\Omega_\infty) = 0$. *This gives* $\Omega_\infty \in ker\mu$.

Theorem 1.2. [14] Let $F : E \to E$ be a continuous function of E which fulfills the condition if for all $B \subset E$ with B bounded, $F(B)$ is bounded and $\hat{\alpha}(FB) \leq k\hat{\alpha}(B), k \in (0,1)$. If $\hat{\alpha}(FB) < \hat{\alpha}(B), \ \forall\ \hat{\alpha}(B) > 0$, then F is called densifying or condensing map. A k-set contraction is condensing but converse is not true.

Theorem 1.3. [15] Let $F : B_\rho \to E$ be a condensing function which fulfills the boundary condition $F(z) = kz, \ for\ some\ z \in \partial B_\rho$ then $k \leq 1$. Then $\mathbf{F}(F)$ in B_ρ is nonempty, where $\mathbf{F}(F)$ is the set of fixed points of F.

1.1 Measure of noncompactness on $C([0,1])$

Consider the space $\mathbf{E} = C(I)$ *which consists of the set of real continuous functions on* I, *where* $I = [0,1]$. *Then* $\mathbf{E}$ *is a Banach space with the norm*

$$\| \varrho \| = \sup\{|\varrho(\varsigma)| : \varsigma \in I\}, \ \varrho \in \mathbf{E}.$$

Let $\phi \neq \varUpsilon \subseteq \mathbf{E}$ *be fixed bounded, for* $\varrho \in \varUpsilon$ *and* $\epsilon > 0$, *denote by* $\omega(\varrho, \epsilon)$ *the modulus of the continuity of* ϱ *i.e.*

$$\omega(\varrho, \epsilon) = \sup\{|\varrho(\varsigma_1) - \varrho(\varsigma_2)| : \varsigma_1, \varsigma_2 \in I, |\varsigma_1 - \varsigma_1| \leq \epsilon\}.$$

Further we define

$$\omega(\varUpsilon, \epsilon) = \sup\{\omega(\varrho, \epsilon) : \varrho \in \varUpsilon\}; \ \omega_0(\varUpsilon) = \lim_{\epsilon\to 0} \omega(\varUpsilon, \epsilon).$$

It is well-known that the function ω_0 *is a MNC in* $\mathbf{E}$ *such that the Hausdorff MNC* χ *is given by* $\chi(\varUpsilon) = \frac{1}{2}\omega_0(\varUpsilon)$ *(see [2]).*

2 Main Result

The Riemann-Liouville [18] fractional integral of function $\varTheta$ *is defined by*

$$\left({}^R I_a^{\varpi} \varTheta\right)(\varsigma) = \frac{1}{\Gamma(\varpi)} \int_a^{\varsigma} (\varsigma - \eta)^{\varpi - 1} \varTheta(\eta) d\eta,$$

where $0 < \varpi < 1, \ 0 < a < \infty, T > 0, \varsigma \in [a, T]$.
In [18], the Hadamard fractional integral of function Θ *is defined by*

$$\left({}^H J_{\varsigma}^{\varpi} \Theta\right)(\varsigma) = \frac{1}{\Gamma(\varpi)} \int_a^{\varsigma} \left(\ln \frac{\varsigma}{\eta}\right)^{\varpi - 1} \frac{\Theta(\eta)}{\eta} d\eta,$$

where $0 < \varpi < 1; 1 < a < \infty, \varsigma \in [a, T]$.
In this part, we study the following fractional integral equation

$$\Theta(\varsigma) = \Lambda(\varsigma, l\left(\varsigma, \Theta(\varsigma)\right), \left({}^{R}I_{\varsigma}^{\varpi}\Theta(\varsigma)\right), \left({}^{H}J_{\varsigma}^{\varpi}\Theta(\varsigma)\right)), \tag{2.1}$$

where $0 \leq \varpi \leq 1,\ \varsigma \in I = [2, 3]$.
Let $B_{d_0} = \{\Theta(s) \in \mathbf{E} : \| \Theta \| \leq d_0\}$.
We consider the following assumptions to solve the Eq. (2.1):

(A) $\Lambda : I \times \mathbb{R}^3 \to \mathbb{R},\ l : I \times \mathbb{R} \to \mathbb{R}$ *be continuous and there exists constants* $\sigma_1,\ \sigma_2\ \sigma_3\ \sigma_4 \geq 0$ *satisfying*

$$\left|\Lambda(\varsigma, l, I_1, I_2) - \Lambda(\varsigma, \bar{l}, \bar{I}_1, \bar{I}_2)\right| \leq \sigma_1 \left|l - \bar{l}\right| + \sigma_2 \left|I_1 - \bar{I}_1\right| + \sigma_3 \left|I_2 - \bar{I}_2\right|,\ \varsigma \in I,\ l, I_1, I_2, \bar{l}, \bar{I}_1, \bar{I}_2 \in \mathbb{R}$$

and

$$|l(\varsigma, J_1) - l(\varsigma, J_2)| \leq \sigma_4 |J_1 - J_2|, J_1, J_2 \in \mathbb{R}.$$

(B) There exists two positive real numbers $\mathbb{M}$ *and* $\mathbb{K}$ *such that*

$$\left|{}^{H}J_{\varsigma_2}^{\varpi}\left[\Theta(\varsigma_2)\right] - {}^{H}J_{\varsigma_1}^{\varpi}\left[\Theta(\varsigma_1)\right]\right| \leq \mathbb{M}\left|\Theta(\varsigma_2) - \Theta(\varsigma_1)\right|$$

and

$$\left|\left({}^{R}I_{a}^{\varpi}\Theta\right)(\varsigma_2) - \left({}^{R}I_{a}^{\varpi}\Theta\right)(\varsigma_1)\right| \leq \mathbb{K}\left|\Theta(\varsigma_2) - \Theta(\varsigma_1)\right|$$

(C) There exists $d_0 > 0$ *satisfying*

$$\bar{\Lambda} = \sup\left\{|\Lambda(\varsigma, l, I_1, I_2)| : \varsigma \in I, l \in [-L, L], I_1 \in [-K_R, K_R], I_2 \in [-K_H, K_H]\right\} \leq d_0$$

and $\sigma_3\mathbb{M} + \sigma_2\mathbb{K} < 1$,

$$L = \sup\left\{|l(\varsigma, \Theta)| : \varsigma \in I,\ \Theta \in [-d_0, d_0]\right\},$$

Also let $B_{r_0} = \{x \in C(I) : \| x \| \leq r_0\}$.

Theorem 2.1. If the conditions (A),(B) and (C) hold then the Equation (2.1) has a solution in $\mathbf{E} = C(I)$.

Proof. The operator $\Upsilon : B_{d_0} \to \mathbf{E}$ is defined as follows

$$(\Upsilon\Theta)(\varsigma) = \Lambda(\varsigma, l\left(\Theta(\varsigma)\right), \left({}^{R}I_{a}^{\varpi}\Theta\right)(\varsigma), \left({}^{H}J_{\varsigma}^{\varpi}\Theta\right)(\varsigma)).$$

Step 1: We prove that Υ is continuous on B_{d_0}.
Let $\epsilon > 0$ and $\Theta, \bar{\Theta} \in B_{d_0}$ such that $\| \Theta - \bar{\Theta} \| < \epsilon$, we have

$$\begin{aligned}
&\left|(\Upsilon\Theta)(\varsigma) - (\Upsilon\bar{\Theta})(\varsigma)\right| \\
&\leq \left|\Lambda(\varsigma, l\left(\Theta(\varsigma)\right), \left({}^{R}I_{a}^{\varpi}\Theta\right)(\varsigma), \left({}^{H}J_{\varsigma}^{\varpi}\Theta\right)(\varsigma)) - \Lambda(\varsigma, l\left(\overline{\Theta}(\varsigma)\right), \left({}^{R}I_{a}^{\varpi}\bar{\Theta}\right)(\varsigma), \left({}^{H}J_{\varsigma}^{\varpi}\overline{\Theta}\right)(\varsigma))\right| \\
&\leq \sigma_1\left|l(\varsigma, \Theta(\varsigma)) - l(\varsigma, \bar{\Theta}(\varsigma))\right| + \sigma_2\left|\left({}^{R}I_{a}^{\varpi}\bar{\Theta}\right)(\varsigma) - \left({}^{R}I_{a}^{\varpi}\bar{\Theta}\right)(\varsigma)\right| + \sigma_3\left|{}^{H}J_{\varsigma}^{\varpi}\Theta - {}^{H}J_{\varsigma}^{\varpi}\left(\bar{\Theta}\right)(\varsigma)\right|.
\end{aligned}$$

Also,

$$\begin{aligned}
&\left|\left({}^{R}I_{a}^{\varpi}\bar{\Theta}\right)(\varsigma) - \left({}^{R}I_{a}^{\varpi}\bar{\Theta}\right)(\varsigma)\right| \\
&= \left|\frac{1}{\Gamma(\varpi)}\int_a^{\varsigma}(\varsigma - \eta)^{\varpi - 1}\left(\Theta(\eta) - \bar{\Theta}(\eta)\right)d\eta\right| \\
&< \frac{\epsilon}{\varpi\Gamma(\varpi)},
\end{aligned}$$

$$\text{implies that}\quad \left|\left({}^{\mathrm{R}}\mathrm{I}_{\mathrm{a}}^{\varpi}\bar{\Theta}\right)(\varsigma) - \left({}^{\mathrm{R}}\mathrm{I}_{\mathrm{a}}^{\varpi}\bar{\Theta}\right)(\varsigma)\right| < \frac{\epsilon}{\varpi\Gamma(\varpi)},$$

and

$$\begin{aligned}
&\left|{}^{H}J_{\varsigma}^{\varpi}\Theta - {}^{H}J_{\varsigma}^{\varpi}\left(\bar{\Theta}\right)(\varsigma)\right| \\
&= \left|\frac{1}{\Gamma(\varpi)}\int_{a}^{\varsigma}\left[[\ln(\frac{\varsigma}{\eta})\right]^{\varpi-1}\frac{\left(\Theta(\eta)-\bar{\Theta}(\eta)\right)}{\eta}d\eta\right| \\
&< \left|\frac{\epsilon}{\Gamma(\varpi)}\int_{a}^{\varsigma}\left[[\ln(\frac{\varsigma}{\eta})\right]^{\varpi-1}\frac{\left(\Theta(\eta)-\bar{\Theta}(\eta)\right)}{\eta}d\eta\right| \\
&= \frac{\epsilon}{\varpi\Gamma(\varpi)}\left[[\ln(\frac{\varsigma}{\eta})\right]^{\varpi} \\
&\le \frac{\epsilon}{\varpi\Gamma(\varpi)}\left[[\ln(\frac{3}{\eta})\right]^{\varpi}
\end{aligned}$$

this gives

$$\left|{}^{H}J_{\varsigma}^{\varpi}(\Theta)(\varsigma) - {}^{H}J_{\varsigma}^{\varpi}\left(\bar{\Theta}\right)(\varsigma)\right| < \frac{\epsilon}{\varpi\Gamma(\varpi)}\left[\ln(\frac{3}{\eta})\right]^{\varpi}.$$

Hence $\| \Theta - \bar{\Theta} \| < \epsilon$ gives

$$\left|(\varUpsilon\Theta)(\varsigma) - \left(\varUpsilon\bar{\Theta}\right)(\varsigma)\right| < \sigma_1\sigma_4\epsilon + \sigma_2\frac{\epsilon}{\varpi\Gamma(\varpi)} + \sigma_3\frac{\epsilon}{\varpi\Gamma(\varpi)}\left[\ln(\frac{3}{\eta})\right]^{\varpi}.$$

As $\epsilon \to 0$ we get $\left|(\varUpsilon\Theta)(\varsigma) - \left(\varUpsilon\bar{\Theta}\right)(\varsigma)\right| \to 0$. This shows that $\varUpsilon$ is continuous on B_{d_0}.

Step 2: An estimate of Δ with respect to ω_0, assume $\phi \neq \varOmega \subseteq B_{r_0}$. For an arbitrary $\epsilon > 0$ and choose $\Theta \in \varOmega$ and $\varsigma_1, \varsigma_2 \in I$ such that $|\varsigma_2 - \varsigma_1| \le \epsilon$ and $\varsigma_2 \ge \varsigma_1$.
Now,

$$\begin{aligned}
&|(\varUpsilon\Theta)(\varsigma_2) - (\varUpsilon\Theta)(\varsigma_1)| \\
&= \left|\Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_2), {}^{H}J_{\varsigma_2}^{\varpi}[\Theta(\varsigma_2)]) - \Lambda(\varsigma_1, l(\varsigma_1,\Theta(\varsigma_1)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)])\right| \\
&\le \left|\Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I^{\varpi}\varTheta\right)(\varsigma_2), {}^{H}J_{\varsigma_2}^{\varpi}[\Theta(\varsigma_2)]) - \Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_2), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)])\right| \\
&+ \left|\Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_2), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)]) - \Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)])\right| \\
&+ \left|\Lambda(\varsigma_2, l(\varsigma_2,\Theta(\varsigma_2)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)]) - \Lambda(\varsigma_2, l(\varsigma_1,\Theta(\varsigma_1)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)])\right| \\
&+ \left|\Lambda(\varsigma_2, l(\varsigma_1,\Theta(\varsigma_1)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)]) - \Lambda(\varsigma_1, l(\varsigma_1,\Theta(\varsigma_1)), \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1), {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)])\right| \\
&\le \sigma_3\left|{}^{H}J_{\varsigma_2}^{\varpi}[\Theta(\varsigma_2)] - {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)]\right| + \sigma_2\left|\left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_2) - \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1)\right| \\
&+ \sigma_1|l(\varsigma_2,\Theta(\varsigma_2)) - l(\varsigma_1,\Theta(\varsigma_1))| + \omega_\Lambda(I,\epsilon),
\end{aligned}$$

where

$$\omega_\Lambda(I,\epsilon) = \sup\{|\Lambda(\varsigma_2, l, I_1, I_2) - \Lambda(\varsigma_1, l, I_1, I_2)| : |\varsigma_2 - \varsigma_1| \le \epsilon; \varsigma_1, \varsigma_2 \in I.\}.$$

Also,

$$\omega_l(I,\epsilon) = \sup\{|l(\varsigma_2,\Theta(\varsigma_2)) - l(\varsigma_1,\Theta(\varsigma_1))| : \varsigma_1,\varsigma_2 \in I; \Theta(\varsigma_1),\Theta(\varsigma_2) \in [-d_0, d_0]\}$$

and

$$\begin{aligned}
&\left|{}^{H}J_{\varsigma_2}^{\varpi}[\Theta(\varsigma_2)] - {}^{H}J_{\varsigma_1}^{\varpi}[\Theta(\varsigma_1)]\right| \\
&\le \mathbb{M}|\Theta(\varsigma_2) - \Theta(\varsigma_1)| \\
&\le \mathbb{M}\omega(\Theta,\epsilon)
\end{aligned}$$

$$\begin{aligned}
&\left|\left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_2) - \left({}^{R}I_a^{\varpi}\varTheta\right)(\varsigma_1)\right| \\
&\le \mathbb{K}|\Theta(\varsigma_2) - \Theta(\varsigma_1)| \\
&\le \mathbb{K}\omega(\Theta,\epsilon).
\end{aligned}$$

Hence,

$$\begin{aligned}&|(\Upsilon\Theta)(\varsigma_2)-(\Upsilon\Theta)(\varsigma_1)|\\&\le \sigma_3\left[\mathbb{M}\omega(\Theta,\epsilon)\right]+\sigma_2\left[\mathbb{K}\omega(\Theta,\epsilon)\right]+\sigma_1\omega_l(I,\epsilon)+\omega_\Lambda(I,\epsilon),\end{aligned}$$

i.e.

$$\omega(\Upsilon\Theta,\epsilon)\le\omega(\Theta,\epsilon)\left[\sigma_3\mathbb{M}+\sigma_2\mathbb{K}\right]+\sigma_1\omega_l(I,\epsilon)+\omega_\Lambda(I,\epsilon).$$

By the uniform continuity of l,Λ, we have as $\epsilon\to 0$ gives $\omega_l(I,\epsilon)\to 0$ and $\omega_\Lambda(I,\epsilon)\to 0$. Taking $\sup\limits_{\Theta\in\Omega}$ and $\epsilon\to 0$ we get,

$$\omega_0(\Upsilon\Omega)\le\left[\sigma_3\mathbb{M}+\sigma_2\mathbb{K}\right]\omega_0(\Omega),$$

which shows that Υ is condensing map.

Now, if $\Theta\in\partial B_{d_0}$ and $\Upsilon\Theta=k\Theta$ then $\|\Upsilon\Theta\|=k\|\Theta\|=kd_0$ and by assumption (B),

$$|(\Upsilon\Theta)(\varsigma)|=\left|\Lambda(\varsigma,l\left(\varsigma,\Theta(\varsigma)\right),\left({}^RI_a^{\varpi}\Theta\right)(\varsigma),{}^HJ_\varsigma^{\varpi}\left[\Theta(\varsigma)\right])\right|\le d_0,\ \varsigma\in I$$

hence $\|\Upsilon\Theta\|\le d_0$ which implies $k\le 1$.

Thus by Petryshyn's fixed point theorem (Theorem 1.3), Υ has a fixed point in $\Omega\subseteq B_{d_0}$ i.e. equation (2.1) has a solution in $\mathbf{E}$. □

3 Example

Example 3.1. Consider the following equation

$$\Theta(\varsigma)=\varsigma^2+\frac{\sqrt{\Theta(\varsigma)}}{7+\varsigma^5+\varsigma^7}+\frac{\left({}^RI_0^{\frac12}\Theta\right)(\varsigma)}{7}+\frac{{}^HI_0^{\frac12}\left[\Theta(\varsigma)\right]}{14}\tag{3.1}$$

for $\varsigma\in[2,3]=I$.

Here

$$\left({}^RI_0^{\frac12}\Theta\right)(\varsigma)=\frac{1}{\Gamma(\frac12)}\int_0^\varsigma(\varsigma-\eta)^{-\frac12}\Theta(\eta)d\eta,$$

and

$$\left({}^HI_0^{\frac12}\Theta\right)(\varsigma)=\frac{1}{\Gamma(\frac12)}\int_0^\varsigma\left(\log\frac{\varsigma}{\eta}\right)^{-\frac12}\frac{\Theta(\eta)}{\eta}d\eta,$$

Also, $\Lambda(\varsigma,l,I_1,I_2)=\varsigma^2+l+\frac{I_1}{7}+\frac{I_2}{14}$ *and* $l(\varsigma,\Theta)=\frac{\sqrt{\Theta(\varsigma)}}{7+\varsigma^5+\varsigma^7}$*. It is trivial that both* $\Lambda,\ l$ *are continuous satisfying*

$$|l(\varsigma,J_1)-l(\varsigma,J_2)|\le\frac{|J_1-J_2|}{7}$$

and

$$\left|\Lambda(\varsigma,l,I_1,I_2)-\Lambda(\varsigma,\bar{l},\bar{I}_1,\bar{I}_2)\right|\le\left|l-\bar{l}\right|+\frac{1}{7}\left|I_1-\bar{I}_1\right|+\frac{1}{14}\left|I_2-\bar{I}_2\right|.$$

Therefore, $\sigma_1=1,\ \sigma_2=\frac17,\ \sigma_3=\frac{1}{14}$*. If* $\|\Theta\|\le d_0$ *then*

$$K_R=\frac{d_0}{\Gamma(\omega)},\ K_H=2d_0,\ L=\frac{d_0}{7}.$$

Further,

$$|\Lambda(\varsigma,l,I_1,I_2)|\le\frac{d_0}{7}+\frac{1}{7}\frac{d_0}{\Gamma(\frac12)}+\frac{2d_0}{14}=\frac{d_0}{7}\left(2+\frac{1}{\Gamma(\frac12)}\right)\le d_0.$$

If we choose $d_0=3$ *then*

$$L=\frac{3}{7},\ K_R=1.70,\ K_H=6$$

which gives

$$\bar{\Lambda}\le 3,\ \sigma_2+\sigma_3\approx 0.214<1.$$

We observe that all the assumption from $(A)-(C)$ *of Theorem 2.1 are satisfied. By Theorem 2.1 we concluded that the equation (3.1) has a solution in* $\mathbf{E}$.

4 Conclusion remarks

In the present article, we have introduced a new fractional integral operator involving both Riemann-Liouville and Hadamard fractional integral equations. Then, we have established the existence solution of a fractional integral equation involving both Riemann-Liouville and Hadamard fractional integral equations by applying Petryshyn's fixed point theorem using the technique of MNC. Finally, we have verified our result by a suitable example.

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