Sz*á*sz-Gamma-Schurer operators linking Adjoint Bernoulli's polynomials

N. Rao, S. Kumar, A. Srivastava, A. K. Yadav, M. Rani and U. R. Prajapati

Corresponding Author: N. Rao

Abstract. This research work introduce a new connection of adjoint Bernoulli's polynomials and gamma function as a new sequence of linear positive operators denoted by $\{\tilde{G}_{r+p,\lambda}(.;.)\}_1^\infty$. Further, convergence properties of these sequences of operators, i.e., $\{\tilde{G}_{r+p,\lambda}(.;.)\}_1^\infty$ are investigated in various functional spaces with the aid of Korovkin theorem, Voronovskaja type theorem, first order of modulus of continuity, second order of modulus of continuity, Peetre's K-functional, Lipschitz condition etc.

1 Introduction

Approximation theory indeed plays a crucial role across various disciplines, providing a framework to represent complex functions with simpler ones. Its applications span from mathematics to engineering, including fields like computational science, data analysis, and computer graphics. In computational aspects, approximation theory aids in describing geometric shapes and solving differential equations. It forms the backbone of numerical analysis, where it helps in devising efficient algorithms for solving mathematical problems numerically. Moreover, in applied mathematics, approximation theory contributes to areas like control theory, where control points and control nets are utilized to study of parametric curves and surfaces. These concepts are fundamental in designing control systems for various engineering applications ([1], [2]). It enables the representation of curves and surfaces using simpler mathematical constructs, facilitating tasks like rendering realistic images and solving symbolic equations efficiently. Many scientists in medical sciences and others are also working in terms of these sequences of ([3], [4], [5]).

In 1715, an English mathematician named as Taylor, who presented a finite or infinite sum of a finitely or infinitely differentiable function respectively which is known as Taylor series to approximate a class functions. The theory of Taylor series approximation was restricted for the finitely or infinitely differentiable function in terms of finite or infinite sum respectively. To remove the above restriction, i.e., differentiability of the function, Weierstrass (1885) [6] introduced an elegant result named as Weierstrass approximation theorem, i.e., every continuous function over the closed interval $[a, b]$ can be uniformly approximated by a polynomial function over $[a, b]$ with the desired accuracy. Several branches of approximation theory as numerical analysis, operator theory, wavelet analysis etc. are generated in the light of this theorem. But, the proof of this theorem was difficult and lengthy to understand. Since this theorem plays a central role in approximation theory. Therefore, several mathematicians worked on the proof of this theorem to make it short and understandable. One of the mathematician, Bernstein (1912) [7] presented a simplest and shortest proof of this theorem via binomial probability distribution

as:

$$B_r(\hbar; \mu) = \sum_{\nu=0}^{r} p_{r,\nu}(\mu)\hbar\left(\frac{\nu}{r}\right), \ r \in \mathbb{N}, \tag{1.1}$$

where $p_{r,\nu}(\mu) = \binom{r}{\nu}\mu^\nu(1-\mu)^{\nu-r}$. He found that $B_r(\hbar; .) \rightrightarrows \hbar$ for every bounded function $\hbar$ defined on $[0,1]$ where $\rightrightarrows$ denotes the convergence is uniform. In the past decade, many mathematicians have constructed various modifications of the operators defined by (1.1) to achieve better flexibility in approximation properties over bounded and unbounded intervals in various functional spaces, e.g., Aslan et al. ([8]-[10]), Ayman-Mursaleen et al. ([12], [13]), Mohiuddine et al. ([14], [15]), Mursaleen et al. ([16], [17]), Khan et al. [18], Nasiruzzaman [19], Braha et al. [20], Rao et al. ([21], [22]), Çetin et al. ([23], [24]) etc. (see also [34]-[39] and references therein).

In view of polynomials classes which is an active field of research as special function field of research, we recall a class of polynomial by Appell [27] termed as Appell polynomials $\{p_\nu(\mu)\}_{\nu=0}^{\infty}$ associated with the generating function as:

$$A(t)e^{\mu t} = \sum_{\nu=0}^{\infty} p_\nu(\mu)\frac{t^\nu}{\nu!}, \tag{1.2}$$

where $A(t) = \sum_{\nu=0}^{\infty} a_\nu \frac{t^\nu}{\nu!}, A(0) = 0$, which is an analytic function at $t = 0$ such that $a_\nu = p_\nu(0)$. Recently, Natalini et al. [28] gave Appell Bernaulli's polynomials choosing $A(t) = \frac{t}{e^t-1}$ in (1.2). Adjoint Bernoulli's polynomials choosing $A(t)$ as $\frac{1}{A(t)}$ in the generating function is denoted as $\{\beta_\nu(\mu)\}_{\nu=0}^{\infty}$ and defined by the exponential type generating function:

$$\frac{e^t - 1}{t}e^{\mu t} = \sum_{\nu=0}^{\infty} \frac{\beta_\nu(\mu)t^\nu}{\nu!}. \tag{1.3}$$

The adjoint Bernoulli's polynomials defined in (1.3) are positive on $[0, \infty)$.

Motivated with the above literature, we construct a new connection of adjoint Bernoulli's polynomials coupling gamma function as a new sequence of positive linear operator as:

$$\tilde{G}_{r+p,\lambda}(\hbar; \mu) = \sum_{\nu=0}^{\infty} g_\nu(a_{r+p}\mu) \int_0^\infty b_{c_{r+p},\nu}(\theta)\hbar(\theta)d\theta, \tag{1.4}$$

where $\{a_{r+p}\}_1^\infty$, $\{c_{r+p}\}_1^\infty$ are increasing sequences of real numbers such that $\lim_{r\to\infty} a_{r+p} = \lim_{r\to\infty} c_{r+p} = \infty$, $\frac{a_{r+p}}{c_{r+p}} = 1 + \frac{1}{c_{r+p}}$, $\hbar \in \mathbb{L}_\beta[0,\infty)$ (a space of Lebesgue measurable and bounded functions) and $g_\nu(a_{r+p}\mu) = \frac{e^{-a_{r+p}\mu}}{e-1}\frac{\beta_\nu(a_{r+p}\mu)}{\nu!}$, $b_{c_{r+p},\nu}^\lambda(\theta) = \frac{c_{r+p}^{\nu+\lambda+1}}{\Gamma(\nu+\lambda)}\theta^{\nu+\lambda}e^{-c_{r+p}\theta}$. Here, we discuss some preliminaries to discuss approximation properties of $\tilde{G}_{r+p,\lambda}(.;.)$ in (1.4) as:

Lemma 1.1. *For $\mu \in [0, \infty)$ and the generating function given by* (1.3)*, we have*

$$\sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} = e^{a_{r+p}\mu}(e-1),$$

$$\sum_{\nu=0}^{\infty} \nu \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} = a_{r+p}\mu e^{a_{r+p}\mu}(e-1) + e^{a_{r+p}\mu},$$

$$\sum_{\nu=0}^{\infty} \nu^2 \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} = a_{r+p}^2\mu^2 e^{a_{r+p}\mu}(e-1) + a_{r+p}\mu e^{a_{r+p}\mu}(e+1) + e^{a_{r+p}\mu}(e-1),$$

$$\sum_{\nu=0}^{\infty} \nu^3 \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} = a_{r+p}^3\mu^3 e^{a_{r+p}\mu}(e-1) + a_{r+p}^2\mu^2 e^{a_{r+p}\mu}3e + a_{r+p}\mu e^{a_{r+p}\mu}(4e-1)$$
$$+ e^{a_{r+p}\mu}(e+1),$$

$$\sum_{\nu=0}^{\infty} \nu^4 \frac{\beta_\nu(a_{r+p}\nu)}{\nu!} = a_{r+p}^4\mu^4 e^{a_{r+p}\mu}(e-1) + a_{r+p}^3\mu^3 e^{a_{r+p}\mu}(6e-2)$$
$$+ a_{r+p}^2\mu^2 e^{a_{r+p}\mu}(13e-1)$$
$$+ a_{r+p}\mu e^{a_{r+p}\mu}(11e+1) + e^{a_{r+p}\mu}(4e-1).$$

Lemma 1.2. *Let $\hbar_j(\theta) = \theta^j$, $j \in \{0, 1, 2, 3, 4\}$. Then, the following equalities hold:*

$$\tilde{G}_{r+p,\lambda}(1; \mu) = 1,$$

$$\tilde{G}_{r+p,\lambda}(\hbar_1, \mu) = \frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right),$$

$$\tilde{G}_{r+p,\lambda}(\hbar_2, \mu) = \left(\frac{a_{r+p}}{c_{r+p}}\right)^2\mu^2 + \frac{a_{r+p}}{c_{r+p}^2}(2e(\lambda+1) - 2\lambda)\mu$$
$$+ \frac{1}{c_{r+p}^2}\left(1 + \frac{2\lambda+1}{e-1} + \lambda^2 + \lambda\right),$$

$$\tilde{G}_{r+p,\lambda}(\hbar_3, \mu) = \left(\frac{a_{r+p}}{c_{r+p}}\right)^3\mu^3 + \frac{a_{r+p}^2}{c_{r+p}^3}\left[3e(\lambda+2) - 3(\lambda+1)\right]\mu^2$$
$$+ \frac{a_{r+p}}{c_{r+p}^3}[3\lambda^2(e-1) + 3\lambda(3e-1)$$
$$+ 3(3e-1)]\mu + \frac{1}{c_{r+p}^3}\left[\lambda^3 + 6\lambda^2 + 8\lambda + 3 + \frac{e+1}{e-1}\right],$$

$$\tilde{G}_{r+p,\lambda}(\hbar_4, \mu) = \left(\frac{a_{r+p}}{c_{r+p}}\right)^4\mu^4 + o\left(\frac{1}{c_{r+p}}\right).$$

Proof. To prove Lemma (1.2), we call operators $\tilde{G}_{r+p,\lambda}(.,.)$ in (1.4) as:

$$\tilde{G}_{r+p,\lambda}(\hbar_j; \mu) = \frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^{\infty} b_{c_{r+p},\nu}(\theta)\theta^j d\theta. \tag{1.5}$$

For $j = 0$, $\int_0^{\infty} b_{c_{r+p},\nu}(\theta)d\theta = 1$ which implies that

$$\tilde{G}_{r+p,\lambda}(\hbar_0; \mu) = \frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} = 1.$$

For $j = 1$,

$$\int_0^{\infty} b_{c_{r+p},\nu}(\theta)\theta d\theta = \frac{c_{r+p}^{\nu+\lambda+1}}{\Gamma(\nu+\lambda)} \int_0^{\infty} \theta^{\nu+\lambda+1}e^{-c_{r+p}\theta}d\theta$$
$$= \frac{c_{r+p}^{\nu+\lambda+1}}{\Gamma(\nu+\lambda)}\frac{\Gamma(\nu+\lambda+1)}{c_{r+p}^{\nu+\lambda+2}} = \frac{\nu+\lambda}{c_{r+p}}. \tag{1.6}$$

Clubbing equation (1.5) and equation (1.6), we yield

$$\tilde{G}_{r+p,\lambda}(\hbar_1; \mu) = \frac{1}{c_{r+p}} \left[\frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\nu \beta_\nu(a_{r+p}\mu)}{\nu!} \right] + \frac{\lambda}{c_{r+p}}.$$

In view of Lemma (1.1), we get

$$\tilde{G}_{r+p,\lambda}(\hbar_1, \mu) = \frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}} \left(\frac{\lambda(e-1)+1}{e-1} \right).$$

For $j = 2$,

$$\int_0^\infty b_{c_{r+p},\nu}(\theta)\theta^2 d\theta = \frac{c_{r+p}^{\nu+\lambda+1}}{\Gamma(\nu+\lambda)} \int_0^\infty \theta^{\nu+\lambda+2} e^{-c_{r+p}\theta} d\theta$$

$$= \frac{c_{r+p}^{\nu+\lambda+1}}{\Gamma(\nu+\lambda)} \frac{\Gamma(\nu+\lambda+2)}{c_{r+p}^{\nu+\lambda+3}} = \frac{(\nu+\lambda+1)(\nu+\lambda)}{c_{r+p}^2}. \tag{1.7}$$

Clubbing equation (1.5) and equation (1.7), we yield

$$\tilde{G}_{a_{r+p},\lambda}(\hbar_2; \mu) = \left[\frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \right] \frac{(\nu+\lambda+1)(\nu+\lambda)}{c_{r+p}^2}.$$

In view of Lemma (1.1), we get

$$\tilde{G}_{r+p,\lambda}(\hbar_2, \mu) = \left(\frac{a_{r+p}}{c_{r+p}} \right)^2 \mu^2 + \frac{a_{r+p}}{c_{r+p}^2}(2e(\lambda+1)-2\lambda)\mu + \frac{1}{c_{r+p}^2} \left(1 + \frac{2\lambda+1}{e-1} + \lambda^2 + \lambda \right).$$

Similarly, rest part of this Lemma can be proved very easily. □

Lemma 1.3. *For the sequence of operators presented by* (1.4) *and* $\hbar_j^\mu(\theta) = (\theta-\mu)^j$, *one has the following equalities:*

$$\tilde{G}_{r+p,\lambda}(\hbar_o^\mu; \mu) = 1,$$

$$\tilde{G}_{r+p,\lambda}(\hbar_1^\mu; \mu) = \frac{1}{c_{r+p}} \left(\mu + \frac{\lambda(e-1)+1}{e-1} \right),$$

$$\tilde{G}_{r+p,\lambda}(\hbar_2^\mu; \mu) = \frac{\mu^2}{c_{r+p}^2} + \frac{a_{r+p}}{c_{r+p}^2} \left(2e(\lambda+1) - 2\lambda - 2 \left(\frac{\lambda(e-1)+1}{e-1} \right) \right) \mu$$

$$+ \frac{1}{c_{r+p}^2} \left(1 + \frac{2\lambda+1}{e-1} + \lambda^2 + \lambda \right),$$

$$\tilde{G}_{r+p,\lambda}(\hbar_4^\mu; \mu) = o\left(\frac{1}{c_{r+p}^2} \right) \mu^2.$$

Proof. In the light of operators (1.4) and linearity property, we get

$$\tilde{G}_{r+p,\lambda}(\hbar_o^\mu; \mu) = \tilde{G}_{r+p,\lambda}(1; \mu) = 1,$$

$$\tilde{G}_{r+p,\lambda}(\hbar_1^\mu; \mu) = \tilde{G}_{r+p,\lambda}(\theta-\mu; \mu) = \tilde{G}_{r+p,\lambda}(\hbar_1; \mu) - \mu\tilde{G}_{r+p,\lambda}(1; \mu),$$

$$\tilde{G}_{r+p,\lambda}(\hbar_2^\mu; \mu) = \tilde{G}_{r+p,\lambda}((\theta-\mu)^2; \mu) = \tilde{G}_{r+p,\lambda}(\hbar_2; \mu) - 2\mu\tilde{G}_{r+p,\lambda}(\hbar_1; \mu)$$

$$+ \mu^2\tilde{G}_{r+p,\lambda}(1; \mu).$$

In this direction, we can arrive at the desired result. □

Remark 1.4. The sequences of operators given in (1.4) are linear, i.e., for all $k_1, k_2 \in \mathbb{R}$ and $\theta_1, \theta_2 \in [0, \infty)$, we have

$$\tilde{G}_{r+p,\lambda}(k_1\theta_1 + k_2\theta_2; \mu) = k_1\tilde{G}_{r+p,\lambda}(\theta_1; \mu) + k_2\tilde{G}_{r+p,\lambda}(\theta_2; \mu).$$

Remark 1.5. The sequences of operators given in (1.4) are positive, i.e., $\tilde{G}_{r+p,\lambda}(\hbar; \mu) \geq 0$ for $\hbar \geq 0$.

To present the approximation properties of the sequences of operators given by (1.4), we draft the present manuscript in some subsequent sections as: uniform rate of convergence, direct approximation properties, weighted approximation properties, bivariate extension of the operators given by (1.4) with their rate of convergence and order of approximations in various functional spaces to achieve the better approximation behaviour in terms of these sequences operators.

2 Approximation Properties-Uniform rate of convergence and order of approximation

Definition 2.1. [30] Let $\hbar \in C_B[0, \infty)$. Then, the modulus of continuity is defined as:

$$\omega(\hbar; \tilde{\delta}) = \sup_{|\mu_1 - \mu_2| \leq \tilde{\delta}} |\hbar(\mu_1) - \hbar(\mu_2)|, \qquad \mu_1, \mu_2 \in [0, \infty), \tag{2.1}$$

and

$$|\hbar(\mu_1) - \hbar(\mu_2)| \leq \left(1 + \frac{|\mu_1 - \mu_2|}{\tilde{\delta}}\right) \omega(\hbar; \tilde{\delta}). \tag{2.2}$$

Theorem 2.2. *Let $\tilde{G}_{r,\lambda}(.;.)$ be given in (1.4), a sequence of operators and for all $\hbar \in C_B[0, \infty)$. Then, $\tilde{G}_{r,\lambda}(\hbar; .) \rightrightarrows \hbar$ on subset (closed and bounded) of $[0, \infty)$ where $\rightrightarrows$ depicts convergence is uniform.*

Proof. On account of the classical Korovkin theorem [31], it is sufficient to show that

$$\tilde{G}_{r,\lambda}(\theta^j; \mu) = \mu^j, \; j \in \{0, 1, 2\},$$

uniformly on each closed and bounded subset of $[0, \infty)$. Using Lemma (1.2), we arrive at the desired result immediately. □

Next result is the study of order of approximation of (1.4) in terms of modulus of continuity in Eq. (2.1) as:

Theorem 2.3. *For $\hbar \in C_B[0, \infty)$ and the sequence of operators $\tilde{G}_{r,\lambda}(.;.)$ in Eq. (1.4), we has*

$$|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| \leq 2\omega(\hbar; \tilde{\delta}),$$

where $\tilde{\delta} = \sqrt{\tilde{G}_{r,\lambda}(\hbar_2^\mu; \mu)}$.

Proof. With the definition of Eq. (1.4), we have

$$
\begin{aligned}
|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| &= \left| \frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{c_{r+p},\nu}(\theta)\{\hbar(\theta) - \hbar(\mu)\}d\theta \right|, \\
&\leq \frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{c_{r+p},\nu}(\theta)|\hbar(\theta) - \hbar(\mu)|d\theta \\
&\leq \frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{c_{r+p},\nu}(\theta)\left(1 + \frac{|\theta - \mu|}{\tilde{\delta}}\right)\omega(\hbar; \tilde{\delta})d\theta \\
&\leq \left\{1 + \frac{1}{\tilde{\delta}}\left(\frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{c_{r+p},\nu}(\theta)|\theta - \mu|d\theta\right)\right\}\omega(\hbar; \tilde{\delta}).
\end{aligned}
$$

On account of Cauchy-Schwarz inequality, we get

$$
\begin{aligned}
|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| &\leq \left\{1 + \frac{1}{\tilde{\delta}}\left(\frac{e^{-a_{r+p}\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{c_{r+p},\nu}(\theta)d\theta\right)^{\frac{1}{2}}\right. \\
&\quad \times \left. \left(\frac{e^{-r\mu}}{e-1} \sum_{\nu=0}^{\infty} \frac{\beta_\nu(a_{r+p}\mu)}{\nu!} \int_0^\infty b_{a_{r+p},\nu}(\theta)(\theta - \mu)^2 d\theta\right)^{\frac{1}{2}}\right\}\omega(\hbar; \tilde{\delta}) \\
&\leq \left\{1 + \frac{\sqrt{\tilde{G}_{c_{r+p},\lambda}(\hbar_2^\mu; \mu)}}{\tilde{\delta}}\right\}\omega(\hbar; \tilde{\delta}).
\end{aligned}
$$

On choosing $\tilde{\delta} = \tilde{G}_{r+p,\lambda}(\hbar_2^\mu; \mu)$, we yield

$$|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| \leq 2\omega(\hbar; \tilde{\delta}).$$

Hence, we prove the above theorem. □

Now, we discuss Voronovskaja type theorem to approximate a class of functions which has first and second order continuous derivatives with the help of the operators given by (1.4) as:

Theorem 2.4. *Let* $\hbar, \hbar', \hbar'' \in C[0,\infty) \bigcap E = \left\{ \hbar : \frac{\hbar(\mu)}{1+\mu^2} \text{ coverges as } \mu \to \infty \right\}$ *and* $\mu \in [0,\infty)$. *Then, we receive*

$$
\begin{aligned}
\lim_{r \to \infty} c_{r+p}(\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)) &= \hbar'(\mu)\left(\mu + \frac{\lambda(e-1)+1}{e-1}\right) \\
&+ \frac{\hbar''(\mu)}{2!}\left(2e(\lambda+1) - 2\lambda - 2\left(\frac{\lambda(e-1)+1}{e-1}\right)\right)\mu.
\end{aligned}
$$

Proof. First, we recall the Taylor series axpansion to approximate the functions

$$
\hbar(\theta) = \hbar(\mu) + \hbar'(\mu)(\theta - \mu) + \hbar''(\mu)\frac{(\theta - \mu)^2}{2!} + \xi(\theta, \mu)(\theta - \mu)^2, \tag{2.3}
$$

where $\xi(\theta,\mu)$ is the Peano remainder with $\xi(\theta,\mu) \in C[0,\infty)\bigcap E$ and $\lim_{\theta \to \mu}\xi(\theta,\mu) = 0$. Operating the operators $\tilde{G}_{r+p,\lambda}(.;.)$ defined in Eq. (1.4) in the equation (2.3)

$$
\tilde{G}_{r+p,\lambda}(\hbar;\mu) = \hbar(\mu) + \hbar'(\mu)\tilde{G}_{r+p,\lambda}(\hbar_1^\mu;\mu) + \frac{\hbar''}{2!}\tilde{G}_{r+p,\lambda}(\hbar_2^\mu;\mu) + \tilde{G}_{r+p,\lambda}(\xi(\theta,\mu)(\theta-\mu)^2;\mu). \tag{2.4}
$$

On applying the limit both the sides in the expression (2.4), we get

$$
\begin{aligned}
\lim_{r \to \infty} c_{r+p}(\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)) &= \hbar'(\mu)\lim_{r \to \infty} c_{r+p}\tilde{G}_{r+p,\lambda}(\hbar_1^\mu;\mu) + \frac{\hbar''}{2!}\lim_{r \to \infty} c_{r+p}\tilde{G}_{r+p,\lambda}(\hbar_2^\mu;\mu) \\
&+ \lim_{r \to \infty} c_{r+p}\tilde{G}_{r+p,\lambda}(\xi(\theta,\mu)(\theta-\mu)^2;\mu) \\
&= \hbar'(\mu)\left(\frac{\lambda(e-1)+1}{e-1}\right) \\
&+ \frac{\hbar''(\mu)}{2!}\left(2e(\lambda+1) - 2\lambda - 2\left(\frac{\lambda(e-1)+1}{e-1}\right)\right)\mu \\
&+ \lim_{r \to \infty} c_{r+p}\tilde{G}_{r+p,\lambda}(\xi(\theta,\mu)(\theta-\mu)^2;\mu). \tag{2.5}
\end{aligned}
$$

With the aid of Cauchy-Schwarz inequality, last term of the equation is as:

$$
c_{r+p}\tilde{G}_{r+p,\lambda}(\xi(\theta,\mu)(\theta-\mu)^2;\mu) \leq \sqrt{c_{r+p}^2\tilde{G}_{r+p,\lambda}((\theta-\mu)^4;\mu)\tilde{G}_{r+p,\lambda}(\xi^2(\theta,\mu);\mu)}. \tag{2.6}
$$

From Eq. (2.5), (2.6), Lemma (1.3) and $\lim_{r \to \infty} \tilde{G}_{r+p,\lambda}(\xi^2(\theta,\mu);\mu) = 0$, we have

$$
\begin{aligned}
\lim_{c_{r+p} \to \infty} r(\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)) &= \hbar'(\mu)\left(\frac{\lambda(e-1)+1}{e-1}\right) \\
&+ \frac{\hbar''(\mu)}{2!}\left(2e(\lambda+1) - 2\lambda - 2\left(\frac{\lambda(e-1)+1}{e-1}\right)\right)\mu.
\end{aligned}
$$

Which proves the required result.   □

3 Local Approximation Results

Consider $\tilde{C}_{\tilde{B}}[0,\infty)$: space of continuous and bounded function and Peetre's K-functional is defined as:

$$
\tilde{K}_2(\hbar,\delta) = \inf_{g \in \tilde{C}_{\tilde{B}}^2[0,\infty)}\left\{ \|\hbar - g\|_{\tilde{C}_{\tilde{B}}[0,\infty)} + \tilde{\delta}\|g''\|_{\tilde{C}_{\tilde{B}}^2[0,\infty)} \right\},
$$

where $\tilde{C}_{\tilde{B}}^2[0,\infty) = \{\hbar \in \tilde{C}_{\tilde{B}}[0,\infty) : \hbar', \hbar'' \in \tilde{C}_{\tilde{B}}[0,\infty)\}$ with the norm $\|\hbar\| = \sup\limits_{0 \leq \mu < \infty} |\hbar(\mu)|$ and Ditzian-Totik modulus of smoothness of second order is given by

$$\tilde{\omega}_2(\hbar; \sqrt{\tilde{\delta}}) = \sup\limits_{0 < k \leq \sqrt{\tilde{\delta}}} \sup\limits_{\mu \in [0,\infty)} |\hbar(\mu + 2k) - 2\hbar(\mu + k) + \hbar(\mu)|.$$

In view of DeVore and Lorentz ([30] page no. 177, Theorem 2.4) as:

$$\tilde{K}_2(\hbar; \tilde{\delta}) \leq \tilde{C}\tilde{\omega}_2(\hbar; \sqrt{\tilde{\delta}}), \tag{3.1}$$

where $\tilde{C}$ represent absolute constant in order to prove the local approximation results, we define the auxiliary operators as:

$$\tilde{G}^*_{r,\lambda}(\hbar; \mu) = \tilde{G}_{r,\lambda}(\hbar; \mu) + \hbar(\mu) - \hbar\left(\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)\right), \tag{3.2}$$

where $\hbar \in \tilde{C}_{\tilde{B}}[0,\infty)$, $\mu \geq 0$. Using Eq. (3.2), we have

$$\tilde{G}^*_{r,\lambda}(1; \mu) = 1, \ \tilde{G}^*_{r,\lambda}(\hbar_1^\mu; \mu) = 0 \text{ and } |\tilde{G}^*_{r,\lambda}(\hbar; \mu)| \leq 3\|\hbar\|. \tag{3.3}$$

Lemma 3.1. *Let the operators given by* (1.4) *and* $\mu \geq 0$*. Then*

$$|\tilde{G}^*_{r,\lambda}(\hbar; \mu) - \hbar(\mu)| \leq \Theta(\mu)\|\hbar''\|,$$

where $\hbar \in \tilde{C}_{\tilde{B}}^2[0,\infty)$ *and* $\Theta(\mu) = \tilde{G}_{r+p,\lambda}(\hbar_2^\mu; \mu) + (\tilde{G}_{r+p,\lambda}(\hbar_1^\mu; \mu))^2$.

Proof. For $\hbar \in \tilde{C}_{\tilde{B}}^2[0,\infty)$ and Taylor's series expansion, we yield

$$\hbar(\theta) = \hbar(\mu) + (\theta - \mu)\hbar'(\mu) + \int\limits_\mu^\theta (\theta - v)\hbar''(v)dv. \tag{3.4}$$

Operating the sequence of operators $\tilde{G}^*_{r,\lambda}(.;.)$ introduced in Eq. (3.2) on both the sides in the Eq. (3.4), we have

$$\tilde{G}^*_{r,\lambda}(\hbar; \mu) - \hbar(\mu) = \hbar'(\mu)\tilde{G}_{r+p,\lambda}(\hbar_1^\mu; \mu) + \tilde{G}^*_{r,\lambda}\left(\int\limits_\mu^\theta (\theta - v)\hbar''(v)dv; \mu\right).$$

Combining the Eqs. (3.3) and (3.4), we find

$$\tilde{G}^*_{r,\lambda}(\hbar; \mu) - \hbar(\mu) = \tilde{G}^*_{r,\lambda}\left(\int\limits_\mu^\theta (\theta - v)\hbar''(v)dv; \mu\right)$$

$$= \tilde{G}^*_{r,\lambda}\left(\int\limits_\mu^\theta (\theta - v)\hbar''(v)dv; \mu\right)$$

$$- \int\limits_\mu^{\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)} \left(\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right) - v\right)\hbar''(v)dv,$$

$$|\tilde{G}^*_{r,\lambda}(\hbar; \mu) - \hbar(\mu)| \leq \left|\tilde{G}^*_{r,\lambda}\left(\int\limits_\mu^\theta (\theta - v)\hbar''(v)dv; \mu\right)\right|$$

$$+ \left|\int\limits_\mu^{\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)} \left(\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right) - v\right)\hbar''(v)dv\right|. \tag{3.5}$$

Since,

$$\left| \int\limits_{\mu}^{\theta} (\theta - v)\hbar''(v)dv \right| \le (\theta - \mu)^2 \parallel \hbar'' \parallel, \tag{3.6}$$

then

$$\left| \int\limits_{\mu}^{\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)} \left(\frac{a_{r+p}}{c_{r+p}}\mu + \frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right) - v\right)\hbar''(v)dv \right| \le \left(\frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)\right)^2 \parallel \hbar'' \parallel. \tag{3.7}$$

In view of (3.5), (3.6) and (3.7), we find

$$|\tilde{G^*}_{r,\lambda}(\hbar; \mu) - \hbar(\mu)| \le \left\{ \tilde{G^*}_{r,\lambda}(\hbar_2^\mu; \mu) + \left(\frac{1}{c_{r+p}}\left(\frac{\lambda(e-1)+1}{e-1}\right)\right)^2 \right\}\|\hbar''\|$$

$$= \Theta(\mu)\|\hbar''\|.$$

Which completes the proof of the above result. □

Theorem 3.2. *Let* $\hbar \in \tilde{C}_{\tilde{B}}^2[0, \infty)$ *and the operators given in Eq.* (1.4)*. Then*

$$| \tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu) | \le \tilde{C}\tilde{\omega}_2\big(\hbar; \sqrt{\Theta(\mu)}\big) + \omega(\hbar; \tilde{G}_{r+p,\lambda}(\hbar_1^\mu; \mu)),$$

where $\tilde{C} \ge 0$ *and* $\Theta(\mu)$ *is introduced in Lemma 3.1.*

Proof. For $\hbar \in \tilde{C}_{\tilde{B}}^2[0, \infty)$, $h \in \tilde{C}_{\tilde{B}}[0, \infty)$ and on account of $\tilde{G}_{r+p,\lambda}(.;.)$ given by Eq. (1.4), we get

$$|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| \le |\tilde{G}_{r+p,\lambda}(\hbar - h; \mu)| + |(\hbar - h)(\mu)| + |\tilde{G}_{r+p,\lambda}(h; \mu) - h(\mu)|$$

$$+ \left| \hbar\Big(\tilde{G}_{r+p,\lambda}(\hbar_1, \mu)\Big) - \hbar(\mu) \right|.$$

In the light of Lemma 3.1 and inequalities in Eq. (3.3), one get

$$|\tilde{G}_{r+p,\lambda}(\hbar; \mu) - \hbar(\mu)| \le 4\|\hbar - h\| + |\tilde{G}_{r+p,\lambda}(h; \mu) - h(\mu)| + \left| \hbar\left(\tilde{G}_{r+p,\lambda}(\hbar_1, \mu)\right) - \hbar(\mu) \right|$$

$$\le 4\|\hbar - h\| + \theta(y)\|h''\| + \omega\Big(\hbar; \tilde{G}_{r,\lambda}((\theta - \mu); \mu)\Big).$$

Using Eq. (3.1), we yield the desired result. □

Now, we recall Lipschitz type space [32], which is defined as:

$$Lip_M^{\varphi_1,\varphi_2}(\tau) := \Big\{ \hbar \in \tilde{C}_{\tilde{B}}[0, \infty) : |\hbar(t) - \hbar(y)| \le \tilde{M}\frac{|t-y|^\tau}{(t+\varphi_1 y+\varphi_2 y^2)^{\frac{\tau}{2}}} : y, t \in (0, \infty) \Big\},$$

where $\tilde{M} > 0$, $0 < \tau \le 1$ and $\varphi_1, \varphi_2 > 0$.

Theorem 3.3. *Let* $\tilde{G}_{r,\lambda}(.;.)$ *be the operator given by* (??)*. Then, for* $\hbar \in Lip_M^{\varphi_1,\varphi_2}(\tau)$*, one has*

$$|\tilde{G}_{r+p,\lambda}(\hbar; y) - \hbar(y)| \le \tilde{M}\left(\frac{\lambda(y)}{\varphi_1 y + \varphi_2 y^2}\right)^{\frac{\tau}{2}}, \tag{3.8}$$

where $0 < \tau \le 1$, $\varphi_1, \varphi_2 \in (0, \infty)$ *and* $\lambda(y) = \tilde{G}_{r,\lambda}(\eta_2; y)$*.*

Proof. For $\tau = 1$ and $y \geq 0$, one yield

$$|\tilde{G}_{r+p,\lambda}(\hbar;y) - \hbar(y)| \leq \tilde{G}_{r,\lambda}(|\hbar(t) - \hbar(y)|;y)$$

$$\leq \tilde{M}\tilde{G}_{r,\lambda}\left(\frac{|t-y|}{(t+\varphi_1 y + \varphi_2 y^2)^{\frac{1}{2}}};y\right).$$

Since $\dfrac{1}{t+\varphi_1 y + \varphi_2 y^2} < \dfrac{1}{\varphi_1 y + \varphi_2 y^2}$, for all $y \in (0,\infty)$, we yield

$$|\tilde{G}_{r+p,\lambda}(\hbar;y) - \hbar(y)| \leq \frac{\tilde{M}}{(\varphi_1 y + \varphi_2 y^2)^{\frac{1}{2}}}(\tilde{G}_{r,\lambda}(\eta_2;y))^{\frac{1}{2}}$$

$$\leq \tilde{M}\left(\frac{\lambda(y)}{\varphi_1 y + \varphi_2 y^2}\right)^{\frac{1}{2}},$$

which implies that Theorem **??** works for $\tau = 1$. Next, we consider for $\tau \in (0,1)$ and in view of Hölder's inequality using $p = \frac{2}{\tau}$ and $q = \frac{2}{2-\tau}$, one get

$$|\tilde{G}_{r+p,\lambda}(\hbar;y) - \hbar(y)| \leq \left(\tilde{G}_{r,\lambda}(|\hbar(t) - \hbar(y)|^{\frac{2}{\tau}};y)\right)^{\frac{\tau}{2}}$$

$$\leq \tilde{M}\left(\tilde{G}_{r,\lambda}\left(\frac{|t-y|^2}{(t+\varphi_1 y + \varphi_2 y^2)};y\right)\right)^{\frac{\tau}{2}}.$$

Since $\dfrac{1}{t+\varphi_1 y + \varphi_2 y^2} < \dfrac{1}{\varphi_1 y + \varphi_2 y^2}$, for all $y \in (0,\infty)$, one get

$$|\tilde{G}_{r+p,\lambda}(\hbar;y) - \hbar(y)| \leq \tilde{M}\left(\frac{\tilde{G}_{r,\lambda}(|t-y|^2;y)}{\varphi_1 y + \varphi_2 y^2}\right)^{\frac{\tau}{2}} \leq \tilde{M}\left(\frac{\lambda(y)}{\varphi_1 y + \varphi_2 y^2}\right)^{\frac{\tau}{2}}.$$

Hence, Theorem **??** is proved.   □

Next, we discuss the approximation result locally in the direction of bth order modulus of continuity. Lipschitz-type maximal function which is given by Lenze [33] as:

$$\widetilde{\omega}_b(\hbar;\mu) = \sup_{t\neq\mu, t\in(0,\infty)} \frac{|\hbar(t) - \hbar(\mu)|}{|t-\mu|^b}, \ \mu \in [0,\infty) \text{ and } b \in (0,1]. \tag{3.9}$$

Theorem 3.4. *Let $\hbar \in \tilde{C}_{\tilde{B}}[0,\infty)$ and $b \in (0,1]$. Then, $\forall \ \mu \in [0,\infty)$, one has*

$$|\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)| \leq \widetilde{\omega}_r(\hbar;\mu)\left(\lambda(\mu)\right)^{\frac{b}{2}}.$$

Proof. It is found that

$$|\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)| \leq \tilde{G}_{r,\lambda}(|\hbar(t) - \hbar(\mu)|;\mu).$$

In account of Eq. (3.9), we have

$$|\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)| \leq \widetilde{\omega}_s(\hbar;y)\tilde{G}_{r,\lambda}(|t-\mu|^b;\mu).$$

Then, in view of Hölder's inequality with $p_1 = \frac{2}{b}$ and $p_2 = \frac{2}{2-b}$, we have

$$|\tilde{G}_{r+p,\lambda}(\hbar;\mu) - \hbar(\mu)| \leq \widetilde{\omega}_b(\hbar;\mu)\left(\tilde{G}_{r,\lambda}(|t-\mu|^2;\mu)\right)^{\frac{b}{2}}.$$

Hence, we arrive at the desired result.   □

4 Conclusion

This research work introduce a new connection of adjoint Bernoulli's polynomials and gamma function as a new sequence of linear positive operators denoted by $\{\tilde{G}_{r+p,\lambda}(.;.)\}_1^\infty$. Further, convergence properties of these sequences of operators, i.e., $\{\tilde{G}_{r+p,\lambda}(.;.)\}_1^\infty$ are investigated in various functional spaces with the aid of Korovkin theorem, Voronovskaja type theorem, first order of modulus of continuity, second order of modulus of continuity, Peetre's K-functional, Lipschitz condition etc. In the last section, we extend our research for bivariate case of these sequences of operators and their uniform rate of approximation and order of approximation are investigated in different functional spaces.

5 Declaration

6 Conflicts of interest

The authors declare that there is no conflicts of interest.

7 Authors Contributions

All authors have contributed equally.

References

[1] K. Khan and D. K. Lobiyal, *Bèzier curves based on Lupaş (p, q)-analogue of Bernstein functions in CAGD*, Comput. Appl. Math., **317**, 458—477 (2017).

[2] K. Khan, D. K. Lobiyal and A. Kilicman, *Bèzier Curves and Surfaces Based on Modified Bernstein Polynomials*, Azerb. J. Math., **9 (1)**, (2019).

[3] A. Izadbakhsh, A. A. Kalat and S. Khorashadizadeh, *Observer-based adaptive control for HIV infection therapy using the Baskakov operator*, Biomed. Signal Process. Control., **65**, 102343, (2021).

[4] H. Uyan, A. O. Aslan, S. Karateke and İ. Büyükyazıcı, *Interpolation for neural network operators activated with a generalized logistic-type function*, (2024).

[5] Q. Zhang, M. Mu and X. Wang, *A Modified Robotic Manipulator Controller Based on Bernstein-Kantorovich-Stancu Operator*, Micromachines, **14 (1)**, (2022).

[6] K. Weierstrass, *Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen*, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, **2**, 633–639, (1885).

[7] S. N. Bernstein, *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités,* Commun. Soc. Math. Kharkow., **13 (2)** 1—2, (1913).

[8] R. Aslan, *Rate of approximation of blending type modified univariate and bivariate λ-Schurer-Kantorovich operators*, Kuwait Journal of Science, **51 (1)**, 100168, (2024).

[9] R. Aslan, *Approximation properties of univariate and bivariate new class -Bernstein–Kantorovich operators and its associated GBS operators.* Comp. Appl. Math. **42**, 34, (2023).

[10] R. Aslan and M. Mursaleen, *Approximation by bivariate Chlodowsky type Szász–Durrmeyer operators and associated GBS operators on weighted spaces.* J Inequal Appl **2022**, 26, (2022).

[11] M. Ayman-Mursaleen, Md. Nasiruzzaman, N. Rao, M. Dilshad and K. S. Nisar, *Approximation by the modified λ-Bernstein-polynomial in terms of basis function,* AIMS Mathematics, **9(2)**, 4409—4426, (2024).

[12] M. Ayman-Mursaleen, Md. Nasiruzzaman and N. Rao, *On the Approximation of Szász-Jakimovski-Leviatan Beta Type Integral Operators Enhanced by Appell Polynomials,* Iran J Sci, **49**, 1013–1022, (2025).

[13] M. Ayman-Mursaleen, M. Heshamuddin, N. Rao, B. K. Sinha and A. K. Yadav, *Hermite polynomials linking Szász–Durrmeyer operators,* Comp. Appl. Math. **43**, 223, (2024). https://doi.org/10.1007/s40314-024-02752-0

[14] S. A. Mohiuddine, T. Acar and A. Alotaibi, *Construction of a new family of Bernstein-Kantorovich operators*, Math. Methods Appl. Sci., **40**, 7749–7759, (2017).

[15] S. A. Mohiuddine, N. Ahmad, F. Özger, A. Alotaibi and B. Hazarika, *Approximation by the parametric generalization of Baskakov–Kantorovich operators linking with Stancu operators,* Iran. J. Sci. Technol. Trans., **45(2)**, 593–605, (2021).

[16] M. Mursaleen, K. J. Ansari and A. Khan, *Approximation properties and error estimation of q-Bernstein shifted operators,* Numer. Algorithms, **84**, 207–227, (2020). .

[17] M. Mursaleen, A. Naaz and A. Khan, *Improved approximation and error estimations by King type (p, q)-Szász-Mirakjan Kantorovich operators,* Appl. Math. Comput.,*348*, 2175–185, (2019).

[18] A. Khan, M. Mansoori, K. Khan, and M. Mursaleen, *Phillips-type q-Bernstein operators on triangles*, J. Funct. Spaces, *2021*, 1–13, (2021).

[19] Md. Nasiruzzaman, N. Rao, A. Alotaibi and S. A. Mohiuddine, *Approximation by Stancu Variant of α-Schurer-Kantorovich Operators*, Filomat, **36 (17)**, 5751–5764, (2022).

[20] N.L. Braha, V. Loku, T. Mansour and M. Mursaleen, *A new weighted statistical convergence and some associated approximation theorems*, Math. Meth.. Appl. Sci., **45 (10)**, 5682–5698, (2022).

[21] N. Rao, P. Malik and M. Rani, *Blending type Approximations by Kantorovich variant of α-Baskakov operator*, Palestine Journal of Mathematics, **11 (3)**, 402—413, (2022).

[22] N. Rao and K. Gulia, *On two dimensional Chlodowsky-Szász operators via Sheffer Polynomials,* Palestine Journal of Mathematics, **12 (3)**, 521-531, (2023).

[23] N. Çetin, *Approximation and geometric properties of complex α -Bernstein operator,* Results Math. **74**, Article number: 40, (2019).

[24] N. Çetin and V.A. Radu, *Approximation by generalized Bernstein-Stancu operators,* Turkish J. Math. **43**, 2032-2048, (2019).

[25] F. Özger, *Weighted statistical approximation properties of univariate and bivariate λ-Kantorovich operators*, Filomat 33 (11), 3473-3486, 2019.

[26] F. Özger, K. J. Ansari, *Statistical convergence of bivariate generalized Bernstein operators via four-dimensional infinite matrices*, Filomat, 36 2, 2022.

[27] P. Appell, *P. Sur une class de polynômes,* Ann. Sci Éc. Norm Supér; Société matheématique de France, Paris, France **2 (a)**, 119-144, (1880).

[28] P. Natalini, E. Ricci, *Adjoint Appell–Euler and first kind Appell–Bernoulli's polynomials,* Appl. Appl. Math. **14 (2)**, (2019), 1112–1122.

[29] M. M. Yilmaz, *Rate of convergence by Kantorovich type operators involving adjoint Bernoulli's polynomials*, Publications De l'Institut Mathématique Nouvelle série, tome, (114 (128)), 51-62, (2023).

[30] R. A. Devore, G. G. Lorentz, *Constructive approximation. Berlın: Springer*, 1993.

[31] F. Altomare, M. Campiti, *Korovkin-type approximation theory and its applications*. De Gruyter Ser Stud in Math. 17:1994, 266–274.

[32] M. A. Özarslan and H. Aktuğlu, *Local approximation for certain King type operators*, Filomat. **27**, 173-181, (2013).

[33] B. Lenze, *On Lipschitz type maximal functions and their smoothness spaces*, Nederl Akad Indag Math. **50**, 53-63, (1988).

[34] S. Berwal, S. A. Mohiuddine, A. Kajla and A. Alotaibi, *Approximation by Riemann-Liouville type fractional α-Bernstein-Kantorovich operators,* Math. Meth. Appl. Sci., **47**, 8275–8288, (2024).

[35] S. A. Mohiuddine, K. K. Singh and A. Alotaibi, *On the order of approximation by modified summation-integral-type operators based on two parameters*, Demonstr. Math., 56: 20220182, (2023).

[36] S. A. Mohiuddine, Z. Ö. Özger, F. Özger and A. Alotaibi, *Construction of a new family of modified Bernstein-Schurer operators of different order for better approximation*, J. Nonlinear Convex Anal. **25(9)**, 2059–2082, (2024).

[37] T. Khan, N. Rao, S. A. Khan and P. Malik *On α-Baskakov-Gamma operators with two shifted nodes,* Palestine journal of Mathematics **11 (2)**, 457–468, (2022).

[38] A. Kajla, S. A. Mohiuddine, S Berwal and A. Alotaibi, *τ-Bézier-Bernstein-integral type operators,* Math. Meth. Appl. Sci. **48 (11)**, 10942–10951, (2025).

[39] F. Özger, H. M. Srivastava, S. A. Mohiuddine, *Approximation of functions by a new class of generalized Bernstein-Schurer operators,* Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. RACSAM 114:173, (2020).

Author information

N. Rao, Department of Mathematics, University Center for Research and Development, Chandigarh University, Mohali-140413, Punjab, India.
E-mail: nadeemrao1990@gmail.com

S. Kumar, Department of Applied Sciences and Humanities, Galgotias College of Engineering and Technology, Greater Noida, G. B. Nagar, U.P. 201310, India.
E-mail: sunny.kumar@galgotiacollege.edu

A. Srivastava, Amity International Business School, Amity University, Noida-201313, U.P., India.
E-mail: anshulsriv@rediffmail.com

A. K. Yadav, Department of Applied Science, Galgotias College of Engg and Technology, Greater Noida, Gautam Buddha Nagar-201310, U.P., India.
E-mail: avinash.yadav@galgotiacollege.edu

M. Rani, World College of Technology and Management, WCTM Campus, Farukh Nagar, Gurgaon, Delhi-NCR, India.
E-mail: mamtakalra84@gmail.com

U. R. Prajapati, Department of Mathematics, Faculty of Engineering and Technology, Veer Bahadur Singh Purvanchal University, Jaipur, U.P., India.
E-mail: udayrajprajapati2011@gmail.com