

# NEW OSCILLATION CRITERIA FOR THIRD-ORDER NONLINEAR NONCANONICAL DELAY DIFFERENCE EQUATIONS USING THE CANONICAL TRANSFORMATION

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**Abstract** New oscillation criteria are presented for the third-order delay difference equation

$$\Delta(a_2(\bar{h})\Delta(a_1(\bar{h})\Delta y(\bar{h}))) + b(\bar{h})y^\alpha(\zeta(\bar{h})) = 0$$

under the noncanonical condition  $\sum_{\bar{h}=\bar{h}_0}^{\infty} \frac{1}{a_j(\bar{h})} < \infty$ ,  $j = 1, 2$ . The analyzed equation is transformed into a canonical form without additional assumptions, reducing the number of nonoscillatory solution types from four to two. By eliminating these two categories of nonoscillatory solutions, we apply the comparison method and summation averaging technique to establish the necessary conditions for the oscillation of all solutions to the given equation. This approach significantly advances the analysis and yields substantially stronger results compared to existing findings. Four examples are provided to illustrate the significance and originality of the main conclusions.

## 1 Introduction

Consider the following third-order nonlinear delay difference equation(DE)

$$D_3y(\bar{h}) + b(\bar{h})y^\alpha(\zeta(\bar{h})) = 0, \quad \bar{h} \geq \bar{h}_0 > 0, \quad (E)$$

where  $D_0y(\bar{h}) = y(\bar{h})$ ,  $D_jy(\bar{h}) = a_j(\bar{h})\Delta(D_{j-1}y(\bar{h}))$ ,  $j = 1, 2$ ,  $D_3y(\bar{h}) = \Delta(D_2y(\bar{h}))$ . We presume throughout the paper that:

(i)  $\{a_1(\bar{h})\}$ ,  $\{a_2(\bar{h})\}$  and  $\{b(\bar{h})\}$  are positive(+ve) real sequences for  $\bar{h} \geq \bar{h}_0$  and

$$\Lambda_j(\bar{h}_0) = \sum_{\bar{h}=\bar{h}_0}^{\infty} \frac{1}{a_j(\bar{h})} < \infty, \quad j = 1, 2; \quad (1.1)$$

(ii)  $\{\zeta(\bar{h})\}$  is an increasing( $\uparrow$ ) sequence of integers with  $\zeta(\bar{h}) \leq \bar{h} - 1$  with  $\lim_{\bar{h} \rightarrow \infty} \zeta(\bar{h}) = \infty$ ;  
 (iii)  $\alpha$  is a ratio of odd +ve integers.

A nontrivial real sequence  $\{y(\bar{h})\}$  that is defined and satisfies (E) for all  $\bar{h} \geq \bar{h}_0$  is called a solution of equation (E). A nontrivial solution  $\{y(\bar{h})\}$  of (E) is said to be nonoscillatory if it is either eventually positive or eventually negative, and oscillatory if it is neither.

Third-order functional difference equations (DEs) frequently arise in models addressing biological, physical, and economic problems. For further details, see [1, 4, 10]. The existence

of oscillatory solutions plays a crucial role in population dynamics and biological applications. Over the past few years, numerous studies have examined the oscillatory and asymptotic behavior of various classes of third-order functional DEs. For example, see [2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 30, 31] and the references therein.

A review of the existing literature reveals that numerous studies have investigated the oscillatory and asymptotic properties of solutions to (E) under various conditions including  $\Lambda_j(\bar{h}_0) = \infty$ ,  $j = 1, 2$ , or  $\Lambda_1(\bar{h}_0) < \infty$  and  $\Lambda_2(\bar{h}_0) = \infty$ , or  $\Lambda_1(\bar{h}_0) = \infty$ , and  $\Lambda_2(\bar{h}_0) < \infty$ , or  $\Lambda_1(\bar{h}_0) < \infty$ , and  $\Lambda_2(\bar{h}_0) < \infty$ ; see for example [2, 3, 6, 8, 11, 12, 14, 15, 20, 21, 23, 24, 27, 31, 30] and the works cited therein for more details.

Recently [15, 20, 21], the authors studied the equation (E) under the condition (1.1) with  $\alpha = 1$ . There are four possible scenarios for the set of nonoscillatory solutions(NOS) of (E), specifically,

$$\begin{aligned} S_0 & : y(\bar{h}) > 0, \Delta y(\bar{h}) < 0, \Delta(a_1(\bar{h})\Delta y(\bar{h})) > 0, \\ S_2 & : y(\bar{h}) > 0, \Delta y(\bar{h}) > 0, \Delta(a_1(\bar{h})\Delta y(\bar{h})) > 0, \\ S_* & : y(\bar{h}) > 0, \Delta y(\bar{h}) > 0, \Delta(a_1(\bar{h})\Delta y(\bar{h})) < 0, \\ S_{**} & : y(\bar{h}) > 0, \Delta y(\bar{h}) < 0, \Delta(a_1(\bar{h})\Delta y(\bar{h})) < 0, \end{aligned}$$

and to get oscillation criteria, one has to empty all these four cases.

On the other hand, we transform (E) into a canonical-type equation without imposing any additional conditions. The key advantage of this approach is the application of the discrete variant of the generalized Kneser’s theorem [1], which enables us to classify the set of potential nonoscillatory solutions (NOS) into two types instead of four.

This method significantly aids in establishing oscillation criteria for all solutions of (E) by eliminating only two categories of NOS rather than four. Consequently, we expect our approach to make a valuable contribution to the study of oscillation theory for noncanonical third-order functional difference equations. The primary results are illustrated through four cases, underscoring their significance and originality.

## 2 Main Results

For  $\bar{h} \geq \bar{h}_* \geq \bar{h}_0$ , we define

$$\begin{aligned} \Lambda_j(\bar{h}) & = \sum_{\iota=\bar{h}}^{\infty} \frac{1}{a_j(\iota)}, \quad j = 1, 2, \\ \Lambda_{12}(\bar{h}) & = \sum_{\iota=\bar{h}}^{\infty} \frac{\Lambda_2(\iota)}{a_1(\iota)}, \quad \Lambda_{21} = \sum_{\iota=\bar{h}}^{\infty} \frac{\Lambda_1(\iota+1)}{a_2(\iota)}, \\ b_1(\bar{h}) & = \frac{a_1(\bar{h})\Lambda_{12}(\bar{h})\Lambda_{12}(\bar{h}+1)}{\Lambda_{21}(\bar{h})}, \quad b_2(\bar{h}) = \frac{a_2(\bar{h})\Lambda_{21}(\bar{h})\Lambda_{21}(\bar{h}+1)}{\Lambda_{12}(\bar{h}+1)}, \\ \Theta(\bar{h}) & = b(\bar{h})\Lambda_{12}^{\alpha}(\zeta(\bar{h}))\Lambda_{21}(\bar{h}+1), \quad z(\bar{h}) = \frac{y(\bar{h})}{\Lambda_{12}(\bar{h})}. \end{aligned}$$

We start with a lemma that will aid in proving our next theorem.

**Lemma 2.1.** *Let (1.1) hold. Then*

- (i)  $\Lambda_1(\bar{h})\Lambda_2(\bar{h}) = \Lambda_{12}(\bar{h}) + \Lambda_{21}(\bar{h})$ ,
- (ii)  $\Lambda_2(\bar{h}) = \frac{1}{a_2(\bar{h})} + \Lambda_2(\bar{h}+1)$ ,
- (iii)  $\Lambda_{12}(\bar{h}) = \frac{\Lambda_2(\bar{h})}{a_1(\bar{h})} + \Lambda_{12}(\bar{h}+1)$ ,
- (iv)  $\Lambda_{21}(\bar{h}) = \frac{\Lambda_1(\bar{h}+1)}{a_2(\bar{h})} + \Lambda_{21}(\bar{h}+1)$ ,
- (v)  $\frac{\Lambda_1(\bar{h}+1)\Lambda_2(\bar{h})-\Lambda_{21}(\bar{h})}{\Lambda_{12}(\bar{h}+1)} = 1$ .

*Proof.* Using the product rule, we see that

$$\Delta(\Lambda_1(\hbar)\Lambda_2(\hbar)) = -\frac{\Lambda_1(\hbar+1)}{a_2(\hbar)} - \frac{\Lambda_2(\hbar)}{a_1(\hbar)}.$$

Summing up the previous equality from  $\hbar$  to  $\infty$ , yields

$$\Lambda_1(\hbar)\Lambda_2(\hbar) = \sum_{i=\hbar}^{\infty} \frac{\Lambda_1(i+1)}{a_2(i)} + \sum_{i=\hbar}^{\infty} \frac{\Lambda_2(i)}{a_1(i)} = \Lambda_{12}(\hbar) + \Lambda_{21}(\hbar).$$

This proves (i). Next, it is easy to see that (ii)–(iv) follows from the definitions of  $\Lambda_2(\hbar)$ ,  $\Lambda_{12}(\hbar)$  and  $\Lambda_{21}(\hbar)$  respectively. Using (ii) and (iv) in the L.H.S. of (v), we see that

$$\text{L.H.S} = \frac{\Lambda_1(\hbar+1)\Lambda_2(\hbar+1) - \Lambda_{21}(\hbar+1)}{\Lambda_{12}(\hbar+1)} = 1$$

by (i). Thus, the proof is complete.  $\square$

Next, we present a closed form canonical representation for the operator  $D_3y$ . The following theorem is adopted from [7, 15] but here we have presented a different proof.

**Theorem 2.2.** *The canonical form of the noncanonical operator  $D_3y$  is given by*

$$D_3y(\hbar) = \frac{1}{\Lambda_{21}(\hbar+1)} \Delta \left( \frac{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)}{\Lambda_{12}(\hbar+1)} \Delta \left( \frac{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)}{\Lambda_{21}(\hbar)} \Delta \left( \frac{y(\hbar)}{\Lambda_{12}(\hbar)} \right) \right) \right). \quad (2.1)$$

*Proof.* Direct calculation shows that

$$\begin{aligned} & \frac{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)}{\Lambda_{12}(\hbar+1)} \Delta \left( \frac{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)}{\Lambda_{21}(\hbar)} \Delta \left( \frac{y(\hbar)}{\Lambda_{12}(\hbar)} \right) \right) \\ &= \frac{1}{\Lambda_{12}(\hbar+1)} [\Lambda_{21}(\hbar)\Lambda_{12}(\hbar+1)a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar)) - \Lambda_{21}(\hbar)\Lambda_2(\hbar)a_2(\hbar)\Delta y(\hbar) \\ & \quad + \Lambda_1(\hbar+1)\Lambda_{12}(\hbar)a_1(\hbar)\Delta y(\hbar) + \Lambda_{21}(\hbar)\Lambda_2(\hbar+1)a_2(\hbar)\Delta y(\hbar) \\ & \quad - \Lambda_{21}(\hbar)y(\hbar) + \Lambda_1(\hbar+1)\Lambda_2(\hbar)y(\hbar)]. \end{aligned}$$

Using (ii) and (iii) of Lemma 2.1, in the last equality, we see that

$$\begin{aligned} \text{L.H.S.} &= \frac{1}{\Lambda_{12}(\hbar+1)} [\Lambda_{21}(\hbar)\Lambda_{12}(\hbar+1)a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar)) - \Lambda_{21}(\hbar)y(\hbar+1) \\ & \quad + \Lambda_1(\hbar+1)\Lambda_2(\hbar)y(\hbar+1) + \Lambda_1(\hbar+1)\Lambda_{12}(\hbar+1)a_1(\hbar)\Delta y(\hbar)] \\ &= \Lambda_{21}(\hbar)a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar)) + \Lambda_1(\hbar+1)a_1(\hbar)\Delta y(\hbar) \\ & \quad + \left( \frac{\Lambda_1(\hbar+1)\Lambda_2(\hbar)}{\Lambda_{12}(\hbar+1)} - \frac{\Lambda_{21}(\hbar)}{\Lambda_{12}(\hbar+1)} \right) y(\hbar+1). \end{aligned}$$

Now using (v) of Lemma 2.1, we get

$$\text{L.H.S.} = \Lambda_{21}(\hbar)a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar)) + \Lambda_1(\hbar+1)a_1(\hbar)\Delta y(\hbar) + y(\hbar+1).$$

Thus,

$$\begin{aligned} & \Delta \left( \frac{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)}{\Lambda_{12}(\hbar+1)} \Delta \left( \frac{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)}{\Lambda_{21}(\hbar)} \Delta \left( \frac{y(\hbar)}{\Lambda_{12}(\hbar)} \right) \right) \right) \\ &= \Lambda_{21}(\hbar+1)\Delta(a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar))) - \Lambda_1(\hbar+1)\Delta(a_1(\hbar)\Delta y(\hbar)) \\ & \quad + \Lambda_1(\hbar+1)\Delta(a_1(\hbar)\Delta y(\hbar)) - \Delta y(\hbar+1) + \Delta y(\hbar+1) \\ &= \Lambda_{21}(\hbar+1)\Delta(a_2(\hbar)\Delta(a_1(\hbar)\Delta y(\hbar))) \end{aligned}$$

which is (2.1). We will now demonstrate that (2.1) can be expressed in canonical form, specifically,

$$\sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_{12}(\hbar+1)}{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)} = \sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_{21}(\hbar)}{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)} = \infty.$$

Using (i) of Lemma 2.1, one can see that

$$\begin{aligned} \sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_{12}(\hbar+1)}{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)} &= \sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_1(\hbar+1)\Lambda_2(\hbar+1) - \Lambda_{21}(\hbar+1)}{a_2(\hbar)\Lambda_{21}(\hbar)\Lambda_{21}(\hbar+1)} \\ &= \sum_{\hbar=\hbar_0}^{\infty} \left[ \Lambda_2(\hbar+1)\Delta\left(\frac{1}{\Lambda_{21}(\hbar)}\right) - \frac{1}{a_2(\hbar)\Lambda_{21}(\hbar)} \right] \\ &= \frac{\Lambda_2(\hbar)}{\Lambda_{21}(\hbar)} \Big|_{\hbar_0}^{\infty} = \infty \end{aligned}$$

since by discrete L'Hospital rule

$$\lim_{\hbar \rightarrow \infty} \frac{\Lambda_2(\hbar)}{\Lambda_{21}(\hbar)} = \lim_{\hbar \rightarrow \infty} \frac{1}{\Lambda_1(\hbar+1)} = \infty.$$

Similarly,

$$\begin{aligned} \sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_{21}(\hbar)}{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)} &= \sum_{\hbar=\hbar_0}^{\infty} \frac{\Lambda_1(\hbar)\Lambda_2(\hbar) - \Lambda_{12}(\hbar)}{a_1(\hbar)\Lambda_{12}(\hbar)\Lambda_{12}(\hbar+1)} \\ &= \sum_{\hbar=\hbar_0}^{\infty} \left[ \Lambda_1(\hbar)\Delta\left(\frac{1}{\Lambda_{12}(\hbar)}\right) - \frac{1}{a_1(\hbar)\Lambda_{12}(\hbar+1)} \right] \\ &= \frac{\Lambda_1(\hbar)}{\Lambda_{12}(\hbar)} \Big|_{\hbar_0}^{\infty} = \infty \end{aligned}$$

where by discrete L'Hospital rule

$$\lim_{\hbar \rightarrow \infty} \frac{\Lambda_1(\hbar)}{\Lambda_{12}(\hbar)} = \lim_{\hbar \rightarrow \infty} \frac{1}{\Lambda_2(\hbar)} = \infty.$$

This ends the proof. □

From Theorem 2.2, the equation (E) can be written in the equivalent canonical form as

$$L_3z(\hbar) + \Theta(\hbar)z^\alpha(\zeta(\hbar)) = 0, \quad \hbar \geq \hbar_0 > 0, \tag{E_c}$$

where  $L_0z(\hbar) = z(\hbar)$ ,  $L_jz(\hbar) = b_j(\hbar)\Delta(L_{j-1}z(\hbar))$ ,  $j = 1, 2$ ,  $L_3z(\hbar) = \Delta(L_2z(\hbar))$ , and the following results are immediate consequences.

**Corollary 2.3.** *The noncanonical DE (E) possess a solution  $\{y(\hbar)\}$  if and only if the canonical equation (E<sub>c</sub>) has the solution  $\{z(\hbar)\}$ .*

**Corollary 2.4.** *The noncanonical DE (E) has an eventually +ve solution(EPS) if and only if the canonical equation (E<sub>c</sub>) has EPS.*

With the canonical equation (E<sub>c</sub>) in hand, we can now investigate the oscillatory behavior of (E). Based on the well-known discrete Kneser theorem [1], we obtain the following result, which includes two sets of NOS instead of four.

**Corollary 2.5.** *Assume that  $\{z(\hbar)\}$  is an EPS of (E<sub>c</sub>), then*

$$z(\hbar) \in NS_0 \iff L_1z(\hbar) < 0, L_2z(\hbar) > 0, L_3z(\hbar) \leq 0,$$

or

$$z(\hbar) \in NS_2 \iff L_1z(\hbar) > 0, L_2z(\hbar) > 0, L_3z(\hbar) \leq 0$$

eventually.

Thus, the set  $NS$  of all positive solutions (PS) of  $(E_c)$  can be seen as

$$NS = NS_0 \cup NS_2.$$

So to find conditions for the oscillation of  $(E)$ , we have to make these two classes empty.

Prior to stating and proving the main results, we adopt the following notations

$$\Upsilon_1(\hbar) = \sum_{\iota=\hbar_1}^{\hbar-1} \frac{1}{b_1(\iota)} = \frac{\Lambda_1(\hbar)}{\Lambda_{12}(\hbar)}, \quad \Upsilon_2(\hbar) = \sum_{\iota=\hbar_1}^{\hbar-1} \frac{1}{b_2(\iota)} = \frac{\Lambda_2(\hbar)}{\Lambda_{21}(\hbar)}$$

and for further use, we define

$$\Upsilon(\hbar) = \sum_{\iota=\hbar_1}^{\hbar-1} \frac{1}{b_1(\iota)} \Upsilon_2(\iota),$$

for all  $\hbar_1 \geq \hbar_0$ .

**Lemma 2.6.** *Let  $\{z(\hbar)\}$  be an EPS of  $(E_c)$  that belongs to  $NS_2$ . Then*

- (i)  $\Delta z(\hbar) \geq \Upsilon_2(\hbar)L_2z(\hbar)/b_1(\hbar)$ ;
- (ii)  $\frac{L_1z(\hbar)}{\Upsilon_2(\hbar)}$  is decreasing;
- (iii)  $z(\hbar) \geq \Upsilon(\hbar)L_1z(\hbar)/\Upsilon_2(\hbar)$ ;
- (iv)  $\frac{z(\hbar)}{\Upsilon(\hbar)}$  is decreasing.

*Proof.* Let  $z(\hbar) \in NS_2$ . Since  $L_2z(\hbar)$  is decreasing, we have

$$L_1z(\hbar) \geq \sum_{\iota=\hbar_1}^{\hbar-1} \frac{L_2z(\iota)}{b_2(\iota)} \geq \Upsilon_2(\hbar)L_2z(\hbar),$$

that is,

$$\Delta z(\hbar) \geq \Upsilon_2(\hbar)L_2z(\hbar)/b_1(\hbar)$$

and hence (i) holds. Now from (i), we have

$$\Delta \left( \frac{L_1z(\hbar)}{\Upsilon_2(\hbar)} \right) = \frac{\Upsilon_2(\hbar)L_2z(\hbar) - L_1z(\hbar)}{b_2(\hbar)\Upsilon_2(\hbar)\Upsilon_2(\hbar+1)} \leq 0.$$

This implies that  $\frac{L_1z(\hbar)}{\Upsilon_2(\hbar)}$  is decreasing and proves (ii).

Since by (ii),

$$z(\hbar) \geq \sum_{\iota=\hbar_1}^{\hbar-1} \frac{L_1z(\iota)\Upsilon_2(\iota)}{b_1(\iota)\Upsilon_2(\iota)} \geq \frac{\Upsilon(\hbar)L_1z(\hbar)}{\Upsilon_2(\hbar)}$$

and so (iii) holds. Finally, using (iii) we have

$$\Delta \left( \frac{z(\hbar)}{\Upsilon(\hbar)} \right) = \frac{\Upsilon(\hbar)L_1z(\hbar) - \Upsilon_2(\hbar)z(\hbar)}{b_1(\hbar)\Upsilon(\hbar)\Upsilon(\hbar+1)} \leq 0$$

and hence  $\frac{z(\hbar)}{\Upsilon(\hbar)}$  is decreasing. This proves (iv) and the theorem is proved. □

**Lemma 2.7.** *Suppose that  $\{z(\hbar)\}$  is a solution of  $(E_c)$  that is eventually +ve and belongs to  $NS_0$ . If there exists a sequence of integers  $\{\xi(\hbar)\}$  such that*

$$\zeta(\hbar) < \xi(\hbar) \leq \hbar - 1, \quad \hbar \geq \hbar_0, \tag{2.2}$$

then

$$z(\zeta(\hbar)) \geq R(\xi(\hbar), \zeta(\hbar))L_2z(\xi(\hbar)) \tag{2.3}$$

for all  $\hbar \geq \hbar_1 \geq \hbar_0$ , where  $R(\xi(\hbar), \zeta(\hbar)) = \sum_{\iota=\zeta(\hbar)}^{\xi(\hbar)-1} \frac{1}{b_1(\iota)} \sum_{\varkappa=\iota}^{\xi(\hbar)-1} \frac{1}{b_2(\varkappa)}$ .

*Proof.* Let  $z(\hbar) \in NS_0$ . It follows from the monotonicity of  $L_2z(\hbar)$ , we have

$$\begin{aligned}
 -L_1z(u) \geq L_1z(v) - L_1z(u) &= \sum_{i=u}^{v-1} \frac{L_2z(i)}{b_2(i)} \\
 &\geq L_2z(v) \sum_{i=u}^{v-1} \frac{1}{b_2(i)}
 \end{aligned}$$

for  $v > u \geq \hbar_1$ . Summing up the last inequality from  $u$  to  $v - 1$ , we get

$$z(u) \geq R(v, u)L_2z(v). \tag{2.4}$$

Now, letting  $u = \zeta(\hbar)$  and  $v = \xi(\hbar)$  in (2.4), one gets (2.3). This ends the proof.  $\square$

**Theorem 2.8.** Let  $\alpha = 1$ . Assume that

$$\sum_{\hbar=\hbar_1}^{\infty} \frac{1}{b_1(\hbar)} \sum_{i=\hbar}^{\infty} \frac{1}{b_2(i)} \sum_{\varkappa=i}^{\infty} \Theta(\varkappa) = \infty \tag{2.5}$$

and

$$\begin{aligned}
 \lim_{\hbar \rightarrow \infty} \sup \left\{ \frac{1}{\Upsilon_2(\zeta(\hbar))} \sum_{i=\hbar_1}^{\zeta(\hbar)-1} \Upsilon_2(i+1) \Theta(i) \Upsilon(\zeta(i)) + \sum_{i=\zeta(\hbar)}^{\hbar-1} \Theta(i) \Upsilon(\zeta(i)) \right. \\
 \left. + \Upsilon_2(\zeta(\hbar)) \sum_{i=\hbar}^{\infty} \frac{\Theta(i) \Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} \right\} > 1,
 \end{aligned} \tag{2.6}$$

for any  $\hbar_1 \geq \hbar_0$ . Then every NOS  $\{y(\hbar)\}$  of (E) fulfills  $\lim_{\hbar \rightarrow \infty} \frac{|y(\hbar)|}{\Lambda_{12}(\hbar)} = 0$ .

*Proof.* Let  $\{y(\hbar)\}$  be an EPS of equation (E). By Corollary 2.4, the corresponding sequence  $\{z(\hbar)\} = \{y(\hbar)/\Lambda_{12}(\hbar)\}$  is a PS of  $(E_c)$  and from Corollary 2.5, we see that either  $z(\hbar) \in NS_0$  or  $z(\hbar) \in NS_2$  for all  $\hbar \geq \hbar_1 \geq \hbar_0$ .

First assume that  $z(\hbar) \in NS_2$ . Using the estimate (iii) of Lemma 2.6 in  $(E_c)$ , we see that  $x(\hbar) = L_1z(\hbar)$  is a +ve  $\uparrow$  solution of the difference inequality

$$\Delta(b_2(\hbar)\Delta x(\hbar)) + \frac{\Theta(\hbar)\Upsilon(\zeta(\hbar))}{\Upsilon_2(\zeta(\hbar))} x(\zeta(\hbar)) \leq 0 \tag{2.7}$$

and by Lemma 2.6(ii), we have  $\left\{ \frac{x(\hbar)}{\Upsilon_2(\hbar)} \right\}$  is decreasing. On the other hand, summing up (2.7) from  $\hbar$  to  $\infty$  gives

$$\Delta x(\hbar) \geq \frac{1}{b_2(\hbar)} \sum_{i=\hbar}^{\infty} \frac{\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)).$$

By summing from  $\hbar_1$  to  $\hbar - 1$  and employing the summation by parts formula, we obtain

$$\begin{aligned}
 x(\hbar) &\geq \sum_{i=\hbar_1}^{\hbar-1} \frac{1}{b_2(i)} \sum_{\varkappa=i}^{\infty} \frac{\Theta(\varkappa)\Upsilon(\zeta(\varkappa))}{\Upsilon_2(\zeta(\varkappa))} x(\zeta(\varkappa)) \\
 &= \sum_{i=\hbar_1}^{\hbar-1} \frac{1}{b_2(i)} \sum_{\varkappa=i}^{\hbar-1} \frac{\Theta(\varkappa)\Upsilon(\zeta(\varkappa))}{\Upsilon_2(\zeta(\varkappa))} x(\zeta(\varkappa)) + \sum_{i=\hbar_1}^{\hbar-1} \frac{1}{b_2(i)} \sum_{\varkappa=\hbar}^{\infty} \frac{\Theta(\varkappa)\Upsilon(\zeta(\varkappa))}{\Upsilon_2(\zeta(\varkappa))} x(\zeta(\varkappa)) \\
 &= \sum_{i=\hbar_1}^{\hbar-1} \frac{\Upsilon_2(i+1)\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)) + \Upsilon_2(\hbar) \sum_{i=\hbar}^{\infty} \frac{\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 x(\zeta(\hbar)) &\geq \sum_{i=\hbar_1}^{\zeta(\hbar)-1} \frac{\Upsilon_2(i+1)\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)) + \Upsilon_2(\zeta(\hbar)) \sum_{i=\zeta(\hbar)}^{\hbar-1} \frac{\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)) \\
 &\quad + \Upsilon_2(\zeta(\hbar)) \sum_{i=\hbar}^{\infty} \frac{\Theta(i)\Upsilon(\zeta(i))}{\Upsilon_2(\zeta(i))} x(\zeta(i)).
 \end{aligned}$$

Using the monotonicity of  $x(\hbar)$  and  $\frac{x(\hbar)}{B_2(\hbar)}$ , we obtain

$$x(\zeta(\hbar)) \geq \frac{x(\zeta(\hbar))}{\Upsilon_2(\zeta(\hbar))} \sum_{\iota=\hbar_1}^{\zeta(\hbar)-1} \Upsilon_2(\iota+1)\Theta(\iota)\Upsilon(\zeta(\iota)) + x(\zeta(\hbar)) \sum_{\iota=\zeta(\hbar)}^{\hbar-1} \Theta(\iota)\Upsilon(\zeta(\iota)) + \Upsilon_2(\zeta(\hbar))x(\zeta(\hbar)) \sum_{\iota=\hbar}^{\infty} \frac{\Theta(\iota)\Upsilon(\zeta(\iota))}{\Upsilon_2(\zeta(\iota))}.$$

That is,

$$1 \geq \frac{1}{\Upsilon_2(\zeta(\hbar))} \sum_{\iota=\hbar_1}^{\zeta(\hbar)-1} \Upsilon_2(\iota+1)\Theta(\iota)\Upsilon(\zeta(\iota)) + \sum_{\iota=\zeta(\hbar)}^{\hbar-1} \Theta(\iota)\Upsilon(\zeta(\iota)) + \Upsilon_2(\zeta(\hbar)) \sum_{\iota=\hbar}^{\infty} \frac{\Theta(\iota)\Upsilon(\zeta(\iota))}{\Upsilon_2(\zeta(\iota))}$$

which contradicts (2.6) as  $\hbar \rightarrow \infty$ . Hence we conclude that  $\{z(\hbar)\}$  does not belong to  $NS_2$ .

Suppose now that  $z(\hbar) \in NS_0$ . Due to the monotonicity of  $z(\hbar)$  that there exists  $\lim_{\hbar \rightarrow \infty} z(\hbar) = d \geq 0$ . We claim that  $d = 0$ . If not, then  $z(\hbar) \geq d > 0$ . Using this in  $(E_c)$  and summing up from  $\hbar$  to  $\infty$  yields

$$L_2 z(\hbar) \geq \sum_{\iota=\hbar}^{\infty} \Theta(\iota)z(\zeta(\iota)) \geq d \sum_{\iota=\hbar}^{\infty} \Theta(\iota).$$

Summing up once more from  $\hbar$  to  $\infty$ , we get

$$-L_1 z(\hbar) \geq d \sum_{\iota=\hbar}^{\infty} \frac{1}{b_2(\iota)} \sum_{\varkappa=\iota}^{\infty} \Theta(\varkappa).$$

Now, Summing the preceding inequality from  $\hbar_1$  to  $\infty$  yields

$$z(\hbar_1) \geq d \sum_{\hbar=\hbar_1}^{\infty} \frac{1}{b_1(\hbar)} \sum_{\iota=\hbar}^{\infty} \frac{1}{b_2(\iota)} \sum_{\varkappa=\iota}^{\infty} \Theta(\varkappa)$$

which contradicts (2.5) and we conclude that

$$\lim_{\hbar \rightarrow \infty} z(\hbar) = \lim_{\hbar \rightarrow \infty} \frac{y(\hbar)}{\Lambda_{12}(\hbar)} = 0.$$

This end the proof. □

**Remark 2.9.** From the conclusion of Theorem 2.8, we see that every NOS  $\{y(\hbar)\}$  of  $(E)$  satisfies

$$|y(\hbar)| \leq M\Lambda_{12}(\hbar),$$

where  $M$  is a positive constant. Further, we have information about asymptotic behavior of  $D_1y(\hbar)$ . By discrete L'Hospital rule (see [1])

$$0 = \lim_{\hbar \rightarrow \infty} \frac{y(\hbar)}{\Lambda_{12}(\hbar)} = - \lim_{\hbar \rightarrow \infty} \frac{a_1(\hbar)\Delta y(\hbar)}{\Lambda_2(\hbar)}$$

and so  $|D_1y(\hbar)| \leq M_1\Lambda_2(\hbar)$ , where  $M_1$  is a positive constant.

Next, we extend the conclusion of Theorem 2.8 without assuming any condition on  $\alpha$ .

**Theorem 2.10.** Assume (2.5) and

$$\sum_{\hbar=\hbar_1}^{\infty} \frac{\Theta(\hbar)\Upsilon^\alpha(\zeta(\hbar))}{\Upsilon_2^\alpha(\zeta(\hbar))} = \infty \tag{2.8}$$

are satisfied. Then the result of Theorem 2.8 still holds.

*Proof.* Let  $\{y(\hbar)\}$  be an EPS of  $(E)$ . Adopting the same reasoning as in the proof of Theorem 2.8, we find that the corresponding sequence  $\{z(\hbar)\} \in NS_0$  or  $NS_2$ .

First assume that  $z(\hbar) \in NS_2$ . From Lemma 2.6(iii), we see that

$$z^\alpha(\zeta(\hbar)) \geq \frac{\Upsilon^\alpha(\zeta(\hbar))}{\Upsilon_2^\alpha(\zeta(\hbar))} (L_1 z(\zeta(\hbar)))^\alpha. \tag{2.9}$$

Since  $L_1 z(\hbar)$  is +ve and  $\uparrow$ , it follows that there exists a constant  $M_1 > 0$  such that  $L_1 z(\zeta(\hbar)) \geq M_1$  for all  $\hbar \geq \hbar_1$ . Using this in (2.9), we get

$$z^\alpha(\zeta(\hbar)) \geq \frac{M_1^\alpha \Upsilon^\alpha(\zeta(\hbar))}{\Upsilon_2^\alpha(\zeta(\hbar))}, \quad \hbar \geq \hbar_1. \tag{2.10}$$

Combining (2.10) with  $(E_c)$  and then summing up from  $\hbar_1$  to  $\infty$ , we get

$$M_1^\alpha \sum_{\hbar=\hbar_1}^\infty \frac{\Theta(\hbar)\Upsilon^\alpha(\zeta(\hbar))}{\Upsilon_2^\alpha(\zeta(\hbar))} \leq L_2 z(\hbar_1) < \infty$$

which contradicts (2.8) and so  $z(\hbar)$  does not belong to  $NS_2$ .

Next assume that  $z(\hbar) \in NS_0$ . Following a similar argument as in the proof of Theorem 2.8, we obtain

$$L_2 z(\hbar) \geq \sum_{\iota=\hbar}^\infty \Theta(\iota) z^\alpha(\zeta(\iota)) \geq d^\alpha \sum_{\iota=\hbar}^\infty \Theta(\iota).$$

The proof proceeds similarly to Theorem 2.8, so the details are not repeated here. This completes the proof.  $\square$

We now establish criteria for the oscillatory behavior of all solutions to equation  $(E)$ .

**Theorem 2.11.** *Let (2.2) hold. Assume that both the first order delay DEs*

$$\Delta\omega(\hbar) + \Theta(\hbar)\Upsilon^\alpha(\zeta(\hbar))\omega^\alpha(\zeta(\hbar)) = 0 \tag{2.11}$$

and

$$\Delta\nu(\hbar) + \Theta(\hbar)R^\alpha(\xi(\hbar), \zeta(\hbar))\nu^\alpha(\xi(\hbar)) = 0 \tag{2.12}$$

are oscillatory. Then equation  $(E)$  is oscillatory.

*Proof.* Assuming  $\{y(\hbar)\}$  is an EPS of  $(E)$ . From Corollary 2.4 and Corollary 2.5, we see that the corresponding sequence  $\{z(\hbar)\}$  is a PS of  $(E_c)$  and either  $z(\hbar) \in NS_0$  or  $z(\hbar) \in NS_2$ .

First assume that  $z(\hbar) \in NS_0$ . Then by Lemma 2.7, we have (2.3) and using this in  $(E_c)$ , we get

$$L_3 z(\hbar) + \Theta(\hbar)R^\alpha(\xi(\hbar), \zeta(\hbar))(L_2 z(\xi(\hbar)))^\alpha \leq 0.$$

Let  $\nu(\hbar) = L_2 z(\hbar) > 0$ . Then  $\{\nu(\hbar)\}$  represents a PS to the first-order delay difference inequality

$$\Delta\nu(\hbar) + \Theta(\hbar)R^\alpha(\xi(\hbar), \zeta(\hbar))\nu^\alpha(\xi(\hbar)) \leq 0.$$

But by Lemma 3 of [13], consequently, it follows that the corresponding DE (2.12) also admits a PS. This contradiction shows that  $z(\hbar)$  does not belong to  $NS_0$ .

Next assume that  $z(\hbar) \in NS_2$ . Summing  $(i)$  of Lemma 2.6 from  $\hbar_1$  to  $\hbar - 1$ , we obtain

$$z(\hbar) \geq \sum_{\iota=\hbar_1}^{\hbar-1} \frac{\Upsilon_2(\iota)}{b_1(\iota)} L_2 z(\iota) \geq \Upsilon(\hbar) L_2 z(\hbar). \tag{2.13}$$

Using (2.13) in  $(E_c)$  and then letting  $\omega(\hbar) = L_2 z(\hbar)$ , we see that  $\{\omega(\hbar)\}$  is a positive solution of the first-order delay difference inequality

$$\Delta\omega(\hbar) + \Theta(\hbar)\Upsilon^\alpha(\zeta(\hbar))\omega^\alpha(\zeta(\hbar)) \leq 0.$$

But by Lemma 3 of [13], consequently, we observe that the corresponding delay DE (2.11) possesses a PS, leading to another contradiction, thus concluding the proof of the theorem.  $\square$

In the following, we present explicit conditions for the oscillation of all solutions of (E).

**Corollary 2.12.** *Let  $\zeta(\hbar) = \hbar - \tau$  and  $\xi(\hbar) = \hbar - \mu$  where  $\tau$  and  $\mu$  are positive integers such that  $\tau > \mu$ . If*

(i)

$$\lim_{\hbar \rightarrow \infty} \inf \sum_{\iota=\hbar-\tau}^{\hbar-1} \Theta(\iota) \Upsilon(\iota - \tau) > \left(\frac{\tau}{\tau + 1}\right)^{\tau+1} \tag{2.14}$$

and

$$\lim_{\hbar \rightarrow \infty} \inf \sum_{\iota=\hbar-\mu}^{\hbar-1} \Theta(\iota) R(\iota - \mu, \iota - \tau) > \left(\frac{\mu}{\mu + 1}\right)^{\mu+1} \tag{2.15}$$

where  $\alpha = 1$ ,

(ii)

$$\sum_{\hbar=\hbar_1}^{\infty} \Theta(\hbar) \Upsilon^\alpha(\hbar - \tau) = \sum_{\hbar=\hbar_1}^{\infty} \Theta(\hbar) R^\alpha(\hbar - \mu, \hbar - \tau) = \infty \tag{2.16}$$

where  $0 < \alpha < 1$ ,

(iii) *there exists a constant  $\lambda_1 > \frac{1}{\tau} \ln \alpha$  such that*

$$\lim_{\hbar \rightarrow \infty} \inf [\Theta(\hbar) \Upsilon^\alpha(\hbar - \tau) \exp(-e^{\lambda_1 \hbar})] > 0 \tag{2.17}$$

and  $\exists$  a constant  $\lambda_2 > \frac{1}{\mu} \ln \alpha$  such that

$$\lim_{\hbar \rightarrow \infty} \inf [\Theta(\hbar) R^\alpha(\hbar - \mu, \hbar - \tau) \exp(-e^{\lambda_2 \hbar})] > 0 \tag{2.18}$$

when  $\alpha > 1$ ,

then equation (E) is oscillatory.

*Proof.* In view of conditions (2.14) and (2.15), along with Theorem 7.6.1 of [16], we conclude that equations (2.11) and (2.12) are oscillatory for  $\alpha = 1$ .

Next, considering (2.16) and Theorem 1 of [28], we see that equations (2.11) and (2.12) are oscillatory for  $0 < \alpha < 1$ .

Finally, based on (2.17) and (2.18), along with Theorem 2 of [28], we again establish that equations (2.11) and (2.12) are oscillatory for  $\alpha > 1$ .

Thus, the result follows immediately from Theorem 2.11, completing the proof. □

We end this section by presenting the following theorem.

**Theorem 2.13.** *Let  $\alpha = 1$ . If  $\zeta(\hbar) = \hbar - \tau$ ,  $\tau$  is a positive integer,*

$$\lim_{\hbar \rightarrow \infty} \sup \sum_{\iota=\zeta(\hbar)}^{\hbar} \frac{1}{b_1(\iota)} \sum_{\varkappa=\iota}^{\hbar} \frac{1}{b_2(\varkappa)} \sum_{j=\varkappa}^{\hbar} \Theta(j) > 1, \tag{2.19}$$

and

$$\lim_{\hbar \rightarrow \infty} \inf \Upsilon_2(\hbar) \sum_{\iota=\hbar}^{\infty} \frac{\Theta(\iota) \Upsilon(\zeta(\iota))}{\Upsilon_2(\iota)} > \frac{1}{4}, \tag{2.20}$$

then equation (E) is oscillatory.

*Proof.* Let  $\{y(\hbar)\}$  be a PS of (E). Following a similar argument as in the proof of Theorem 2.8, we observe that the corresponding sequence  $\{z(\hbar)\} \in NS_0$  or  $z(\hbar) \in NS_2$ .

First assume that  $z(\bar{h}) \in NS_0$ . Summing up  $(E_c)$  from  $u$  to  $\bar{h}$  and using the monotonicity of  $z(\bar{h})$ , we have

$$\begin{aligned} L_2 z(u) \geq L_2 z(u) - L_2 z(\bar{h} + 1) &= \sum_{\iota=u}^{\bar{h}} \Theta(\iota) z(\zeta(\iota)) \\ &\geq z(\zeta(\bar{h})) \sum_{\iota=u}^{\bar{h}} \Theta(\iota), \end{aligned}$$

that is,

$$\Delta(L_1 z(u)) \geq \frac{z(\zeta(\bar{h}))}{b_2(u)} \sum_{\iota=u}^{\bar{h}} \Theta(\iota).$$

Repeating the above summation from  $u$  to  $\bar{h} (> u)$  twice, we get

$$z(u) \geq z(\zeta(\bar{h})) \sum_{\iota=u}^{\bar{h}} \frac{1}{b_1(\iota)} \sum_{\varkappa=\iota}^{\bar{h}} \frac{1}{b_2(\varkappa)} \sum_{j=\varkappa}^{\bar{h}} \Theta(j).$$

Setting  $u = \zeta(\bar{h})$  in the last inequality, we obtain a contradiction with (2.19).

Next, assume that  $z(\bar{h}) \in NS_2$ . Now, using (iii) of Lemma 2.6 in  $(E_c)$ , we obtain

$$L_3 z(\bar{h}) + \Theta(\bar{h}) \frac{\Upsilon(\zeta(\bar{h}))}{\Upsilon_2(\zeta(\bar{h}))} L_1 z(\zeta(\bar{h})) \leq 0. \tag{2.21}$$

Since by Lemma 2.6(ii), we see that  $\frac{L_1 z(\bar{h})}{\Upsilon_2(\bar{h})} \leq \frac{L_1 z(\zeta(\bar{h}))}{\Upsilon_2(\zeta(\bar{h}))}$  and using this in (2.21), we have

$$L_3 z(\bar{h}) + \Theta(\bar{h}) \frac{\Upsilon(\zeta(\bar{h}))}{\Upsilon_2(\bar{h})} L_1 z(\bar{h}) \leq 0.$$

Let  $x(\bar{h}) = L_1 z(\bar{h})$ . Then the latter inequality reduced to

$$\Delta(b_2(\bar{h})\Delta x(\bar{h})) + \frac{\Theta(\bar{h})\Upsilon(\zeta(\bar{h}))}{\Upsilon_2(\bar{h})} x(\bar{h}) \leq 0. \tag{2.22}$$

Define

$$f(\bar{h}) = \frac{b_2(\bar{h})\Delta x(\bar{h})}{x(\bar{h})} > 0.$$

Then, using (2.22), we obtain

$$\Delta f(\bar{h}) \leq -\frac{\Theta(\bar{h})\Upsilon(\zeta(\bar{h}))}{\Upsilon_2(\bar{h})} - \frac{f(\bar{h})f(\bar{h} + 1)}{b_2(\bar{h})}.$$

By summing the last inequality over all integers from  $\bar{h}$  to infinity, we find that

$$f(\bar{h}) \geq \sum_{\iota=\bar{h}}^{\infty} \frac{\Theta(\iota)\Upsilon(\zeta(\iota))}{\Upsilon_2(\iota)} + \sum_{\iota=\bar{h}}^{\infty} \frac{f(\iota)f(\iota + 1)}{b_2(\iota)},$$

and so

$$\Upsilon_2(\bar{h})f(\bar{h}) \geq \Upsilon_2(\bar{h}) \sum_{\iota=\bar{h}}^{\infty} \frac{\Theta(\iota)\Upsilon(\zeta(\iota))}{\Upsilon_2(\iota)} + \Upsilon_2(\bar{h}) \sum_{\iota=\bar{h}}^{\infty} \frac{\Upsilon_2(\iota)f(\iota)\Upsilon_2(\iota + 1)f(\iota + 1)}{b_2(\iota)\Upsilon_2(\iota)\Upsilon_2(\iota + 1)}. \tag{2.23}$$

Let  $\lim_{\bar{h} \rightarrow \infty} \inf \Upsilon_2(\bar{h})f(\bar{h}) = M > 0$ , then from (2.23) and (2.20) we obtain

$$M > \frac{1}{4} + M^2$$

since  $\Upsilon_2(\bar{h}) \sum_{\iota=\bar{h}}^{\infty} \frac{1}{\Upsilon_2(\iota)\Upsilon_2(\iota + 1)b_2(\iota)} = 1$ . This is not possible for  $M > 0$ . This concludes the proof.  $\square$

### 3 Examples

Four examples are presented in this section to demonstrate the importance of our main results.

**Example 3.1.** Consider a noncanonical linear third-order delay DE

$$\Delta((\hbar + 2)(\hbar + 3)\Delta((\hbar + 1)\hbar\Delta y(\hbar))) + q_0 \frac{(\hbar - 1)(\hbar - 2)}{(\hbar + 1)}y(\hbar - 2) = 0, \quad \hbar \geq 3, \quad (3.1)$$

where  $q_0 > 0$  is a constant. By a simple calculation, the equation (3.1) transformed into the equation

$$\Delta^3 z(\hbar) + \frac{q_0}{(\hbar + 1)(\hbar + 2)(\hbar + 3)}z(\hbar - 2) = 0, \quad \hbar \geq 3$$

which is in canonical form. Further, we see that

$$\Upsilon_1(\hbar) \approx \hbar, \quad \Upsilon_2(\hbar) \approx \hbar, \quad \Upsilon(\hbar) \approx \frac{\hbar(\hbar - 1)}{2} \quad \text{and} \quad \alpha = 1.$$

The condition (2.5) becomes

$$\sum_{\hbar=3}^{\infty} \sum_{\iota=\hbar}^{\infty} \sum_{\varkappa=\iota}^{\infty} \frac{q_0}{(\varkappa + 1)(\varkappa + 2)(\varkappa + 3)} = \frac{q_0}{2} \sum_{\hbar=3}^{\infty} \frac{1}{(\hbar + 1)} = \infty,$$

that is, condition (2.5) holds. The condition (2.6) becomes

$$\begin{aligned} \lim_{\hbar \rightarrow \infty} \sup \left\{ \frac{1}{(\hbar - 2)} \sum_{\iota=3}^{\hbar-3} \frac{q_0(\iota - 2)(\iota - 3)}{(\iota + 2)(\iota + 3)} + \sum_{\iota=\hbar-2}^{\hbar-1} \frac{q_0(\iota - 2)(\iota - 3)}{(\iota + 1)(\iota + 2)(\iota + 3)} \right. \\ \left. + (\hbar - 2) \sum_{\iota=\hbar}^{\infty} \frac{q_0(\iota - 3)}{(\iota + 1)(\iota + 2)(\iota + 3)} \right\} = 2q_0 > 1, \end{aligned}$$

that is, condition (2.6) holds if  $q_0 > \frac{1}{2}$ . Therefore by Theorem 2.8, every NOS  $\{y(\hbar)\}$  of (3.1) satisfies  $\lim_{\hbar \rightarrow \infty} \hbar(\hbar + 1)|y(\hbar)| = 0$ .

**Example 3.2.** Consider the third-order noncanonical linear delay DE

$$\Delta((\hbar + 2)(\hbar + 3)\Delta(\hbar(\hbar + 1)\Delta y(\hbar))) + q_0(\hbar - 1)(\hbar - 2)(\hbar + 2)^2y(\hbar - 2) = 0, \quad \hbar \geq 3, \quad (3.2)$$

where  $q_0 > 0$  is a positive constant. By a simple calculation, we see that

$$\Lambda_1(\hbar) = \frac{1}{\hbar}, \quad \Lambda_2(\hbar) = \frac{1}{\hbar + 2}, \quad \Lambda_{12}(\hbar) = \frac{1}{2\hbar(\hbar + 1)}, \quad \Lambda_{21}(\hbar) = \frac{1}{2\hbar(\hbar + 1)(\hbar + 2)}, \quad \text{and} \quad \alpha = 1.$$

Now, the transformed equation is

$$\Delta^3 z(\hbar) + q_0 \frac{\hbar + 2}{\hbar + 3}z(\hbar - 2) = 0, \quad \hbar \geq 3$$

which is clearly in canonical form. Further calculation shows that

$$\Upsilon_1(\hbar) \approx \hbar, \quad \Upsilon(\hbar) \approx \frac{\hbar(\hbar - 1)}{2} \quad \text{and} \quad \Theta(\hbar) = q_0 \frac{(\hbar + 2)}{(\hbar + 3)}.$$

Choosing  $\xi(\hbar) = \hbar - 1$ , we see that condition (2.2) holds and  $R(\hbar - 1, \hbar - 2) = 1$ . The condition (2.14) becomes

$$\lim_{\hbar \rightarrow \infty} \inf \sum_{\iota=\hbar-2}^{\hbar-1} q_0 \frac{(\iota + 2)(\iota - 2)(\iota - 3)}{(\iota + 3)} = \infty > \frac{8}{27},$$

that is, condition (2.14) holds if  $q_0 > 0$ . The condition (2.15) becomes

$$\lim_{\hbar \rightarrow \infty} \inf \sum_{\iota=\hbar-1}^{\hbar-1} q_0 \frac{(\iota + 2)}{(\iota + 3)} = q_0 > \frac{1}{4},$$

that is, condition (2.15) holds if  $q_0 > \frac{1}{4}$ . Hence, by Corollary 2.12(i), equation (3.2) is oscillatory if  $q_0 > \frac{1}{4}$ .

**Example 3.3.** Consider a third-order delay DE with noncanonical structure and sublinear characteristics

$$\Delta((\hbar + 2)(\hbar + 3)\Delta(\hbar(\hbar + 1)\Delta y(\hbar))) + q_0(\hbar - 1)^{\frac{1}{3}}(\hbar - 2)^{\frac{1}{3}}(\hbar + 2)y^{\frac{1}{3}}(\hbar - 2) = 0, \quad \hbar \geq 3, \quad (3.3)$$

where  $q_0 > 0$  is a constant. By a simple calculation, we see that

$$\Lambda_1(\hbar) = \frac{1}{\hbar}, \quad \Lambda_2(\hbar) = \frac{1}{\hbar + 2}, \quad \Lambda_{12}(\hbar) = \frac{1}{2\hbar(\hbar + 1)}, \quad \Lambda_{21} = \frac{1}{2\hbar(\hbar + 1)(\hbar + 2)}, \quad \text{and} \quad \alpha = \frac{1}{3}.$$

Choosing  $\xi(\hbar) = \hbar - 1$ , we see that condition (2.2) holds. Now, the transformed equation is

$$\Delta^3 z(\hbar) + \frac{2^{\frac{2}{3}}q_0}{(\hbar + 3)}z^{\frac{1}{3}}(\hbar - 2) = 0, \quad \hbar \geq 3,$$

which is in canonical form. Further calculation shows that

$$\Upsilon_1(\hbar) \approx \hbar, \quad \Upsilon(\hbar) \approx \frac{\hbar(\hbar - 1)}{2}, \quad \Theta(\hbar) = \frac{2^{\frac{2}{3}}q_0}{(\hbar + 3)} \quad \text{and} \quad R(\hbar - 1, \hbar - 2) = 1.$$

The condition (2.16) becomes

$$\sum_{\hbar=3}^{\infty} \frac{2^{\frac{2}{3}}q_0}{(\hbar + 3)} \frac{(\hbar - 2)^{\frac{1}{3}}(\hbar - 3)^{\frac{1}{3}}}{2} = \infty,$$

and

$$\sum_{\hbar=3}^{\infty} \frac{2^{\frac{2}{3}}q_0}{(\hbar + 3)} = \infty,$$

that is, condition (2.16) holds if  $q_0 > 0$ . Consequently, by virtue of Corollary 2.12(ii), we deduce that equation (3.3) is oscillatory if  $q_0 > 0$ .

**Example 3.4.** Consider a noncanonical superlinear third-order delay DE

$$\Delta((\hbar + 2)(\hbar + 3)\Delta(\hbar(\hbar + 1)\Delta y(\hbar))) + q_0(\hbar - 1)^3(\hbar - 2)^3 \exp(5^{\hbar})y^3(\hbar - 2) = 0, \quad \hbar \geq 3, \quad (3.4)$$

where  $q_0 > 0$  is a constant. The transformed equation is

$$\Delta^3 z(\hbar) + \frac{q_0}{16} \frac{\exp(5^{\hbar})}{(\hbar + 2)^3(\hbar + 3)^3} z^3(\hbar - 2) = 0, \quad \hbar \geq 3$$

which is in canonical form. Choose  $\xi(\hbar) = \hbar - 1$ , the condition (2.2) holds. By a simple calculation, we see that

$$\Upsilon_1(\hbar) \approx \hbar, \quad \Upsilon(\hbar) \approx \frac{\hbar(\hbar - 1)}{2}, \quad \alpha = 3, \quad \Theta(\hbar) = \frac{q_0}{16} \frac{\exp(5^{\hbar})}{(\hbar + 2)^3(\hbar + 3)^3} \quad \text{and} \quad R(\hbar - 1, \hbar - 2) = 1.$$

Since  $\tau = 2$  and  $\alpha = 3$ , by taking  $\lambda_1 = 1$  we see that  $1 > \frac{1}{2} \ln 3$ , and so the condition (2.17) becomes

$$\liminf_{\hbar \rightarrow \infty} \left[ \frac{q_0 \exp(9^{\hbar})(\hbar - 2)^3(\hbar - 3)^3 \exp(-e^{\hbar})}{16 (\hbar + 2)^3(\hbar + 3)^3} \right] > \frac{q_0}{16} \exp(9^{\hbar} - e^{\hbar}) > 0,$$

that is, condition (2.17) holds. Further  $\mu = 1$ ,  $\alpha = 3$ , and so by taking  $\lambda_2 = 1.5$ , we see that  $1.5 > \ln 3$ , and the condition (2.18) becomes

$$\liminf_{\hbar \rightarrow \infty} \left[ \frac{q_0 \exp(9^{\hbar}) \exp(-e^{1.5\hbar})}{16 (\hbar + 2)^3(\hbar + 3)^3} \right] > \lim_{\hbar \rightarrow \infty} \left[ \frac{q_0 \exp(9^{\hbar} - e^{1.5\hbar})}{16 (\hbar + 3)^6} \right] > 0,$$

that is, condition (2.17) holds. Thus, it follows from Corollary 2.12(iii) that equation (3.4) exhibits oscillatory behavior.

**Remark 3.5.** One should observe that

- (1) The findings presented in [23] exclusively indicate that any solution of (E) is either oscillatory or converges to zero.
- (2) The results reported in [15, 20, 21] are not applicable to Example 3.3 and Example 3.4 due to the nonlinearity of the equations.

## 4 Conclusion

This study presents new oscillation criteria for third-order delay difference equations. Our methodology involves transforming the examined noncanonical third-order nonlinear delay difference equation into a canonical form without imposing additional constraints. This transformation significantly reduces the classification of NOS from four to two.

Subsequently, the techniques developed for canonical-type equations can be applied to derive oscillation conditions for noncanonical third-order delay difference equations. Thus, the results of this study provide novel insights and enhance the current understanding of oscillation theory related to third-order delay difference equations.

Exploring this approach to analyze the oscillatory behavior of solutions in various third-order noncanonical neutral-type difference equations may yield promising outcomes in the future.

## Author Contributions

All authors have equally and significantly contributed to the contents of this manuscript.

## Declaration of Interest Statement

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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