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SOME INSIGHTS INTO WEIGHTED IDEAL STATISTICAL CONVERGENCE OF FUZZY VARIABLES IN CREDIBILITY SPACES

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Abstract This paper investigates different forms of weighted ideal statistical convergence for double sequences of fuzzy variables within the framework of credibility. Furthermore, illustrative examples are provided to analyze the relationships among these fuzzy variable double sequences. Lastly, the concept of a weighted ideal statistically Cauchy sequence of fuzzy variable double sequences is introduced to establish its connection with weighted ideal statistically convergent fuzzy variable double sequences.

1 Introduction

In 1965, Zadeh introduced the concept of fuzzy sets in the domain of fuzzy logic by employing the membership degree function [45]. To evaluate fuzzy events, he also formulated the ideas of possibility measure and necessity measure. These measures are recognized as normal, non-negative, and monotonic. However, they do not conform to the principles of excluded middle and contradiction, nor do they uphold the standard of truth preservation. This behavior stems from their inability to satisfy the self-duality property, a crucial aspect for both theoretical insights and practical implementations.

To resolve this limitation, Liu and Liu introduced the concept of the credibility measure, which adheres to the self-duality property [26]. Unlike possibility and necessity measures, the credibility measure ensures that the complement of a fuzzy event holds the same credibility as the event itself. By satisfying the principles of excluded middle and contradiction, it provides a more robust and consistent framework for managing uncertainty and ambiguity across various applications.

The development of the credibility measure by Liu and Liu significantly enriched the theory of fuzzy logic and broadened its practical applications. It established a strong basis for reasoning under uncertainty, enabling more precise modeling and informed decision-making in contexts with vague or incomplete information. The self-duality property embedded in the credibility measure ensures logical coherence and consistency within fuzzy systems, thereby enhancing their reliability and applicability.

The introduction of credibility theory by Liu and Liu in 2006 [22], along with its further advancements in 2007 [23], marked the emergence of a novel branch in mathematical science. Since then, credibility theory has rapidly evolved and found extensive application across a wide range of disciplines [2, 21, 24, 25, 27, 41, 44].

Credibility theory and uncertainty theory are two distinct approaches employed in the study of sequence spaces. While credibility theory focuses on sequences of fuzzy variables, uncertainty theory deals with sequences of uncertain variables. The fundamental difference lies in the nature of these variables: a fuzzy variable is defined on a possibility space and represents imprecise information, whereas an uncertain variable is based on an uncertainty space and describes belief-based uncertainty. Despite this distinction, both theories explore sequence convergence through similar principles. Specifically, a fuzzy variable is a mapping from a possibility space to the set of real numbers [33], while an uncertain variable is a function from an uncertainty space to the set of real numbers [23].

Simultaneously, the notion of statistical convergence for sequences of real numbers was introduced by Fast in [9]. Kostyrko et al. [20] expanded on the concept of $\mathcal{I}$-convergence from the perspective of sequence spaces, establishing its connection with summability theory. Subsequently, this idea has been extended in various directions. Das et al. [6] introduced $\mathcal{I}$-convergence for double sequences within a metric space and examined several of its characteristics. Further exploration of statistical convergence, $\mathcal{I}$-convergence, and their applications is available in studies [1, 3, 8, 10, 11, 13, 14, 16, 18, 19, 28, 29, 35, 36, 43].

Meanwhile, Savaş et al. [37] pioneered the study of statistical convergence within the structure of credibility spaces, establishing the groundwork for examining this concept in the framework of credibility theory. Important studies on fuzzy variable sequences in credibility spaces can also be examined in references [4, 5, 17, 38, 39].

The exploration of statistical and ideal convergence in the realms of credibility and uncertainty theories has provided significant insights and methodologies for studying the behavior and characteristics of sequences. These developments have enhanced our comprehension of convergence and have practical relevance across a broad spectrum of mathematical and applied fields. For some related study see [7, 12, 30, 34, 40, 42].

The objective of this paper is to explore a novel type of convergence for sequences of fuzzy variables. The structure of the paper is as follows: Section 1 provides a review of the relevant literature within the introduction. The primary results are presented in Section 2, where the foundational definitions and notations that form the basis of the paper are introduced. Section 3 focuses on the investigation of weighted ideal statistical convergence for double sequences of fuzzy variables within the credibility framework, along with the development of key properties of deferred statistical convergence in credibility. Finally, Section 4 concludes by summarizing the findings derived from the results.

2 Preliminary concepts

Before exploring the concept of lacunary ideal convergence, it is essential to lay the groundwork with preliminary definitions and theorems related to fuzzy variable sequences and credibility space. In this section, we provide the necessary background to ensure a thorough understanding of the subsequent analysis. We present the bounded convergence theorem by Liu and Wang [27], along with results regarding the interrelationships between various convergence concepts of fuzzy variable sequences in a credibility space.

The definitions and properties required for this paper are presented in Li and Liu [21], Liu [25], Liu and Liu [26], and Wang and Liu [41].

Mathematically, a fuzzy variable ξ is defined as a function:

$$\xi : \Omega \to \mathbb{R}$$

where Ω is a possibility space, and the possibility distribution of ξ describes the degree of possibility for different real values. Unlike probabilistic approaches, which rely on probability distributions, fuzzy variables use possibility measures to handle uncertainty.

In the subsequent analysis, we delve into the foundational investigation of ideals and filters.

Let's assume that $Y \neq \emptyset$. We define $\mathcal{I} \subset 2^Y$ as an ideal on Y if it satisfies the following properties: (a) for any S and T in $\mathcal{I}$, their union $S \cup T$ is also in $\mathcal{I}$, and (b) for any S in $\mathcal{I}$ and any subset T contained in S, T is also in $\mathcal{I}$.

Let us assume that $Y \neq \emptyset$. We define $\mathcal{F} \subset 2^Y$ as a filter on Y if it satisfies the following properties: (a) for any S and T in $\mathcal{F}$, their intersection $S \cap T$ is also in $\mathcal{F}$, and (b) for any S in $\mathcal{F}$ and any set T that contains S, T is also in $\mathcal{F}$.

We say that $\mathcal{I}$ is non-trivial if it satisfies two conditions: Y is not an element of $\mathcal{I}$ and $\mathcal{I}$ is not an empty set. An ideal $\mathcal{I}$ is referred to as an admissible ideal in Y if it contains all singleton sets $\{w\}$ for $w \in Y$. In this case, the associated filter $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{Y - S : S \in \mathcal{I}\}$ is called the filter connected with the ideal.

A nontrivial ideal $\mathcal{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathcal{I}_2$ for each $i\in\mathbb{N}$.

The notion of weighted statistical convergence was introduced by Karakaya and Chishti [15] as follows:

Let (p_n) be a sequence of positive real numbers such that $\mathfrak{U}_\sigma=\mathfrak{p}_1+\mathfrak{p}_2+\ldots+\mathfrak{p}_\sigma\to\infty$ as $\sigma\to\infty$ and $\mathfrak{p}_\sigma\neq 0$ and $\mathfrak{p}_0>0$.

A sequence $x=(x_\mathfrak{f})$ is said to be weighted statistical convergent if for every $\varkappa>0$

$$\lim_{\sigma\to\infty}\frac{1}{\mathfrak{U}_\sigma}\left|\{\mathfrak{f}\leq\sigma:\mathfrak{p}_\mathfrak{f}\left|x_\mathfrak{f}-x_0\right|\geq\varkappa\}\right|=0.$$

In this case we write $S_{\overline{N}}-\lim x=x_0$.

Mursaleen et al. [32] modified the definition of weighted statistical convergence as follows:

A sequence $x=(x_\mathfrak{f})$ is said to be weighted statistical convergent if for every $\varkappa>0$

$$\lim_{\sigma\to\infty}\frac{1}{\mathfrak{U}_\sigma}\left|\{\mathfrak{f}\leq\mathfrak{U}_\sigma:\mathfrak{p}_\mathfrak{f}\left|x_\mathfrak{f}-x_0\right|\geq\varkappa\}\right|=0.$$

A double sequence $x=(x_{\mathfrak{f},\mathfrak{h}})$ is said to be convergent in the Pringsheim sense if for every $\varepsilon>0$ there exists $N\in\mathbb{N}$ such that $|x_{\mathfrak{f},\mathfrak{h}}-x_0|<\varepsilon$, whenever $\mathfrak{f},\mathfrak{h}>N$. In this case we write $P-\lim x=x_0$.

Let $\mathfrak{F}\subseteq\mathbb{N}\times\mathbb{N}$ and $\mathfrak{F}(\sigma,\rho)=\{(\mathfrak{f},\mathfrak{h}),\mathfrak{f}\leq\sigma \text{ and } \mathfrak{h}\leq\rho\}$. The double natural density of $\mathfrak{F}$ is defined by

$$\delta_2(\mathfrak{F})=P-\lim_{\sigma,\rho\to\infty}\frac{1}{\sigma\rho}|K(\sigma,\rho)|,\ \text{if the limit exists.}$$

A double sequence $x=(x_{\mathfrak{f},\mathfrak{h}})$ is said to be statistically convergent to x_0 if for every $\varkappa>0$ the set

$$\{(\mathfrak{f},\mathfrak{h}),\mathfrak{f}\leq\sigma \text{ and } \mathfrak{h}\leq\rho:|x_{\mathfrak{f},\mathfrak{h}}-x_0|\geq\varkappa\}$$

has double natural density zero [31]. In this case we write $st_2-\lim x=x_0$.

Let $\mathfrak{p}=\{\mathfrak{p}_\mathrm{j}\}_{\mathrm{j}=0}^{\infty}$ and $\mathfrak{q}=\{\mathfrak{q}_\mathrm{i}\}_{\mathrm{i}=0}^{\infty}$ be sequences of non-negative numbers that are not all zero and let $\mathfrak{R}_\rho=\mathfrak{q}_1+\mathfrak{q}_2+\mathfrak{q}_3+\ldots+\mathfrak{q}_\rho$, $\mathfrak{q}_1>0$ and $\mathfrak{U}_\sigma=\mathfrak{p}_1+\mathfrak{p}_2+\ldots+\mathfrak{p}_\sigma$, $\mathfrak{p}_1>0$.

Definition 2.1. Let $\mathfrak{F}$ be a subset of $\mathbb{N}\times\mathbb{N}$. We define the double weighted density of $\mathfrak{T}$ by

$$\delta_{\bar{N}_2}(\mathfrak{F})=\lim_{\sigma,\rho\to\infty}\frac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\mathfrak{F}_{\mathfrak{U}_\sigma\mathfrak{R}_\rho}(\sigma,\rho)\right|,\ \text{provided the limit exists,}$$

where

$$\mathfrak{F}_{\mathfrak{U}_\sigma\mathfrak{R}_\rho}(\sigma,\rho)=\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\left|x_{\mathfrak{f},\mathfrak{h}}-x_0\right|\geq\varkappa\},$$

$\liminf\mathfrak{p}_\sigma>0, \liminf\mathfrak{q}_\rho>0$. We say that a double sequence $x=(x_{\mathfrak{f},\mathfrak{h}})$ is said to be weighted statistically convergent (or $S_{\bar{N}_2}$-convergent) to x_0 if for every $\varkappa>0$

$$\lim_{\sigma,\rho\to\infty}\frac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\left|x_{\mathfrak{f},\mathfrak{h}}-x_0\right|\geq\varkappa\}\right|=0.$$

This notion is expressed as $S_{\bar{N}_2}-\lim x=x_0$.

3 Main results

Building on the foundational concepts discussed earlier, we now present the main results of this study. By exploring various forms of weighted ideal statistical convergence for double sequences of fuzzy variables, we uncover significant relationships within the framework of credibility. The following results provide deeper insights into these convergence properties, supported by illustrative examples and a discussion on weighted ideal statistically Cauchy sequences.

Definition 3.1. Let $\varpi=(\varpi_{\mathfrak{f},\mathfrak{h}})$ be a sequence of fuzzy variables in a credibility space $(\Omega,P(\Omega),\mathrm{Cr})$. A sequence $\varpi=(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in almost surely to a fuzzy variable ϖ_0 if there exists a set $Q\in P(\Omega)$ with unit credibility measure such that

$$\left\{(\sigma,\rho)\in\mathbb{N}^2:\frac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi_0(\varsigma)\right|\geq\varkappa\}\right|\geq\varrho\right\}\in\mathcal{I}_2,$$

for any $\varsigma \in Q$ and $\varkappa, \varrho > 0$. This is represented as $WS^{\mathcal{I}_2}(\Omega_{a.s}) - \lim(\varpi_{\mathfrak{f},\mathfrak{h}}) = \varpi$ or $\varpi_{\mathfrak{f},\mathfrak{h}} \overset{WS^{\mathcal{I}_2}(\Omega_{a.s})}{\rightarrow} \varpi$.

Definition 3.2. A sequence $\varpi = (\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in credibility to ϖ_0 if for any preassigned $\varkappa, \varrho, \sigma > 0$

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\} \in \mathcal{I}_2.$$

This is expressed as $WS^{\mathcal{I}_2}(\Omega_{\mathrm{Cr}}) - \lim(\varpi_{\mathfrak{f},\mathfrak{h}}) = \varpi$ or $\varpi_{\mathfrak{f},\mathfrak{h}} \overset{WS^{\mathcal{I}_2}(\Omega_{\mathrm{Cr}})}{\rightarrow} \varpi$.

Definition 3.3. A sequence $\varpi = (\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in mean to ϖ_0 if all fuzzy variables $\varpi_0, \varpi_{\mathfrak{f},\mathfrak{h}}$ have finite expected values, and for any given $\varkappa, \varrho > 0$

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{E}[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\|] \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2.$$

Definition 3.4. Let $\varpi = (\varpi_{\mathfrak{f},\mathfrak{h}})$ represent a sequence of fuzzy variables in a credibility space $(\Omega, P(\Omega), \mathrm{Cr})$ and $\Phi_{\mathfrak{f},\mathfrak{h}}$ denote the credibility distribution functions corresponding to the fuzzy variables $\varpi_{\mathfrak{f},\mathfrak{h}}$. Then, $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is called weighted ideal statistically convergent in distribution to ϖ_0 whose credibility distribution function is Φ if for any given $\varkappa, \varrho > 0$

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} |\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{y}) - \Phi(\mathfrak{y})| \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2,$$

where $\mathfrak{y}$ is any real number where Φ is continuous.

Definition 3.5. The sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ of fuzzy variables in the space $(\Omega, P(\Omega), \mathrm{Cr})$ is called weighted ideal statistically convergent with respect to uniformly almost surely to a ϖ_0 if there exists some events $A_{\mathfrak{f},\mathfrak{h}}$ $(\mathfrak{f}, \mathfrak{h} \in \mathbb{N})$ each of whose credibility measure approaches zero such that the sequence is weighted uniformly statistically converges to the same limit in the sense of ideal. In this case

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} |\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)| \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I},$$

for all $\varsigma \in \Omega - A_{\mathfrak{f},\mathfrak{h}}$, and $\varkappa, \varrho > 0$.

Remark 3.6. The concepts of weighted ideal statistical convergence in terms of credibility and almost sure convergence are independent. This assertion is illustrated through the following sequential examples.

Example 3.7. Weighted ideal statistical convergence in almost surely does not imply weighted ideal statistical convergence in terms of credibility. Consider the credibility space $(\Omega, P(\Omega), \mathrm{Cr})$ with $P(\Omega) = \{\varsigma_1, \varsigma_2, ...\}$ having a credibility measurable function as follows:

$$\mathrm{Cr}\{Y\} = \begin{cases} \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y} \frac{\mathfrak{f}+\mathfrak{h}}{2(\mathfrak{f}+\mathfrak{h})+1}, & \text{if } \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y} \frac{\mathfrak{f}+\mathfrak{h}}{2(\mathfrak{f}+\mathfrak{h})+1} < \frac{1}{2}; \\ 1 - \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y^c} \frac{\mathfrak{f}+\mathfrak{h}}{2(\mathfrak{f}+\mathfrak{h})+1}, & \text{if } \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y^c} \frac{\mathfrak{f}+\mathfrak{h}}{2(\mathfrak{f}+\mathfrak{h})+1} < \frac{1}{2}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Now, let us define fuzzy variable by

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) = \begin{cases} \mathfrak{f}+\mathfrak{h}, & \text{if } \varsigma = \varsigma_{\mathfrak{f},\mathfrak{h}} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. Then we obtain

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} |\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)| \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2,$$

where $\mathfrak{p}_\mathfrak{f} = \mathfrak{q}_\mathfrak{h} = 1$ and for any $\varsigma \in Q \in P(\Omega)$ and $\varkappa, \varrho > 0$ and $\mathfrak{U}_\sigma, \mathfrak{R}_\rho \geq 2$,

$$\begin{aligned}&\left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \tfrac{1}{2}\right\}\\ &= \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\varsigma : |\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)| \geq \varkappa\} \geq \varrho\}\right| \geq \tfrac{1}{2}\right\}\\ &= \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\varpi_{\mathfrak{f},\mathfrak{h}}\}\}\right| \geq \tfrac{1}{2}\right\} \in \mathcal{I}_2,\end{aligned}$$

where $\mathfrak{p}_\mathfrak{f} = \mathfrak{q}_\mathfrak{h} = 1$. Thus, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is not weighted ideal statistically convergent in terms of credibility to ϖ_0.

Example 3.8. On the other hand, weighted ideal statistical convergence in almost surely does not imply weighted ideal statistical convergence in terms of credibility. For example, consider, $P(\Omega) = \{\varsigma_1, \varsigma_2, ...\}$, with $\mathrm{Cr}\{\varsigma_\mathfrak{t}\} = (\mathfrak{t}-1)/\mathfrak{t}$ for $\mathfrak{t} = 1, 2, ...$ The fuzzy variables are defined as follows:

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma_\mathfrak{t}) = \begin{cases} \mathfrak{f}+\mathfrak{h}, & \text{if } \mathfrak{t} = \mathfrak{f}+\mathfrak{h} \\ 0, & \text{if not} \end{cases}$$

where $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. As a result, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in almost surely to ϖ. However, for any small $\varkappa, \varrho > 0$ and $\sigma \in \left(0, \frac{1}{2}\right)$, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ does not exhibit weighted ideal statistical convergence in terms of credibility.

Example 3.9. Weighted ideal statistical convergence in credibility does not imply weighted ideal statistical convergence in almost surely. We examine the credibility space for fuzzy variables $(\Omega, P(\Omega), \mathrm{Cr})$ with $P(\Omega) = \{\varsigma_1, \varsigma_2, ...\}$ using Lebesgue measure and Borel algebra. For $\mathfrak{f}, \mathfrak{h} \in \mathbb{Z}^+$, there exists integers t_1, t_2 such that $\mathfrak{f} = 2^{t_1} + \mathfrak{u}$ and $\mathfrak{h} = 2^{t_2} + \mathfrak{u}$, where $\mathfrak{u}$ is an integer between 0 and $\min\{2^{t_1}, 2^{t_2}\} - 1$. The fuzzy variables are defined by

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) = \begin{cases} 1, & \text{if } \frac{\mathfrak{u}}{2^{t_1+t_2}} \leq \varsigma \leq \frac{\mathfrak{u}+1}{2^{t_1+t_2}} \\ 0, & \text{if not,} \end{cases}$$

for $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. For some given $\varkappa, \varrho, \sigma > 0$, one has

$$\begin{aligned}&\left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\}\\ &= \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\varsigma : |\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\}\\ &= \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\varpi_{\mathfrak{f},\mathfrak{h}}\}\}\right| \geq \sigma\right\} \in \mathcal{I}_2,\end{aligned}$$

where $\mathfrak{p}_\mathfrak{f} = \mathfrak{q}_\mathfrak{h} = 1$. Consequently, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in credibility. Furthermore,

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{E}\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\|\right] \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2.$$

Therefore, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in mean to ϖ_0. For any $\varsigma \in [0, 1]$, there exists an infinite number of closed intervals of the form $\left[\frac{\mathfrak{u}}{2^{t_1+t_2}}, \frac{\mathfrak{u}+1}{2^{t_1+t_2}}\right]$ containing ς. Thus, $\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)$ does not weighted statistically converge to zero. In other words, $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is not weighted ideal statistically convergent in almost surely to ϖ_0.

Example 3.10. Conversely, weighted ideal statistical convergence in terms of credibility does not imply weighted ideal statistical convergence in almost surely, too. For example, consider, $\Omega = \{\varsigma_1, \varsigma_2, ...\}$, with $\mathrm{Cr}\{\varsigma_\mathfrak{t}\} = 1/\mathfrak{t}$ for $\mathfrak{t} = 1, 2, ...$ The fuzzy variables are defined as follows:

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma_\mathfrak{t}) = \begin{cases} (\mathfrak{t}+1)/\mathfrak{t}, & \text{when } \mathfrak{t} = \mathfrak{f}+\mathfrak{h},\ \mathfrak{f}+\mathfrak{h}+1,\ \mathfrak{f}+\mathfrak{h}+2, ... \\ 0, & \text{if not} \end{cases} \tag{3.1}$$

for $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. For any small number $\varkappa, \varrho > 0$ and $\sigma \in \left[\frac{1}{2}, 1\right)$,

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\} \in \mathcal{I}_2,$$

which states that $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ weighted ideal statistical converges in credibility to ϖ. But, it is obvious that $\varpi_{\mathfrak{f},\mathfrak{h}} \overset{WS^{\mathcal{I}_2}(\Omega_{a.s})}{\nrightarrow} \varpi$.

Remark 3.11. The concepts of weighted ideal statistical convergence almost surely and in mean do not imply each other. This claim is demonstrated through the following two successive cases.

Example 3.12. Weighted ideal statistical convergence in mean does not imply weighted ideal statistical convergence in almost surely. Let us consider the credibility space $(\Omega, P(\Omega), \mathrm{Cr})$ with $P(\Omega) = \{\varsigma_1, \varsigma_2, ...\}$ and the credibility measure for the events is determined by $\mathrm{Cr}\{Y\} = \sum\limits_{\varsigma_{\mathfrak{f}}, \varsigma_{\mathfrak{h}} \in Y} \frac{1}{2^{\mathfrak{f}+\mathfrak{h}}}$. The fuzzy variables are defined by

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) = \begin{cases} 2^{\mathfrak{f}+\mathfrak{h}}, & \text{if } \varsigma = \varsigma_{\mathfrak{f}+\mathfrak{h}} \\ 0, & \text{if not} \end{cases}$$

for $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. Thus, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in almost surely to ϖ. Consider

$$\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{y}) = \begin{cases} 0, & \text{if } \mathfrak{y} < 0 \\ 1 - \frac{1}{2^{\mathfrak{f}+\mathfrak{h}}}, & \text{if } 0 \le \mathfrak{y} < 2^{\mathfrak{f}+\mathfrak{h}} \\ 1, & \text{if } \mathfrak{y} \ge 2^{\mathfrak{f}+\mathfrak{h}}. \end{cases}$$

Here, $\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{y})$ is the credibility distribution of $\|\varpi_{\mathfrak{f},\mathfrak{h}}\|$. Then, we have

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_{\mathfrak{f}} \mathfrak{q}_{\mathfrak{h}} \mathrm{E}\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\|\right] \ge 1\}\right| \ge \varrho\right\} \in \mathcal{I}_2.$$

So, the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ does not weighted ideal statistically converge in mean to ϖ.

Example 3.13. Weighted ideal statistical convergence in mean does not necessarily lead to weighted ideal statistical convergence almost surely. Consider the fuzzy variables as defined in (3.1), which fail to exhibit weighted ideal statistical convergence almost surely to ϖ. However, we observe that

$$E\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi\|\right] = \frac{(\mathfrak{f}+\mathfrak{h}) + 1}{2(\mathfrak{f}+\mathfrak{h})^2} \to 0.$$

Consequently, for any $\varkappa, \varrho > 0$, we obtain

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_{\mathfrak{f}} \mathfrak{q}_{\mathfrak{h}} \mathrm{E}\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\|\right] \ge \varkappa\}\right| \ge \varrho\right\} \in \mathcal{I}_2,$$

which indicates that $\{\varpi_{klm}\}$ is weighted ideal statistically convergent in mean to ϖ.

Example 3.14. Weighted ideal statistical convergence almost surely does not imply weighted ideal statistical convergence in mean. To illustrate this, consider $\Theta = \{\theta_1, \theta_2, ...\}$ with $\mathrm{Cr}\{\varsigma_{\mathfrak{t}}\} = 1/\mathfrak{t}$ for $\mathfrak{t} = 1, 2, ...$ and let the fuzzy variables be defined as

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma_{\mathfrak{t}}) = \begin{cases} \mathfrak{f}+\mathfrak{h}, & \text{if } \mathfrak{t} = \mathfrak{f}+\mathfrak{h} \\ 0, & \text{if not} \end{cases} \tag{3.2}$$

for $\mathfrak{f}, \mathfrak{h} = 1, 2, ...$ and $\varpi = 0$. Under this setup, the sequence $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ is weighted ideal statistically convergent almost surely to ϖ. However, for any $\varkappa > 0$ and $\varrho \in \left(0, \frac{1}{2}\right)$,

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_{\mathfrak{f}} \mathfrak{q}_{\mathfrak{h}} \mathrm{E}\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\|\right] \ge \varkappa\}\right| \ge \varrho\right\} \notin \mathcal{I}_2.$$

This indicates that the sequence $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ does not exhibit weighted ideal statistical convergence in mean to ϖ.

Remark 3.15. Weighted ideal statistical convergence in credibility does not imply weighted ideal statistical convergence in mean.

Example 3.16. Consider the fuzzy variables defined by (3.2) which do not exhibit weighted ideal statistical convergence in mean to ϖ. However, for any $\varkappa, \varrho > 0$ and $\sigma \in \left[\frac{1}{2}, 1\right)$, we have

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f} \mathfrak{q}_\mathfrak{h} \mathrm{Cr}\left\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \ge \varkappa\right\} \ge \varrho\}\right| \ge \sigma\right\} \in \mathcal{I}_2.$$

This shows that the sequence $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ is weighted ideal statistically convergent in credibility to ϖ.

Remark 3.17. Weighted ideal statistically convergence in distribution does not imply weighted ideal statistically convergence in credibility. Let us now consider an example to illustrate this.

Example 3.18. Assume that $\Omega = \{\varsigma_1, \varsigma_2, ...\}$ represents the credibility space $(\Omega, P(\Omega), \mathrm{Cr})$ with $\mathrm{Cr}(\varsigma_1) = 1$ and $\mathrm{Cr}(\varsigma_2) = -1$. We establish the fuzzy variable as follows:

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) = \begin{cases} 1, & \text{if } \varsigma = \varsigma_1 \\ -1, & \text{if } \varsigma = \varsigma_2. \end{cases}$$

We also take $\{\varpi_{\mathfrak{f},\mathfrak{h}}\} = -\varpi$ for $\mathfrak{f}, \mathfrak{h} \in \mathbb{N}$. Then, $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ and ϖ have the same distribution and $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ weighted ideal statistically converges in distribution to ϖ. However, for any given $\varkappa, \varrho, \sigma > 0$

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f} \mathfrak{q}_\mathfrak{h} \mathrm{Cr}\left\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \ge \varkappa\right\} \ge \varrho\}\right| \ge \sigma\right\} \notin \mathcal{I}_2.$$

Hence, the sequence $\{\varpi_{\mathfrak{f},\mathfrak{h}}\}$ does not weighted ideal statistically converge in terms of credibility to ϖ_0.

The following example shows the existence of a weighted ideal statistically convergent with respect to almost surely in a given credibility space.

Example 3.19. Let us examine the credibility space $(\Omega, P(\Omega), \mathrm{Cr})$ with credibility function as follows:

$$\mathrm{Cr}\{Y\} = \begin{cases} \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y} \frac{1}{(\mathfrak{f}+\mathfrak{h})^2+1}, & \text{if } \sup_{\varsigma_{\mathfrak{f},\mathfrak{h}} \in Y} \frac{1}{(\mathfrak{f}+\mathfrak{h})^2+1} < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Now, let us define fuzzy variables by

$$\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) = \begin{cases} (\mathfrak{f}+\mathfrak{h})^2 + 1, & \text{if } \varsigma = \varsigma_{\mathfrak{f},\mathfrak{h}}, \\ 0, & \text{otherwise.} \end{cases}$$

Consider $\mathfrak{p}_\mathfrak{f} = \mathfrak{q}_\mathfrak{h} = 1$, we get

$$\begin{aligned} &\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)\right| = \int_0^{+\infty} \mathrm{Cr}\left\{\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)\right| \ge \varkappa\right\} d\varkappa \\ &- \int_{-\infty}^{0} \mathrm{Cr}\left\{\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)\right| \ge \varkappa\right\} d\varkappa \\ &= \int_0^{\frac{1}{2}} \mathrm{Cr}\left\{\varsigma_{\mathfrak{f},\mathfrak{h}}\right\} d\varkappa = \frac{1}{2\left((\mathfrak{f}+\mathfrak{h})^2+1\right)}, \end{aligned}$$

therefore,

$$\left\{(\sigma, \rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f}, \mathfrak{h}) : \mathfrak{f} \le \mathfrak{U}_\sigma, \mathfrak{h} \le \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_0(\varsigma)\right| \ge \varkappa\}\right| \ge \varrho\right\} \in \mathcal{I}_2.$$

for any $\varsigma \in Q$ and $\varkappa > 0$ and $\varrho \in \left(0, \frac{1}{2}\right)$.
Thus, $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent to ϖ in almost surely.

Theorem 3.20. *If a sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent in mean to ϖ, then it is weighted ideal statistically convergent in credibility to ϖ.*

Proof. Let the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ be weighted ideal statistically convergent to ϖ in mean, then from Markov's inequality, for any $\varkappa > 0$, we get

$$\mathrm{Cr}\left\{\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi_0(\varsigma)\right|\geq\varkappa\right\}\leq\frac{\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{E}\left[\left\|\varpi_{\mathfrak{f},\mathfrak{h}}-\varpi_0\right\|\right]}{\varkappa}.$$

So, for any arbitrary $\varrho, \sigma > 0$

$$\begin{aligned}
&\left\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\left\|\varpi_{\mathfrak{f},\mathfrak{h}}-\varpi_0\right\|\geq\varkappa\right\}\geq\varrho\right\}\right|\geq\sigma\right\}\\
&=\left\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi_0(\varsigma)\right|\geq\varkappa\right\}\geq\varrho\right\}\right|\geq\sigma\right\}\\
&\subseteq\left\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\left(\tfrac{\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{E}\left[\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi_0(\varsigma)\right|\right]}{\varkappa}\right)\geq\varrho\right\}\right|\geq\sigma\right\}\\
&=\left\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\left(\tfrac{\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{E}\left[\left\|\varpi_{\mathfrak{f},\mathfrak{h}}-\varpi_0\right\|\right]}{\varkappa}\right)\geq\varrho\right\}\right|\geq\sigma\right\}\in\mathcal{I}_2.
\end{aligned}$$

Thus, $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted ideal statistically convergent to ϖ in credibility. □

Theorem 3.21. *Suppose that $\Phi, \Phi_1, \Phi_2, \ldots$ are the credibility distributions of the fuzzy variables $\varpi, \varpi_1, \varpi_2, \ldots$, respectively. If the sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ converges weighted ideal statistically in credibility to ϖ_0, then $(\varpi_{\mathfrak{f},\mathfrak{h}})$ converges weighted ideal statistically in distribution to ϖ_0.*

Proof. For credibility distribution Φ, let $\mathfrak{u}$ represent any given continuity point. On the one hand, for any $\mathfrak{v} > \mathfrak{u}$, we get

$$\begin{aligned}
\left\{\varsigma:\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)\leq\mathfrak{u}\right\}&=\left\{\varsigma:\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)\leq\mathfrak{u},\varpi(\varsigma)\leq\mathfrak{v}\right\}\cup\left\{\varsigma:\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)\leq\mathfrak{u},\varpi(\varsigma)>\mathfrak{v}\right\}\\
&\cup\left\{\varsigma:\varpi(\varsigma)\leq\mathfrak{v}\right\}\cup\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{v}-\mathfrak{u}\right\},
\end{aligned}$$

which implies that

$$\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{u})=\Phi(\mathfrak{v})+\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{v}-\mathfrak{u}\right\}.$$

Since $(\varpi_{\mathfrak{f},\mathfrak{h}})$ converges weighted ideal statistically in credibility to ϖ_0, we get

$$\begin{aligned}
&\Big\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\right.\right.\\
&\qquad\left.\left.\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{v}-\mathfrak{u}\right\}\right\}\geq\varrho\right|\geq\sigma\Big\}\in\mathcal{I}_2,
\end{aligned}$$

for any $\varrho > 0$.
Thus, we obtain

$$\left\{(\sigma,\rho)\in\mathbb{N}^2:\frac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\sup_{\mathfrak{f},\mathfrak{h}}\left|\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{u})-\Phi(\mathfrak{u})\right|\geq\varrho\right\}\right|\right\}\in\mathcal{I}_2, \tag{3.3}$$

for all $\varrho > 0$ as $\mathfrak{v} \to \mathfrak{u}$.
On the other hand, for any $\mathfrak{q} < \mathfrak{u}$, we have

$$\begin{aligned}
\left\{\varsigma:\varpi(\varsigma)\leq\mathfrak{q}\right\}&=\left\{\varsigma:\varpi(\varsigma)\leq\mathfrak{q},\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)\leq\mathfrak{u}\right\}\cup\left\{\varsigma:\varpi(\varsigma)\leq\mathfrak{q},\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)>\mathfrak{u}\right\}\\
&\cup\left\{\varsigma:\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)\leq\mathfrak{u}\right\}\cup\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{u}-\mathfrak{q}\right\},
\end{aligned}$$

which implies that

$$\Phi(\mathfrak{q})=\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{u})+\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{u}-\mathfrak{q}\right\}.$$

Since

$$\begin{aligned}
&\Big\{(\sigma,\rho)\in\mathbb{N}^2:\tfrac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\right.\right.\\
&\left.\left.\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\mathrm{Cr}\left\{\varsigma:\left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma)-\varpi(\varsigma)\right|\geq\mathfrak{u}-\mathfrak{q}\right\}\right\}\right|\geq\varrho\Big\}\in\mathcal{I}_2,
\end{aligned}$$

we obtain

$$\left\{(\sigma,\rho)\in\mathbb{N}^2:\frac{1}{\mathfrak{U}_\sigma\mathfrak{R}_\rho}\left|\left\{(\mathfrak{f},\mathfrak{h}):\mathfrak{f}\leq\mathfrak{U}_\sigma,\mathfrak{h}\leq\mathfrak{R}_\rho:\mathfrak{p}_{\mathfrak{f}}\mathfrak{q}_{\mathfrak{h}}\inf_{\mathfrak{f},\mathfrak{h}}\left|\Phi_{\mathfrak{f},\mathfrak{h}}(\mathfrak{u})-\Phi(\mathfrak{u})\right|\geq\varrho\right\}\right|\right\}\in\mathcal{I}_2, \tag{3.4}$$

for all $\varrho > 0$ as $\mathfrak{q} \to \mathfrak{u}$. It follows from (3.3) and (3.4) that $(\varpi_{\mathfrak{f},\mathfrak{h}})$ converges weighted ideal statistically in distribution to ϖ_0. □

Definition 3.22. The fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is said to be weighted $\mathcal{I}_2$-statistically Cauchy in credibility if for every $\varkappa, \varrho, \sigma > 0$, there exist $\mathfrak{u}_1, \mathfrak{v}_1 \in \mathbb{N}$ such that for all $\mathfrak{f},\mathfrak{a} \geq \mathfrak{u}_1$ and $\mathfrak{g},\mathfrak{b} \geq \mathfrak{v}_1$,

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\} \in \mathcal{I}_2.$$

Definition 3.23. The fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is said to be weighted $\mathcal{I}_2$-statistically Cauchy in mean if for every $\varkappa, \varrho > 0$, there exist $\mathfrak{u}_1, \mathfrak{v}_1 \in \mathbb{N}$ such that for all $\mathfrak{f},\mathfrak{a} \geq \mathfrak{u}_1$ and $\mathfrak{g},\mathfrak{b} \geq \mathfrak{v}_1$,

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{E}\left[\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\|\right] \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2.$$

Definition 3.24. The fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is said to be weighted $\mathcal{I}_2$-statistically Cauchy with respect to almost surely if for any positive $\varkappa, \varrho > 0$, there exists a set $Q \in P(\Omega)$ with unit credibility measure and $\mathfrak{u}_1, \mathfrak{v}_1 \in \mathbb{N}$ such that

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_{\mathfrak{a},\mathfrak{b}}(\varsigma)\right| \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2,$$

$\mathfrak{f},\mathfrak{a} \geq \mathfrak{u}_1$ and $\mathfrak{g},\mathfrak{b} \geq \mathfrak{v}_1$, for any $\varsigma \in Q$ and $\varkappa, \varrho > 0$.

Definition 3.25. The fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is said to be weighted $\mathcal{I}_2$-statistically Cauchy with respect to uniformly almost surely if for any positive $\varkappa, \varrho > 0$, there exists a sequence of events $\{T_\mathfrak{m}\}$ approaching to credibility measure zero and natural numbers $\mathfrak{u}_1, \mathfrak{v}_1$ with $\mathfrak{f},\mathfrak{a} \geq \mathfrak{u}_1$ and $\mathfrak{g},\mathfrak{b} \geq \mathfrak{v}_1$ such that

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \left\|\varpi_{\mathfrak{f},\mathfrak{h}}(\varsigma) - \varpi_{\mathfrak{a},\mathfrak{b}}(\varsigma)\right\| \geq \varkappa\}\right| \geq \varrho\right\} \in \mathcal{I}_2,$$

for all $\varsigma \in P(\Omega) - \{T_\mathfrak{m}\}$.

Theorem 3.26. *A fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent in credibility if and only if $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy in credibility.*

Proof. Let the fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent in credibility to ϖ_0. Then, for any preassigned $\varkappa, \varrho, \sigma > 0$

$$\left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\} \in \mathcal{I}_2.$$

Let us chose two natural numbers $\mathfrak{a}, \mathfrak{b} \in \mathbb{N}$ such that $\mathrm{Cr}\{\|\varpi_{\mathfrak{a},\mathfrak{b}} - \varpi_0\| \geq \varkappa\} \geq \varrho$. Let us take three sets

$$\mathfrak{L} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\},$$
$$\mathfrak{R} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\},$$
$$\mathfrak{T} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \tfrac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{a},\mathfrak{b}} - \varpi_0\| \geq \varkappa\} \geq \varrho\}\right| \geq \sigma\right\}.$$

Obviously $\mathfrak{L} \subseteq \mathfrak{R} \cup \mathfrak{T}$. Therefore, $\delta_2^{\mathcal{I}_2}(\mathfrak{L}) \leq \delta_2^{\mathcal{I}_2}(\mathfrak{R}) + \delta_2^{\mathcal{I}_2}(\mathfrak{T}) = 0$, since $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent in credibility to ϖ_0. Hence, $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy in credibility.

Conversely, let the fuzzy variable sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy in credibility. Then, $\delta_2^{\mathcal{I}_2}(\mathfrak{L}) = 0$. So for the set

$$\mathfrak{U} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\| \geq \varkappa\} < \varrho\}\right| < \sigma\right\},$$

we have $\delta_2^{\mathcal{I}_2}(\mathfrak{U}) = 1$.

Now for each $\nu > 0$, there exists some $0 < \nu' \leq \frac{\nu}{2}$, such that

$$\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\| \geq \nu\} \leq 2\mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h} \mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \nu'\} < \varrho. \tag{3.5}$$

Moreover, if $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is not weighted $\mathcal{I}_2$-statistically convergent in credibility, then $\delta_2^{\mathcal{I}_2}(\mathfrak{N}) = 1$. Consequently, for the set

$$\mathfrak{B} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_0\| \geq \varkappa\} < \varrho\}\right| < \sigma\right\},$$

we have $\delta_2^{\mathcal{I}_2}(\mathfrak{B}) = 0$. Thus, from the Eq. (3.5), for the set

$$\mathfrak{C} = \left\{(\sigma,\rho) \in \mathbb{N}^2 : \frac{1}{\mathfrak{U}_\sigma \mathfrak{R}_\rho} \left|\{(\mathfrak{f},\mathfrak{h}) : \mathfrak{f} \leq \mathfrak{U}_\sigma, \mathfrak{h} \leq \mathfrak{R}_\rho : \mathfrak{p}_\mathfrak{f}\mathfrak{q}_\mathfrak{h}\mathrm{Cr}\{\|\varpi_{\mathfrak{f},\mathfrak{h}} - \varpi_{\mathfrak{a},\mathfrak{b}}\| \geq \varkappa\} < \varrho\}\right| < \sigma\right\},$$

we have $\delta_2^{\mathcal{I}_2}(\mathfrak{C}) = 0$, which implies that $\delta_2^{\mathcal{I}_2}(\mathfrak{L}) = 1$ and thus it arises a contradiction that $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is a weighted $\mathcal{I}_2$-statistically Cauchy sequence in credibility. □

The results mentioned above hold for almost surely, mean, distribution, and uniformly almost surely as well. The proofs can be validated by employing similar techniques used for establishing the preceding two theorems. These results are stated as follows:

Theorem 3.27. *A fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent almost surely if and only if $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy with respect to almost surely.*

Theorem 3.28. *A fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent in mean if and only if $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy in mean.*

Proof. The proof can be established by considering credibility expected value operator instead of credibility measure like in the Theorem 3.26. □

Theorem 3.29. *A fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent with respect to uniformly almost surely if and only if $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy with respect to uniformly almost surely.*

Proof. This can be demonstrated by applying the techniques utilized in Theorem 3.27, focusing on the events ς within the collection $P(\Omega) - \{T_\mathfrak{m}\}$, where $T_\mathfrak{m}$ represents events whose credibility measures approach zero. □

Theorem 3.30. *A fuzzy variable double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically convergent in distribution if and only if the double sequence $(\varpi_{\mathfrak{f},\mathfrak{h}})$ is weighted $\mathcal{I}_2$-statistically Cauchy in distribution.*

Proof. The proof can be done by considering fuzzy credibility distribution function in the Theorem 3.26. □

4 Conclusion remarks

This study delves into the intricate concept of weighted ideal statistical convergence for double sequences of fuzzy variables in the context of credibility. By presenting engaging examples, we uncover the subtle interconnections between these sequences, offering new insights into their behavior. Additionally, we introduce the novel idea of a weighted ideal statistically Cauchy sequence, paving the way for a deeper understanding of its relationship with weighted ideal statistical convergence, thus advancing the field of fuzzy variable sequences.

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