

# $K$ - $g$ -frames in Hilbert modules over locally- $C^*$ -algebras

R. El Jazzar, M. Rossafi and M. Mouniane

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**Corresponding Author: M. Rossafi**

**Abstract** This paper investigates the concept of  $K$ - $g$ -frames in locally  $C^*$ -algebras, showing that they constitute a proper generalization of  $g$ -frames. Motivated by their applications in fields such as signal processing and operator theory, we first introduce the notion of a  $g$ -orthonormal basis and employ it to define the associated  $g$ -operator, which plays a central role in the construction of  $K$ - $g$ -frames in locally  $C^*$ -algebras. We then establish a relationship between  $g$ -frames and  $K$ - $g$ -frames and introduce the notion of a  $K$ -dual  $g$ -frame, together with its fundamental properties. Finally, we provide characterizations of  $K$ - $g$ -frames through two related concepts.

## 1 Introduction and preliminaries

The concept of locally  $C^*$ -algebras, also referred to as pro- $C^*$ -algebras, represents an extension of the classical  $C^*$ -algebra framework. The notion of locally  $C^*$ -algebras was first introduced in the scholarly literature in 1971 by A. Inoue [7]. These algebras have been studied under various terminologies, including pro- $C^*$ -algebras (as mentioned by D. Voiculescu and N. C. Philips), multi-normed  $C^*$ -algebras (as described by A. Dosiev), and  $LMC^*$ -algebras (as identified in the works of G. Lassner and K. Schmüdgen).

Hilbert pro- $C^*$ -modules can be viewed as analogues of Hilbert spaces, with the distinction that their inner product takes values in a pro- $C^*$ -algebra rather than in the field of complex numbers. For a comprehensive overview, readers are referred to [3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14].

Frames and their generalizations, such as  $g$ -frames, have been extensively studied in Hilbert spaces and in  $C^*$ -Hilbert modules due to their numerous applications in operator theory and signal processing. In particular, Haddadzadeh [6] introduced and studied the concept of  $g$ -frames in the context of Hilbert pro- $C^*$ -modules, establishing several fundamental properties and dual relations. However, these results do not address situations in which a bounded operator  $K$  plays a role in the structure of the frame. This motivates the study of  $K$ - $g$ -frames, which provide a more flexible framework allowing reconstruction with respect to a given operator  $K$ .

A pro- $C^*$ -algebra is a complete Hausdorff complex topological  $*$ -algebra, denoted by  $\mathcal{A}$ , whose topology is defined by a family of continuous  $C^*$ -seminorms. Specifically, a net  $\{a_\alpha\}$  is said to converge to 0 if and only if  $p(a_\alpha) \rightarrow 0$  for every continuous  $C^*$ -seminorm  $p$  on  $\mathcal{A}$ . Furthermore, it is characterized by the following property:

- 1)  $p_\alpha(\gamma\beta) \leq p_\alpha(\gamma)p_\alpha(\beta)$
- 2)  $p_\alpha(\gamma^*\gamma) = (p_\alpha(\gamma))^2$ .

For every  $\gamma, \beta \in \mathcal{A}$ , the term  $\text{sptr}(\gamma)$  refers to the spectrum of  $\gamma$ , defined as:  $\text{sptr}(\gamma) = \{\lambda \in \mathbb{C} : \lambda 1_{\mathcal{A}} - \gamma \text{ is not invertible, for all } \gamma \in \mathcal{A}$ . Here,  $\mathcal{A}$  represents a unital pro- $C^*$ -algebra with unity  $1_{\mathcal{A}}$ .

The notation  $Se(\mathcal{A})$  is used to denote the set of all continuous  $C^*$ -seminorms on  $\mathcal{A}$ . Furthermore,  $\mathcal{A}^+$  signifies the set of all positive elements in  $\mathcal{A}$ , which forms a closed convex subset under the  $C^*$ -seminorms on  $\mathcal{A}$ .

$\mathcal{H}_{\mathcal{A}}$  denotes the set of all sequences  $(\gamma_n)_n$  with  $\gamma_n \in \mathcal{A}$  such that  $\sum \gamma_n^* \gamma_n$  converges in  $\mathcal{A}$ .

**Example 1.1.** Every  $C^*$ -algebra qualifies as a locally  $C^*$ -algebra.

**Definition 1.2** ([13]). A pre-Hilbert module over a locally  $C^*$ -algebra  $\mathcal{A}$  is defined as a complex vector space  $\mathcal{U}$  that simultaneously functions as a left  $\mathcal{A}$ -module, aligning with the complex algebra structure. This module is endowed with an  $\mathcal{A}$ -valued inner product  $\langle \cdot, \cdot \rangle : \mathcal{U} \times \mathcal{U} \rightarrow \mathcal{A}$ , which exhibits  $\mathbb{C}$ -linearity and  $\mathcal{A}$ -linearity in its first argument and fulfills the following criteria:

- 1) For all  $\xi, \eta \in \mathcal{U}$ , it holds that  $\langle \xi, \eta \rangle^* = \langle \eta, \xi \rangle$ .
- 2) For every  $\xi \in \mathcal{U}$ , the condition  $\langle \xi, \xi \rangle \geq 0$  is satisfied.
- 3) The equality  $\langle \xi, \xi \rangle = 0$  is true if and only if  $\xi = 0$ .

The space  $\mathcal{U}$  is referred to Hilbert  $\mathcal{A}$ -module (or Hilbert pro- $C^*$ -module over  $\mathcal{A}$ ) when it achieves completeness in relation to the topology that emerges from the family of seminorms defined by:

$$\bar{p}_{\mathcal{U}}(\xi) = \sqrt{p_{\alpha}(\langle \xi, \xi \rangle)} \quad \forall \xi \in \mathcal{U}, p \in Se(\mathcal{A}). \quad (1.1)$$

Throughout the remainder of this document, we consider  $\mathcal{A}$  as a pro- $C^*$ -algebra. Furthermore,  $\mathcal{U}$  and  $\mathcal{V}$  are designated as Hilbert modules over the algebra  $\mathcal{A}$ , with  $I$  and  $J$  representing countably infinite sets of indices.

An operator from  $\mathcal{U}$  to  $\mathcal{V}$  is defined as any bounded  $\mathcal{A}$ -module map from  $\mathcal{U}$  to  $\mathcal{V}$ . The collection of all such operators from  $\mathcal{U}$  to  $\mathcal{V}$  is denoted by  $Hom_{\mathcal{A}}(\mathcal{U}, \mathcal{V})$ .

**Definition 1.3** ([2]). Consider  $\mathfrak{F} : \mathcal{U} \rightarrow \mathcal{V}$  as an  $\mathcal{A}$ -module map. The operator  $\mathfrak{F}$  is termed adjointable if there exists a map  $\mathfrak{F}^* : \mathcal{V} \rightarrow \mathcal{U}$  satisfying  $\langle \mathfrak{F}\xi, \eta \rangle = \langle \xi, \mathfrak{F}^*\eta \rangle$  for all  $\xi \in \mathcal{U}, \eta \in \mathcal{V}$ .  $\mathfrak{F}$  is considered bounded if, for all  $p \in Se(\mathcal{A})$ , there exists  $N_p > 0$  such that

$$\bar{p}_{\alpha_{\mathcal{V}}}(\mathfrak{F}\xi) \leq N_p \bar{p}_{\alpha_{\mathcal{U}}}(\xi), \quad \forall \xi \in \mathcal{U}. \quad (1.2)$$

The set of all adjointable operators from  $\mathcal{U}$  to  $\mathcal{V}$  is represented by  $Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V})$ , and  $Hom_{\mathcal{A}}^*(\mathcal{U}) = Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{U})$ .

**Definition 1.4** ([1]). Consider the operator  $\mathfrak{F} : \mathcal{U} \rightarrow \mathcal{V}$ . An operator is said to be bounded below if there exists a constant  $A > 0$  such that for each  $p_{\alpha} \in Se(\mathcal{A})$ ,

$$\bar{p}_{\alpha_{\mathcal{V}}}(\mathfrak{F}\xi) \leq A \bar{p}_{\alpha_{\mathcal{U}}}(\xi), \quad \forall \xi \in \mathcal{U}.$$

Similarly, it is termed uniformly bounded above if a constant  $A' > 0$  ensures that for every  $p \in Se(\mathcal{A})$ ,

$$\bar{p}_{\alpha_{\mathcal{V}}}(\mathfrak{F}\xi) \geq A' \bar{p}_{\alpha_{\mathcal{U}}}(\xi), \quad \forall \xi \in \mathcal{U}$$

$$\|\mathfrak{F}\|_{\infty} = \inf\{N : N \text{ is an upper bound for } \mathfrak{F}\}$$

$$\hat{p}_{\mathcal{V}}(\mathfrak{F}) = \sup\{\bar{p}_{\alpha_{\mathcal{V}}}(\mathfrak{F}(\xi)) : \xi \in \mathcal{U}, \bar{p}_{\alpha_{\mathcal{U}}}(\xi) \leq 1\}.$$

We obtain that,  $\hat{p}(\mathfrak{F}) \leq \|\mathfrak{F}\|_{\infty}$  for all  $p \in Se(\mathcal{A})$ .

**Proposition 1.5** ([2]). Assume  $\mathfrak{F}$  belongs to  $Hom_{\mathcal{A}}^*(\mathcal{U})$  and is an invertible operator, with both  $\mathfrak{F}$  and its inverse  $\mathfrak{F}^{-1}$  being uniformly bounded. element  $\eta \in \mathcal{U}$ , the following inequality holds

$$\|\mathfrak{F}^{-1}\|_{\infty}^{-2} \langle \eta, \eta \rangle \leq \langle \mathfrak{F}\eta, \mathfrak{F}\eta \rangle \leq \|\mathfrak{F}\|_{\infty}^2 \langle \eta, \eta \rangle. \quad (1.3)$$

Here, the notation  $Ran(T)$  denotes the range of the operator  $T$ , that is

$$Ran(T) = \{T\xi : \xi \in U\}.$$

**Lemma 1.6** ([15]). In the setting where  $\mathcal{U}$  is a Hilbert  $\mathcal{A}$ -module within a pro- $C^*$ -algebra  $\mathcal{A}$ , and given  $T, Z \in Hom_{\mathcal{A}}^*(\mathcal{U})$ , the following conditions are equivalent when the range of  $Z$ , denoted as  $Ran(Z)$ , is closed:

- (i)  $\text{Ran}(T) \subseteq \text{Ran}(Z)$ .
- (ii) The relation  $TT^* \leq \alpha^2 ZZ^*$  is valid for some non-negative  $\alpha$ .
- (iii) An operator  $U$ , uniformly bounded and within  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , exists such that  $T = ZU$ .

**Definition 1.7** ([6]). We define a sequence  $\Gamma = \{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)\}_{i \in I}$  as a  $g$ -frame for  $\mathcal{U}$  relative to the set  $\{\mathcal{V}_i\}_{i \in I}$  if there exist positive constants  $A$  and  $B$  satisfying the following condition for every  $\eta \in \mathcal{U}$ :

$$A\langle \eta, \eta \rangle \leq \sum_{i \in I} \langle \Gamma_i \eta, \Gamma_i \eta \rangle \leq B\langle \eta, \eta \rangle. \quad (1.4)$$

Here,  $A$  and  $B$  are referred to as the  $g$ -frame bounds for  $\Gamma$ . The  $g$ -frame is termed tight if  $A = B$ , and Parseval if  $A = B = 1$ . If only the upper bound is satisfied in 1.4,  $\Gamma$  is identified as a  $g$ -Bessel sequence.

**Lemma 1.8** ([6]). A sequence  $\{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)\}_{i \in I}$  forms a  $g$ -frame for  $\mathcal{U}$  in relation to  $\{\mathcal{V}_i\}_{i \in I}$  if and only if

$$Q : \{g_i\}_{i \in I} \mapsto \sum_{i \in I} \Gamma_i^* g_i,$$

can be characterized as a well-defined and bounded linear operator mapping from  $\bigoplus_{i \in I} \mathcal{V}_i$  onto  $\mathcal{U}$ .

## 2 $K$ - $g$ -frame in Hilbert Modules over Pro- $C^*$ -Algebras

**Definition 2.1.** Given  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , a sequence  $\Gamma = \{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)\}_{i \in I}$  is defined as a  $K$ - $g$ -frame for  $\mathcal{U}$  with regard to  $\{\mathcal{V}_i\}_{i \in I}$  if there exist two positive constants  $C$  and  $D$  such that for all  $\xi \in \mathcal{U}$ , the following inequality is satisfied:

$$C\langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle \leq D\langle \xi, \xi \rangle. \quad (2.1)$$

The constants  $C$  and  $D$  are known as the lower and upper bounds of the  $K$ - $g$ -frame, respectively. If  $C = D$ , the  $K$ - $g$ -frame is termed tight, and it is known as a Parseval frame when  $C = D = 1$ .

**Definition 2.2.** Consider  $\Gamma = \{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)\}_{i \in I}$  constituting a  $K$ - $g$ -frame for  $\mathcal{U}$  in relation to  $\{\mathcal{V}_i\}_{i \in I}$ , equipped with an analysis operator  $T$ . The frame operator  $S : \mathcal{U} \rightarrow \mathcal{U}$  is then defined as follows:

$$S\xi = T^*T\xi = \sum_{i \in I} \Gamma_i^* \Gamma_i \xi, \quad \xi \in \mathcal{U}. \quad (2.2)$$

**Definition 2.3.** A sequence  $\{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  is termed a  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  if it fulfills the following conditions:

$$\begin{aligned} \langle \Gamma_i^* g_i, \Gamma_j^* g_j \rangle &= \delta_{ij} \langle g_i, g_j \rangle, \quad \forall i, j \in I, g_i \in \mathcal{V}_i, g_j \in \mathcal{V}_j, \\ \sum_{i \in I} \bar{p}_{\alpha \mathcal{U}}(\Gamma_i \xi)^2 &= \bar{p}_{\alpha \mathcal{U}}(\xi)^2, \quad \forall \xi \in \mathcal{U}. \end{aligned}$$

**Definition 2.4.** Given  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and a  $K$ - $g$ -frame  $\{\Gamma_i\}_{i \in I}$  for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , a  $g$ -frame sequence  $\{\Xi_i\}_{i \in I}$  for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  is considered a  $K$ -dual  $g$ -frame sequence of  $\{\Gamma_i\}_{i \in I}$  if the following holds true:

$$K\xi = \sum_{i \in I} \Gamma_i^* \Xi_i \xi, \quad \forall \xi \in \mathcal{U}. \quad (2.3)$$

**Lemma 2.5.** Assume  $\{\Gamma_i\}_{i \in I}$  as a  $g$ -Bessel sequence within  $\mathcal{U}$ , associated with  $\{\mathcal{V}_i\}_{i \in I}$ . The sequence  $\{\Gamma_i\}_{i \in I}$  forms a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  if there is a constant  $C > 0$  such that the following condition is met:  $S \geq CKK^*$ , with  $S$  being the frame operator for  $\{\Gamma_i\}_{i \in I}$ .

*Proof.* The sequence  $\{\Gamma_i\}_{i \in I}$  qualifies as a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  with bounds  $C, D$  if and only if the following condition is satisfied:

$$C\langle K^*\xi, K^*\xi \rangle \leq \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle \leq D\langle \xi, \xi \rangle, \forall \xi \in \mathcal{U}, \quad (2.4)$$

which can be equivalently expressed as:

$$\langle CKK^*\xi, \xi \rangle \leq \langle S\xi, \xi \rangle \leq \langle D\xi, \xi \rangle, \forall \xi \in \mathcal{U}, \quad (2.5)$$

where  $S$  denotes the frame operator of the  $K$ - $g$ -frame  $\{\Gamma_i\}_{i \in I}$ . Thus, the assertion is substantiated.  $\square$

**Definition 2.6.** Consider  $\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)$ . The sequence  $\{\Gamma_i\}_{i \in I}$  is deemed  $g$ -complete if the set  $\{\xi \in \mathcal{U} : \Gamma_i \xi = 0 \text{ for all } i \in I\} = \{0\}$ .

**Proposition 2.7.** The set  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  attains  $g$ -completeness if and only if

$$\overline{span} \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I} = \mathcal{U}. \quad (2.6)$$

*Proof.* Assume  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  to be  $g$ -complete. Given that  $\overline{span} \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I} \subseteq \mathcal{U}$ , it suffices to show that if  $\zeta \in \mathcal{U}$  and  $\zeta \in span \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I}$ , then  $\zeta = 0$ . Let  $\zeta \in \mathcal{U}$  and  $\zeta \perp span \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I}$ . For any  $i \in I, \zeta \perp \Gamma_i^* \Gamma_i(\zeta)$  implies that for all  $i \in I$ ,

$$p_{\alpha \mathcal{U}}(\Gamma_i \zeta)^2 = p_{\alpha}(\langle \zeta, \Gamma_i^* \Gamma_i(\zeta) \rangle) = 0. \quad (2.7)$$

Therefore, by  $g$ -completeness of  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ , it follows that  $\zeta = 0$ .

Conversely, if  $\overline{span} \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I} = \mathcal{U}$ , and for a given  $\zeta \in \mathcal{U}, \Gamma_i \zeta = 0$  for all  $i \in I$ , then for each  $\theta \in \mathcal{V}_i$

$$\langle \Gamma_i \zeta, \theta \rangle = \langle \zeta, \Gamma_i^* \theta \rangle = 0, \quad (2.8)$$

leading to  $\zeta \perp span \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I}$ . Thus,  $\zeta \perp \overline{span} \{\Gamma_i^*(\mathcal{V}_i)\}_{i \in I} = \mathcal{U}$ . This implies that  $\zeta = 0$ , establishing the  $g$ -completeness of  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ .  $\square$

**Lemma 2.8.** Suppose  $\{\mathfrak{E}_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  constitutes a  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i : i \in I\}$ . In this case,  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  will be a  $g$ -frame sequence with respect to  $\{\mathcal{V}_i : i \in I\}$  if and only if a singular bounded operator  $Q : \mathcal{U} \rightarrow \mathcal{U}$  exists such that  $\Gamma_i = \mathfrak{E}_i Q^*$  for each  $i \in I$ .

*Proof.*  $\Rightarrow$  If  $\{\mathfrak{E}_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  forms a  $g$ -orthonormal basis for  $\mathcal{U}$ , then for any element  $\xi \in \mathcal{U}$ , the set  $\{\mathfrak{E}_i \xi : i \in I\}$  is part of

$$\left( \sum_{i \in I} \oplus \mathcal{V}_i \right)_{\mathcal{H}_{\mathcal{A}}}$$

Assuming  $\{\Gamma_i \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  to be a  $g$ -Bessel sequence, Lemma 1.8 ensures that the operator  $Q$ , defined as

$$Q : \mathcal{U} \longrightarrow \mathcal{U}, \quad Q\xi = \sum_{i \in I} \Gamma_i^* \mathfrak{E}_i \xi \quad (2.9)$$

is both well-defined and bounded. Utilizing the  $g$ -orthonormal basis definition, it's clear that  $\mathfrak{E}_i \mathfrak{E}_j^* \eta = \delta_{ij} \eta$ . Thus,

$$Q \mathfrak{E}_j^* \eta = \sum_{i \in I} \Gamma_i^* \mathfrak{E}_i \mathfrak{E}_j^* \eta = \Gamma_j^* \mathfrak{E}_j \mathfrak{E}_j^* \eta = \Gamma_j^* \eta \quad (2.10)$$

for all  $\eta \in \mathcal{V}_j, j \in I$ . Consequently,  $Q \mathfrak{E}_j^* = \Gamma_j^*$ , leading to  $\mathfrak{E}_j Q^* = \Gamma_j$  for each  $j \in I$ . Now, assume  $Q_1, Q_2 \in Hom_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i)$  where  $\mathfrak{E}_i Q_1^* = \mathfrak{E}_i Q_2^* = \Gamma_i$  for all  $i \in I$ . For any  $\xi \in \mathcal{U}, \eta_i \in \mathcal{V}_i$ , it holds that  $\langle \mathfrak{E}_i Q_1^* \xi, \eta_i \rangle = \langle \mathfrak{E}_i Q_2^* \xi, \eta_i \rangle$ , implying  $\langle Q_1^* \xi, \mathfrak{E}_i^* \eta_i \rangle = \langle Q_2^* \xi, \mathfrak{E}_i^* \eta_i \rangle$ . Since  $\overline{span} \{\mathfrak{E}_i^*(\mathcal{V}_i)\}_{i \in I} = \mathcal{U}$  (Proposition 2.7), it follows that  $Q_1^* \xi = Q_2^* \xi$ , establishing  $Q_1 = Q_2$ . Therefore,  $Q$  is unique.

⇐ Given  $\Gamma_i = \mathfrak{E}_i Q^*$  for all  $i \in I$ , for any  $\xi \in \mathcal{U}$ , we have

$$\sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle = \sum_{i \in I} \langle \mathfrak{E}_i Q^* \xi, \mathfrak{E}_i Q^* \xi \rangle = \langle Q^* \xi, Q^* \xi \rangle. \quad (2.11)$$

Proposition 2.2 in [1] demonstrates that  $Q^*$  is bounded below, hence invertible. Following Theorem 3.2 in [2], we obtain:

$$\left\| (Q^*)^{-1} \right\|_{\infty}^{-2} \langle \xi, \xi \rangle \leq \langle Q^* \xi, Q^* \xi \rangle \leq \|Q^*\|_{\infty}^2 \langle \xi, \xi \rangle. \quad (2.12)$$

□

**Remark 2.9.** When the set  $\{\mathfrak{E}_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  forms a  $g$ -orthonormal basis, the operator  $Q$  described in Lemma 2.8 is recognized as the  $g$ -operator corresponding to the sequence  $\{\Gamma_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ .

**Theorem 2.10.** Given  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and  $\{\Gamma_i\}_{i \in I}$  as a  $g$ -frame sequence for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , if  $Q$  is the  $g$ -operator corresponding to  $\{\Gamma_i\}_{i \in I}$ , then  $\{\Gamma_i\}_{i \in I}$  qualifies as a  $K$ - $g$ -frame if and only if  $\text{Ran}(K) \subset \text{Ran}(Q)$ .

*Proof.* ⇒ Assume that  $\{\Gamma_i\}_{i \in I}$  is a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . This means there exists a positive constant  $C$  such that

$$C \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle, \quad \forall \xi \in \mathcal{U}. \quad (2.13)$$

By Lemma 2.8,  $\Gamma_i = \mathfrak{E}_i Q^*$ , where  $\{\mathfrak{E}_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  is the  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Therefore, we obtain

$$C \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \mathfrak{E}_i Q^* \xi, \mathfrak{E}_i Q^* \xi \rangle = \langle Q^* \xi, Q^* \xi \rangle, \quad \forall \xi \in \mathcal{U}, \quad (2.14)$$

which implies that  $CKK^* \leq QQ^*$ . So, by Lemma 1.6, we have  $\text{Ran}(K) \subset \text{Ran}(Q)$ .

⇐ If  $\text{Ran}(K) \subset \text{Ran}(Q)$ , Lemma 1.6 suggests the existence of a constant  $\alpha > 0$  such that  $KK^* \leq \alpha QQ^*$ . Then for all  $\xi \in \mathcal{U}$ ,

$$\left\langle \frac{1}{\alpha} KK^* \xi, \xi \right\rangle \leq \langle QQ^* \xi, \xi \rangle,$$

that is,

$$\frac{1}{\alpha} \langle K^* \xi, K^* \xi \rangle \leq \langle Q^* \xi, Q^* \xi \rangle.$$

Assuming  $\{\Gamma_i\}_{i \in I}$  is a  $g$ -frame and  $\{\mathfrak{E}_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  is the  $g$ -orthonormal basis for  $\mathcal{U}$  in relation to  $\{\mathcal{V}_i\}_{i \in I}$ , then by Lemma 2.8

$$\langle Q^* \xi, Q^* \xi \rangle = \sum_{i \in I} \langle \mathfrak{E}_i Q^* \xi, \mathfrak{E}_i Q^* \xi \rangle = \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle \quad (2.15)$$

Hence

$$\frac{1}{\alpha} \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle, \quad \forall \xi \in \mathcal{U} \quad (2.16)$$

Thus,  $\{\Gamma_i\}_{i \in I}$  is a  $K$ - $g$ -frame. □

**Theorem 2.11.** Assume  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and consider  $\{\Gamma_i\}_{i \in I}$  as a  $g$ -frame sequence for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Let  $\{\mathfrak{E}_i \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  be the  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , and let  $Q$  be the  $g$ -operator corresponding to  $\{\Gamma_i\}_{i \in I}$ . The theorem posits that  $Q$  functions as a co-isometry if and only if the sequence  $\{\Gamma_i K^*\}_{i \in I}$  establishes a Parseval  $K$ - $g$ -frame.

*Proof.* By utilizing the properties of a  $g$ -orthonormal basis, we have the following equality for any  $\xi \in \mathcal{U}$ :

$$\sum_{i \in I} \langle \Gamma_i K^* \xi, \Gamma_i K^* \xi \rangle = \sum_{i \in I} \langle U_i Q^* K^* \xi, U_i Q^* K^* \xi \rangle = \langle Q^* K^* \xi, Q^* K^* \xi \rangle, \quad (2.17)$$

which clearly demonstrates the conclusion.  $\square$

**Theorem 2.12.** Consider  $K \in \text{Hom}_A^*(\mathcal{U})$  and let  $\{\mathfrak{E}_i \in \text{Hom}_A(\mathcal{U}, \mathcal{V}_i) : i \in I\}$  constitute a  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Suppose  $\{\Gamma_i\}_{i \in I}$  serves as a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , with  $Q$  being the corresponding  $g$ -operator. If  $P$  is the  $g$ -operator for the  $g$ -frame sequence  $\{\Xi_i\}_{i \in I}$  and is invertible, with  $P^{-1}$  being the inverse of  $Q$ , then the sequence  $\{T_i\}_{i \in I}$  qualifies as a  $K$ - $g$ -frame.

*Proof.* Under the assumptions above, there exists a positive constant  $C$  such that  $\forall \xi \in \mathcal{X}$

$$C \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle = \sum_{i \in I} \langle U_i Q^* \xi, U_i Q^* \xi \rangle = \sum_{i \in I} \langle U_i P^* (P^*)^{-1} Q^* \xi, U_i P^* (P^*)^{-1} Q^* \xi \rangle.$$

Since  $P$  is invertible and  $QP^{-1} = I$ , we obtain  $(P^*)^{-1} Q^* = I$ , so

$$C \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle U_i P^* \xi, U_i P^* \xi \rangle = \sum_{i \in I} \langle \Xi_i \xi, \Xi_i \xi \rangle, \quad \forall \xi \in \mathcal{X}. \quad (2.18)$$

$\square$

**Theorem 2.13.** Assume  $K \in \text{Hom}_A^*(\mathcal{U})$ . Let us consider a set  $\{\mathfrak{E}_i \in \text{Hom}_A^*(\mathcal{U}, \mathcal{V}_i) : i \in I\}$ , constituting a  $g$ -orthonormal foundation for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Given that  $\{\Gamma_i\}_{i \in I}$  establishes a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  with  $Q$  denoting the associated  $g$ -operator, and considering  $P$  as the  $g$ -operator linked to the  $g$ -frame sequence  $\{T_i\}_{i \in I}$ , it follows that  $\{T_i\}_{i \in I}$  forms the  $K$ -dual  $g$ -frame sequence of  $\{\Gamma_i\}_{i \in I}$  if and only if the relation  $K = QP^*$  is satisfied.

*Proof.* Assume that  $\{T_i\}_{i \in I}$  is the  $K$ -dual  $g$ -frame sequence of  $\{\Gamma_i\}_{i \in I}$ . Then for every  $\xi \in \mathcal{U}$ , it follows that  $K\xi = \sum_{i \in I} \Gamma_i^* T_i \xi$ . Given that  $\{\mathfrak{E}_i\}_{i \in I}$  forms the  $g$ -orthonormal basis for  $\mathcal{U}$ , applying Lemma 2.8 yields that for all  $\xi \in \mathcal{U}$ ,

$$K\xi = \sum_{i \in I} (\mathfrak{E}_i Q^*)^* (\mathfrak{E}_i P^*) \xi = Q \sum_{i \in I} \mathfrak{E}_i^* \mathfrak{E}_i P^* \xi = QP^* \xi. \quad (2.19)$$

The arbitrariness of  $\xi$  leads to the conclusion that  $K = QP^*$ .

Conversely, acknowledging that  $\{\Gamma_i\}_{i \in I}$  and  $\{T_i\}_{i \in I}$  are both  $g$ -frame sequences, Lemma 2.8 indicates the existence of bounded operators  $Q$  and  $P$  such that

$$\Gamma_i = \mathfrak{E}_i P^*, \quad T_i = \mathfrak{E}_i Q^*.$$

Therefore, for each  $\xi \in \mathcal{U}$ ,

$$\sum_{i \in I} \Gamma_i^* T_i \xi = \sum_{i \in I} (\mathfrak{E}_i Q^*)^* (\mathfrak{E}_i P^*) \xi = Q \sum_{i \in I} \mathfrak{E}_i^* \mathfrak{E}_i P^* \xi = QP^* \xi = K\xi. \quad (2.20)$$

This demonstrates that  $\{T_i\}_{i \in I}$  is indeed the  $K$ -dual  $g$ -frame sequence of  $\{\Gamma_i\}_{i \in I}$ .  $\square$

**Theorem 2.14.** Consider  $K \in \text{Hom}_A^*(\mathcal{U})$  and let  $\{\mathcal{V}_i\}_{i \in I}$  represent the  $K$ -dual  $g$ -frame sequence for  $\{\Gamma_i\}_{i \in I}$ . Suppose  $\Xi \in \text{Hom}_A^*(\mathcal{U})$  is a co-isometry, then the sequence  $\{\Xi^* \mathcal{V}_i\}_{i \in I}$  forms the  $K$ -dual  $g$ -frame sequence for  $\{\Xi^* \Gamma_i\}_{i \in I}$ .

*Proof.* Given that  $\Xi$  is a co-isometry, it implies  $\Xi \Xi^* = I$ . Thus, for every element  $\xi \in \mathcal{U}$ , the following equation holds true:

$$\sum_{i \in I} (\Xi^* \Gamma_i)^* (\Xi^* \mathcal{V}_i) \xi = \sum_{i \in I} \Gamma_i^* \Xi \Xi^* \mathcal{V}_i \xi = \sum_{i \in I} \Gamma_i^* \mathcal{V}_i \xi = K\xi, \quad (2.21)$$

which means that the sequence  $\{\Xi^* \mathcal{V}_i\}_{i \in I}$  constitutes the  $K$ -dual  $g$ -frame sequence of  $\{\Xi^* \Gamma_i\}_{i \in I}$ .  $\square$

**Theorem 2.15.** Assume  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  has a closed range, and  $\{\Gamma_i\}_{i \in I}$  forms a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . In this case, the sequence  $\left\{ \Gamma_i \pi_{S(\text{Ran}(K))} \left( S_{\Gamma}^{-1} \right)^* K \right\}_{i \in I}$  acts as the  $K$ -dual  $g$ -frame sequence for  $\left\{ \Gamma_i \pi_{\text{Ran}(K)} \right\}_{i \in I}$ , where  $S_{\Gamma}$  is the operator defined by

$$S_{\Gamma} : \text{Ran}(K) \rightarrow S(\text{Ran}(K)).$$

*Proof.* Given that  $S_{\Gamma} : \text{Ran}(K) \rightarrow S(\text{Ran}(K))$  is a bounded operator, it's clear that

$$\left\{ \Gamma_i \pi_{S(\text{Ran}(K))} \left( S_{\Gamma}^{-1} \right)^* K \right\}_{i \in I}$$

forms a  $g$ -frame sequence for  $\mathcal{U}$ . Additionally, considering that  $S_{\Gamma}$  possesses the properties of being both self-adjoint and invertible, for every  $\xi \in \mathcal{U}$ , the following holds true:

$$\begin{aligned} K\xi &= \left( S_{\Gamma}^{-1} S_{\Gamma} \right)^* K\xi = S_{\Gamma}^* \left( S_{\Gamma}^{-1} \right)^* K\xi \\ &= S_{\Gamma} \pi_{S(\text{Ran}(K))} \left( S_{\Gamma}^{-1} \right)^* K\xi \\ &= \sum_{i \in I} \Gamma_i^* \Gamma_i \pi_{S(\text{Ran}(K))} \left( S_{\Gamma}^{-1} \right)^* K\xi \\ &= \sum_{i \in I} (\Gamma_i)^* \left( \Gamma_i \pi_{S(\text{Ran}(K))} \left( S_{\Gamma}^{-1} \right)^* K \right) \xi, \end{aligned}$$

which completes the proof, demonstrating the proposed relationship.  $\square$

**Theorem 2.16.** Assume  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and  $\{\mathcal{V}_i\}_{i \in I}$  forms the  $K$ -dual  $g$ -frame sequence of  $\{\Gamma_i\}_{i \in I}$ . Given  $P$  as the  $g$ -operator for  $\{\Gamma_i\}_{i \in I}$  and  $Q$  as the  $g$ -operator of the  $g$ -frame sequence  $\{\Xi_i\}_{i \in I}$ , the equality  $PQ^* = 0$  is both necessary and sufficient for the sequence  $\{\mathcal{V}_i + \Xi_i\}_{i \in I}$  to act as the  $K$ -dual  $g$ -frame sequence for  $\{\Gamma_i\}_{i \in I}$ .

*Proof.* If  $PQ^* = 0$  holds, and considering  $\{\mathfrak{E}_i\}_{i \in I}$  as the  $g$ -orthonormal basis for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , then for every element  $\xi \in \mathcal{U}$ , it follows that:

$$\sum_{i \in I} \Gamma_i^* \Xi_i \xi = \sum_{i \in I} (\mathfrak{E}_i P^*)^* (\mathfrak{E}_i Q^*) \xi = P \sum_{i \in I} \mathfrak{E}_i^* \mathfrak{E}_i Q^* \xi = PQ^* \xi = 0. \quad (2.22)$$

Consequently, for all  $\xi \in \mathcal{U}$ , we have:

$$\sum_{i \in I} \Gamma_i^* (\mathcal{V}_i + \Xi_i) \xi = \sum_{i \in I} \Gamma_i^* \mathcal{V}_i \xi = K\xi. \quad (2.23)$$

The reverse implication is derived by following the same logical steps in reverse order, arriving at the initial condition.  $\square$

**Theorem 2.17.** Let  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , and consider two  $K$ -dual generalized frame sequences  $\{\Phi_i\}_{i \in I}$  and  $\{\Xi_i\}_{i \in I}$  corresponding to the sequence  $\{\Gamma_i\}_{i \in I}$ , respectively. Suppose  $T_1$  and  $T_2$  are linear operators on  $\mathcal{U}$  satisfying  $T_1 + T_2 = I$ . In this case, the sequence  $\{\Phi_i T_1 + \Xi_i T_2\}_{i \in I}$  forms the  $K$ -dual generalized frame sequence for  $\{\Gamma_i\}_{i \in I}$ .

*Proof.* Given that  $T_1 + T_2 = I$ , it follows that for every  $\xi \in \mathcal{U}$ ,

$$\begin{aligned} \sum_{i \in I} \Gamma_i^* (\Phi_i T_1 + \Xi_i T_2) \xi &= \sum_{i \in I} \Gamma_i^* \Phi_i T_1 \xi + \sum_{i \in I} \Gamma_i^* \Xi_i T_2 \xi \\ &= KT_1 \xi + KT_2 \xi = K(T_1 + T_2) \xi = K\xi. \end{aligned}$$

This concludes the proof.  $\square$

**Corollary 2.18.** Suppose  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  is invertible. Let  $\{\Phi_i\}_{i \in I}$  and  $\{\Xi_i\}_{i \in I}$  be the  $K$ -dual generalized frame sequences of  $\{\Gamma_i\}_{i \in I}$ , respectively. If  $T_1$  and  $T_2$  are linear operators on  $\mathcal{U}$ , then the sequence  $\{\Phi_i T_1 + \Xi_i T_2\}_{i \in I}$  constitutes the  $K$ -dual generalized frame sequence of  $\{\Gamma_i\}_{i \in I}$  if and only if  $T_1 + T_2 = I$ .

*Proof.* Assume  $T_1 + T_2 = I$ . From Theorem 2.17,  $\{\Phi_i T_1 + \Xi_i T_2\}_{i \in I}$  forms the  $K$ -dual generalized frame. Now, we establish the sufficiency condition.

If  $\{\Phi_i T_1 + \Xi_i T_2\}_{i \in I}$  is the  $K$ -dual generalized frame sequence for  $\{\Gamma_i\}_{i \in I}$ , then for any  $\xi \in \mathcal{U}$ , we have

$$\begin{aligned} K\xi &= \sum_{i \in I} \Gamma_i^* (\Phi_i T_1 + \Xi_i T_2) \xi = \sum_{i \in I} \Gamma_i^* \Phi_i T_1 \xi + \sum_{i \in I} \Gamma_i^* \Xi_i T_2 \xi \\ &= K T_1 \xi + K T_2 \xi = K (T_1 + T_2) \xi \end{aligned}$$

Therefore,

$$K = K(T_1 + T_2)$$

As  $K$  is invertible, it follows that  $T_1 + T_2 = I$ .  $\square$

**Theorem 2.19.** *Let  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  such that the range of  $K$ ,  $\text{Ran}(K)$ , is closed. If  $\{\Gamma_i\}_{i \in I}$  forms a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  with a frame operator  $S$ , and  $C$  and  $D$  are its lower and upper bounds, respectively, then*

$$CKK^* \leq S \leq D I_{\mathcal{U}} \quad (2.24)$$

*Proof.* Given that  $\{\Gamma_i\}_{i \in I}$  is a  $K$ - $g$ -frame with the frame operator  $S$ , it follows that

$$\langle S\xi, \xi \rangle = \left\langle \sum_{i \in I} \Gamma_i^* \Gamma_i \xi, \xi \right\rangle = \sum_{i \in I} \langle \Gamma_i \xi, \Gamma_i \xi \rangle \quad (2.25)$$

In line with equation 2.25, we have

$$C \langle K^* \xi, K^* \xi \rangle \leq \langle S\xi, \xi \rangle \leq D \langle \xi, \xi \rangle, \quad \forall \xi \in \mathcal{U}, \quad (2.26)$$

which implies

$$\langle CKK^* \xi, \xi \rangle \leq \langle S\xi, \xi \rangle \leq \langle B\xi, \xi \rangle. \quad (2.27)$$

$\square$

**Corollary 2.20.** *Suppose  $K$  is an element of  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , and consider  $\{\Gamma_i\}_{i \in I}$  as a  $g$ -Bessel sequence within  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  with the corresponding frame operator denoted as  $S$ . The sequence  $\{\Gamma_i\}_{i \in I}$  qualifies as a  $K$ - $g$ -frame for  $\mathcal{U}$  if and only if  $K = S^{\frac{1}{2}}Q$  for a certain  $Q \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ .*

*Proof.* Referencing Lemma 2.5, the sequence  $\{\Gamma_i\}_{i \in I}$  can be identified as a  $K$ - $g$ -frame for  $\mathcal{U}$ , with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , if and only if there exists a positive constant  $C$  that ensures

$$CKK^* \leq S = S^{\frac{1}{2}} \left( S^{\frac{1}{2}} \right)^*.$$

Following this, as per Lemma 1.6, it is concluded that a bounded operator  $U$  can be found such that

$$K = S^{\frac{1}{2}}U.$$

$\square$

**Theorem 2.21.** *Consider a mapping  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , and  $\{\Gamma_i\}_{i \in I}$  a  $K$ - $g$ -frame for  $\mathcal{U}$ , with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , having bounds  $C$  and  $D$  as its lower and upper limits, respectively. Assume  $Q$  is an element of  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  with the property of a closed range and the condition  $QK = KQ$  being met. Under these circumstances, the following are observed:*

- (i) *The sequence  $\{\Gamma_i Q^*\}_{i \in I}$  establishes itself as a  $K$ - $g$ -frame for the range of  $Q$  (denoted as  $\text{Ran}(Q)$ ) with respect to  $\{\mathcal{V}_i\}_{i \in I}$ , bounded below and above by  $C \|Q^\dagger\|_\infty^{-2}$  and  $D \|Q\|_\infty^2$ , respectively.*

(ii) The frame operator  $S$  corresponding to  $\{\Gamma_i Q^*\}_{i \in I}$  fulfills the relation  $S = Q S_\Gamma Q^*$ , with  $S_\Gamma$  being the frame operator for  $\{\Gamma_i\}_{i \in I}$ .

*Proof.* Considering that  $Q$  has a closed range, it consequently possesses a pseudo-inverse  $Q^\dagger$ , fulfilling  $Q Q^\dagger = I_{\text{Ran}(Q)}$ , where the subscript  $\text{Ran}(Q)$  of  $Q^\dagger$  is omitted. Now,  $I_{\text{Ran}(Q)} = I_{\text{Ran}(Q)}^* = (Q^\dagger)^* (Q^*)$ . Thus, for every  $\xi \in \text{Ran}(Q)$ ,

$$K^* \xi = (Q^\dagger)^* Q^* K^* \xi$$

leads to,

$$\begin{aligned} \langle K^* \xi, K^* \xi \rangle &= \langle (Q^\dagger)^* Q^* K^* \xi, (Q^\dagger)^* Q^* K^* \xi \rangle \\ &\leq \left\| (Q^\dagger)^* \right\|_\infty^2 \langle Q^* K^* \xi, Q^* K^* \xi \rangle \\ &= \left\| (Q^\dagger)^* \right\|_\infty^2 \langle K^* Q^* \xi, K^* Q^* \xi \rangle. \end{aligned}$$

Consequently,

$$\left\| (Q^\dagger)^* \right\|_\infty^{-2} \langle K^* \xi, K^* \xi \rangle \leq \langle K^* Q^* \xi, K^* Q^* \xi \rangle.$$

For each  $\xi \in \text{Ran}(Q)$ , we have

$$\sum_{i \in I} \langle \Gamma_i Q^* \xi, \Gamma_i Q^* \xi \rangle \geq C \langle K^* Q^* \xi, K^* Q^* \xi \rangle \geq C \left\| (Q^\dagger)^* \right\|_\infty^{-2} \langle K^* \xi, K^* \xi \rangle \quad (2.28)$$

Moreover,  $\{\Gamma_i Q^*\}_{i \in I}$  is a  $g$ -Bessel sequence, as evidenced by:

$$\sum_{i \in I} \langle \Gamma_i Q^* \xi, \Gamma_i Q^* \xi \rangle \leq D \langle Q^* \xi, Q^* \xi \rangle \leq B \|Q\|_\infty^2 \langle \xi, \xi \rangle, \quad (2.29)$$

From 2.28 and 2.29, point 1 is established. For point 2, it is clear that:

$$\begin{aligned} S\xi &= \sum_{i \in I} (\Gamma_i Q^*)^* \Gamma_j Q^* \xi \\ &= Q \sum_{i \in I} \Gamma_i^* \Gamma_i Q^* \xi \\ &= Q S_\Lambda Q^* \xi. \end{aligned}$$

This concludes the proof of Theorem 2.21.  $\square$

**Theorem 2.22.** Assume we have a mapping  $K \in \text{Hom}_A^*(U)$  and  $\{\Gamma_i\}_{i \in I}$  as a  $K$ - $g$ -frame for  $U$ , with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Provided that  $Q$ , also a member of  $\text{Hom}_A^*(U)$ , functions as a co-isometry and satisfies the commutative relation  $QK = KQ$ , it follows that the set  $\{\Gamma_i Q^*\}_{i \in I}$  forms a  $K$ - $g$ -frame for  $U$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ .

*Proof.* Given that  $Q$  is a co-isometry and satisfies  $QK = KQ$ , for any  $\xi \in U$ , it follows that

$$\langle K^* Q^* \xi, K^* Q^* \xi \rangle = \langle Q^* K^* \xi, Q^* K^* \xi \rangle = \langle K^* \xi, K^* \xi \rangle. \quad (2.30)$$

Assuming  $C$  and  $D$  are the respective lower and upper bounds of  $\{\Gamma_i Q^*\}_{i \in I}$ , then for every  $\xi \in U$ , we have

$$\sum_{i \in I} \langle \Gamma_i Q^* \xi, \Gamma_i Q^* \xi \rangle \geq C \langle K^* Q^* \xi, K^* Q^* \xi \rangle = C \langle K^* \xi, K^* \xi \rangle. \quad (2.31)$$

Conversely,

$$\sum_{i \in I} \langle \Gamma_i Q^* \xi, \Gamma_i Q^* \xi \rangle \leq D \langle Q^* \xi, Q^* \xi \rangle \leq B \|Q\|_\infty^2 \langle \xi, \xi \rangle. \quad (2.32)$$

Therefore,  $\{\Gamma_i Q^*\}_{i \in I}$  constitutes a  $K$ - $g$ -frame for  $U$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ .  $\square$

**Corollary 2.23.** Assume  $K$  is a part of  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and  $\{\Gamma_i\}_{i \in I}$  constitutes a  $K$ - $g$ -frame for  $\mathcal{U}$ , with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . If  $Q$  operates as an isometry, then the set  $\{Q\Gamma_i\}_{i \in I}$  also forms a  $K$ - $g$ -frame for  $\mathcal{U}$ , with respect to  $\{\mathcal{E}\mathcal{V}_i\}_{i \in I}$ , maintaining the same frame bounds as  $\{\Gamma_i\}_{i \in I}$ .

*Proof.* Suppose that  $C$  and  $D$  are, respectively, the lower and upper bounds for the set  $\{\Gamma_i\}_{i \in I}$ . Consequently, it follows that

$$C\langle K^*\xi, K^*\xi \rangle \leq \sum_{i \in I} \langle Q\Gamma_i\xi, Q\Gamma_i\xi \rangle \leq D\langle \xi, \xi \rangle. \quad (2.33)$$

It is important to note here that  $Q$  functions as an isometry. □

**Theorem 2.24.** Consider  $K$  as an element of  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$  and  $\{\Gamma_i\}_{i \in I}$  as a  $g$ -Bessel sequence for  $\mathcal{U}$ , with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Suppose that  $T$  represents the synthesis operator for  $\{\Gamma_i\}_{i \in I}$ . The condition  $\text{Ran}(K) = \text{Ran}(T)$  holds true if and only if  $\{\Gamma_i\}_{i \in I}$  forms a tight  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ .

*Proof.* Let us start by assuming  $\text{Ran}(K) = \text{Ran}(T)$ . Based on Lemma 1.6, there exists a constant  $A > 0$  such that

$$AKK^* = TT^*,$$

leading to

$$A\langle K^*\xi, K^*\xi \rangle = \langle T^*\xi, T^*\xi \rangle = \sum_{i \in I} \langle \Gamma_i\xi, \Gamma_i\xi \rangle.$$

Conversely, if  $\{\Gamma_i\}_{i \in I}$  is recognized as a tight  $K$ - $g$ -frame with a frame bound of  $A$ , then it follows that

$$A\langle K^*\xi, K^*\xi \rangle = \sum_{i \in I} \langle \Gamma_i\xi, \Gamma_i\xi \rangle = \langle T^*\xi, T^*\xi \rangle,$$

which simplifies to

$$AKK^* = TT^*.$$

Therefore, invoking Lemma 1.6 again, it can be concluded that  $\text{Ran}(K) = \text{Ran}(T)$ . □

**Theorem 2.25.** Suppose  $\{\Gamma_i\}_{i \in I}$  and  $\{\Xi_i\}_{i \in I}$  represent two  $g$ -Bessel sequences for  $\mathcal{U}$ , each associated with  $\{\mathcal{V}_i\}_{i \in I}$  and having bounds  $A$  and  $B$ , respectively. Let  $T_1$  and  $T_2$  be the synthesis operators for  $\{\Gamma_i\}_{i \in I}$  and  $\{\Xi_i\}_{i \in I}$ , respectively. If it holds that  $T_1T_2^* = I_{\mathcal{U}}$ , then both  $\{\Gamma_i\}_{i \in I}$  and  $\{\Xi_i\}_{i \in I}$  qualify as  $K$ - $g$ -frames for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ .

### 3 Characterizing the $K$ - $g$ Frames by Other Concepts

In this section, we discuss the concept of identity resolution and explore the idea of quotients derived from bounded operators. These concepts will subsequently be utilized in the construction of  $K$ - $g$ -frames.

**Definition 3.1.** Let  $I$  be an indexing set. A collection of bounded operators  $\{\Psi_i\}_{i \in I}$  on  $\mathcal{U}$  is defined as an (unconditional) resolution of the identity on  $\mathcal{U}$  if, for every element  $\xi \in \mathcal{U}$ , it satisfies:

$$\xi = \sum_{i \in I} \Psi_i(\xi)$$

(where the series converges unconditionally for each  $\xi \in \mathcal{U}$ ). It can be readily shown that if  $\{\Theta_i\}_{i \in I}$  and  $\{\Psi_i\}_{i \in I}$  are both resolutions of the identity on  $\mathcal{U}$ , then the combined set  $\{\Theta_i\Psi_i\}_{i \in I}$  also constitutes a resolution of the identity on  $\mathcal{U}$ .

**Theorem 3.2.** Consider an element  $K$  in  $\text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , which is invertible and where both  $K$  and its inverse  $K^{-1}$  are uniformly bounded. Assume that the collection  $\{\Psi_i\}_{i \in I}$  forms the identity resolution of  $\mathcal{U}$  and constitutes a  $g$ -Bessel sequence with a bound of  $D$ . Under these conditions,  $\{\Psi_i\}_{i \in I}$  is identified as a  $K$ - $g$ -frame for  $\mathcal{U}$  in relation to  $\{\mathcal{V}_i\}_{i \in I}$ .

*Proof.* Given that  $\{\Psi_i\}_{i \in I}$  constitutes an identity resolution for the space  $\mathcal{U}$ , we observe that for every element  $\xi$  within  $\mathcal{U}$ , the expression

$$\xi = \sum_{i \in I} \Psi_i(\xi)$$

is valid. Consequently, the following relationship is established:

$$K^* \xi = \sum_{i \in I} \Psi_i K^* \xi,$$

leading to the inference that

$$\langle K^* \xi, K^* \xi \rangle = \left\langle \sum_{i \in I} \Psi_i K^* \xi, \sum_{i \in I} \Psi_i K^* \xi \right\rangle \leq \sum_{i \in I} \langle \Psi_i K^* \xi, \Psi_i K^* \xi \rangle \leq D \langle K^* \xi, K^* \xi \rangle \leq D \|K\|_\infty^2 \langle \xi, \xi \rangle. \quad (3.1)$$

□

Let  $\mathcal{F}, \mathcal{T} \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ . The quotient  $[\mathcal{F}/\mathcal{T}]$  is a map from  $\text{Ran}(\mathcal{F})$  to  $\text{Ran}(\mathcal{T})$  defined by  $\mathcal{T}x \mapsto \mathcal{F}x$ . Similar to the case in Hilbert spaces,  $Q = [\mathcal{F}/\mathcal{T}]$  is a linear operator on  $\mathcal{U}$  if and only if  $\text{Ker}(\mathcal{T}) \subset \text{Ker}(\mathcal{F})$ .

**Theorem 3.3.** Suppose  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{U})$ , and consider  $\{\Psi_i\}_{i \in I}$ , which constitutes a  $g$ -Bessel sequence in the context of  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ . Given that  $S$  represents the frame operator for  $\{\Psi_i\}_{i \in I}$ , it follows that  $\{\Psi_i\}_{i \in I}$  forms a  $K$ - $g$ -frame with respect to  $\{\mathcal{V}_i\}_{i \in I}$  if and solely if the operator defined by  $\left[ K^*/S^{\frac{1}{2}} \right]$  demonstrates boundedness.

*Proof.* Given that  $\{\Psi_i\}_{i \in I}$  constitutes a  $K$ - $g$ -frame, we can ascertain the existence of a constant  $C > 0$ . This ensures that for any  $\xi \in \mathcal{U}$ , the following inequality holds:

$$C \langle K^* \xi, K^* \xi \rangle \leq \sum_{i \in I} \langle \Psi_i \xi, \Psi_i \xi \rangle = \langle S e \xi, \xi \rangle = \langle S e^{\frac{1}{2}} \xi, S e^{\frac{1}{2}} \xi \rangle \quad (3.2)$$

holds. This implies that

$$C \langle K^* \xi, K^* \xi \rangle \leq \langle S e^{\frac{1}{2}} \xi, S e^{\frac{1}{2}} \xi \rangle. \quad (3.3)$$

Define the operator  $\Xi : \text{Ran}(S e^{\frac{1}{2}}) \rightarrow \text{Ran}(K^*)$  by  $\Xi(S e^{\frac{1}{2}} \xi) = K^* \xi$  for all  $\xi \in \mathcal{U}$ . The operator  $\Xi$  is well-defined since  $\text{Ker}(S e^{\frac{1}{2}}) \subseteq \text{Ker}(K^*)$ . Consequently, we obtain

$$\langle \Xi S e^{\frac{1}{2}} \xi, \Xi S e^{\frac{1}{2}} \xi \rangle = \langle K^* \xi, K^* \xi \rangle \leq \frac{1}{\sqrt{C}} \langle S e^{\frac{1}{2}} \xi, S e^{\frac{1}{2}} \xi \rangle. \quad (3.4)$$

Thus,  $\Xi$  is bounded. By the concept of quotient of bounded operators,  $\Xi$  is expressed as  $\left[ K^*/S e^{\frac{1}{2}} \right]$ .

On the other hand, if  $\left[ K^*/S e^{\frac{1}{2}} \right]$  is bounded, there exists a constant  $D > 0$  such that for all  $\xi \in \mathcal{U}$ ,

$$\langle K^* \xi, K^* \xi \rangle \leq D \langle S e^{\frac{1}{2}} \xi, S e^{\frac{1}{2}} \xi \rangle, \quad (3.5)$$

implying

$$\frac{1}{D} \langle K^* \xi, K^* \xi \rangle^2 \leq \langle S e^{\frac{1}{2}} \xi, S e^{\frac{1}{2}} \xi \rangle = \sum_{i \in I} \langle \Psi_i \xi, \Psi_i \xi \rangle. \quad (3.6)$$

Therefore,  $\{\Psi_i\}_{i \in I}$  is a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\Theta_i\}_{i \in I}$ . □

**Theorem 3.4.** Let  $K \in \text{Hom}_{\mathcal{A}}^*(\mathcal{X})$  and  $\{\Psi_i\}_{i \in I}$  be a  $g$ -Bessel sequence for  $\mathcal{X}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$  with frame operator  $S$ . If  $Q \in \text{Hom}_{\mathcal{A}}^*(\mathcal{X})$ , then the following statements are equivalent:

(i)  $\left[ K^*/S^{\frac{1}{2}}Q \right]$  is bounded.

(ii)  $\{\Psi_i Q\}_{i \in I}$  is a  $K$ - $g$ -frame for  $\mathcal{X}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ ;

**Theorem 3.5.** Consider  $K \in \text{Hom}_A^*(\mathcal{X})$  and let  $\{\Psi_i\}_{i \in I}$  constitute a  $g$ -Bessel sequence in the context of  $\mathcal{U}$ , associated with  $\{\mathcal{V}_i\}_{i \in I}$ , and possessing the frame operator  $S$ . Given another element  $Q$  within  $\text{Hom}_A^*(\mathcal{V})$ , the following conditions are equivalent:

(i) The operator  $\left[ K^*/S^{\frac{1}{2}}Q \right]$  exhibits boundedness.

(ii)  $\{\Psi_i Q\}_{i \in I}$  forms a  $K$ - $g$ -frame for  $\mathcal{U}$  with respect to  $\{\mathcal{V}_i\}_{i \in I}$ .

*Proof.* (2)  $\Rightarrow$  (1) Assume  $C$  is the lower bound constant for the  $K$ - $g$ -frame  $\{\Psi_i Q\}_{i \in I}$  for  $\mathcal{U}$  with respect to  $\{\Theta_i\}_{i \in I}$ . Then,

$$C\langle K^*\xi, K^*\xi \rangle \leq \sum_{i \in I} \langle \Psi_i Q\xi, \Psi_i Q\xi \rangle.$$

Furthermore,

$$\sum_{i \in I} \langle \Psi_i Q\xi, \Psi_i Q\xi \rangle = \langle SQ\xi, Q\xi \rangle = \langle S^{\frac{1}{2}}Q\xi, S^{\frac{1}{2}}Q\xi \rangle, \tag{3.7}$$

since  $\{\Psi_i\}_{i \in I}$  is a  $g$ -Bessel sequence for  $\mathcal{U}$  and  $S$  its frame operator. Therefore,

$$C\langle K^*\xi, K^*\xi \rangle \leq \langle S^{\frac{1}{2}}Q\xi, S^{\frac{1}{2}}Q\xi \rangle, \tag{3.8}$$

indicating that  $\left[ K^*/S^{\frac{1}{2}}Q \right]$  is bounded.

(1)  $\Rightarrow$  (2) If  $\left[ K^v/S^{\frac{1}{2}}Q \right]$  is bounded, there exists a constant  $\beta > 0$  such that

$$\langle K^*\xi, K^*\xi \rangle \leq \beta \langle S^{\frac{1}{2}}Q\xi, S^{\frac{1}{2}}Q\xi \rangle. \tag{3.9}$$

Assuming that the  $g$ -Bessel sequence  $\{\Psi_i Q\}_{i \in I}$  for  $\mathcal{U}$  with respect to  $\{\Theta_i\}_{i \in I}$  has an upper bound constant  $D$ , it follows

$$\frac{1}{\beta} \langle K^*\xi, K^*\xi \rangle \leq \langle S^{\frac{1}{2}}Q\xi, S^{\frac{1}{2}}Q\xi \rangle = \sum_{i \in I} \langle \Psi_i Q\xi, \Psi_i Q\xi \rangle \leq D \|\mathfrak{E}\|_\infty^2 \langle \xi, \xi \rangle, \tag{3.10}$$

leading to the conclusion. □

### Conclusion

To conclude, we have developed the theory of  $K$ - $g$ -frames in Hilbert locally  $C^*$ -algebras as an extension of the classical  $g$ -frame concept. By introducing the  $g$ -orthonormal basis and the associated  $g$ -operator, we established several fundamental relationships between  $g$ -frames and  $K$ - $g$ -frames. Furthermore, we provided characterizations of  $K$ - $g$ -frames through equivalent conditions and studied the notion of the  $K$ -dual  $g$ -frame along with its essential properties. The results obtained in this work open up several directions for further research. In particular, one may investigate perturbation and stability results for  $K$ - $g$ -frames, as well as explore their potential applications in operator theory, functional analysis, and signal reconstruction problems over locally  $C^*$ -algebras.

### References

[1] L. Alizadeh, M. Hassani, On frames for countably generated Hilbert modules over locally  $C^*$ -algebras, Commun. Korean Math. Soc. **33** (2018) 527-533. <https://doi.org/10.4134/CKMS.c170194>.  
 [2] M. Azhini, N. Haddadzadeh, Fusion frames in Hilbert modules over pro- $C^*$ -algebras, Int. J. Ind. Math. **5** (2013) 109-118.  
 [3] R. El Jazzar, M. Rossafi, On frames in Hilbert modules over locally  $C^*$ -algebras, Int. J. Anal. Appl. **21** (2023) 130. <https://doi.org/10.28924/2291-8639-21-2023-130>.

- [4] M. Fragoulopoulou, *Tensor products of enveloping locally  $C^*$ -algebras*, *Schriftenreihe, Univ. Münster* (1997) 1-81.
- [5] M. Fragoulopoulou, *An introduction to the representation theory of topological  $*$ -algebras*, *Schriftenreihe, Univ. Münster* **48** (1988) 1-81.
- [6] N. Haddadzadeh, *g-frames in Hilbert pro- $C^*$ -modules*, *Int. J. Pure Appl. Math.* **105** (2015) 727-743. <https://doi.org/10.12732/ijpam.v105i4.13>.
- [7] A. Inoue, *Locally  $C^*$ -algebras*, *Mem. Fac. Sci. Kyushu Univ. Ser. A Math.* **25** (1972) 197-235. <https://doi.org/10.2206/kyushumfs.25.197>.
- [8] S. Jahan, V. Kumar, C. Shekhar, *Cone associated with frames in Banach spaces*, *Palestine J. Math.* **7** (2) (2018) 641–649.
- [9] A. Karara, M. Rossafi, M. Klilou, S. Kabbaj, *Construction of continuous  $K$ -g-Frames in Hilbert  $C^*$ -Modules*, *Palest. J. Math.* **13**(4) (2024), 198–209.
- [10] K. M. Krishna, P. S. Johnson, *Dilation theorem for  $p$ -approximate Schauder frames for separable Banach spaces*, *Palestine J. Math.* **11** (2) (2022) 384–394.
- [11] A. Mallios, *Topological Algebras: Selected Topics*, North Holland, Elsevier, 2011.
- [12] N. C. Phillips, *Inverse limits of  $C^*$ -algebras*, *J. Oper. Theory* **19** (1988) 159-195.
- [13] N. C. Phillips, *Representable  $K$ -theory for  $\sigma$ - $C^*$ -algebras*, *K-Theory* **3** (1989) 441–478.
- [14] S. Ramesan, K. T. Ravindran, *Scalability and  $K$ -frames*, *Palestine J. Math.* **12** (1) (2023) 493–500.
- [15] M. Rossafi, R. El Jazzar and R. Mohapatra, *Douglas' factorization theorem and atomic systems in Hilbert pro- $C^*$ -modules*, *Sahand Commun. Math. Anal.* **21** (2024), 25–49. <https://doi.org/10.22130/scma.2023.2001846.1318>

### Author information

R. El Jazzar, *Laboratory Analysis, Geometry and Applications, University of Ibn Tofail, Kenitra, Morocco, Morocco.*

E-mail: [roumaissae.eljazzar@uit.ac.ma](mailto:roumaissae.eljazzar@uit.ac.ma)

M. Rossafi, *Laboratory Analysis, Geometry and Applications, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco, Morocco.*

E-mail: [rossafimohamed@gmail.com](mailto:rossafimohamed@gmail.com); [mohamed.rossafi1@uit.ac.ma](mailto:mohamed.rossafi1@uit.ac.ma)

M. Mouniane, *LIRAMEF, Higher School of Education and Training, University of Ibn Tofail, Kenitra, Morocco, Morocco.*

E-mail: [mohammed.mouniane@uit.ac.ma](mailto:mohammed.mouniane@uit.ac.ma)

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