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COMMON NEIGHBORHOOD LAPLACIAN AND SIGNLESS LAPLACIAN SPECTRA AND ENERGIES OF COMMUTING GRAPHS

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Abstract. In this paper, we compute common neighbourhood Laplacian spectrum, common neighbourhood signless Laplacian spectrum and their respective energies of commuting graph of some finite non-abelian groups including some AC-groups, groups whose central quotients are isomorphic to $Sz(2)$, $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m}. Our findings lead us to conclude that these graphs are CNL (CNSL)-integral. Additionally, we characterize the aforementioned groups such that their commuting graphs are CNL (CNSL)-hyperenergetic.

1 Introduction

Let G be a finite non-abelian group with center $Z(G)$. The commuting graph of G, denoted by Γ_G is a simple undirected graph whose vertex set is $G \setminus Z(G)$ and two vertices x and y are adjacent if they commute. This graph was originated from the work of Brauer and Fowler [4] published in the year 1955. However, after Neumann's work [21] on its complement (that is non-commuting graph) in 1976, the commuting graph became popular. Various aspect of commuting graphs of finite non-abelian groups can be found in [1, 6, 7, 16, 19]. Recently, people have become interested on the spectral aspects of Γ_G [8, 9, 10, 11, 12, 22].

In 2011, Alwardi et al. [2] introduced the concepts of common neighbourhood spectrum and energy (abbreviated as CN-spectrum and CN-energy) of a graph. Let $\mathcal{G}$ be a simple graph with vertex set $V(\mathcal{G}) = \{v_1, v_2, \dots, v_n\}$. The common neighbourhood matrix of $\mathcal{G}$, denoted by $CN(\mathcal{G})$, is a matrix of size n whose (i, j)-th entry is given by

$$\text{CN}(\mathcal{G})_{i,j} = \begin{cases} |C(v_i, v_j)|, & \text{if } i \neq j \\ 0, & \text{otherwise,} \end{cases}$$

where $C(v_i, v_j) = \{x \in V(\mathcal{G}) : x \neq v_i, v_j \text{ and adjacent to both } v_i, v_j\}$. The CN-energy of $\mathcal{G}$ is defined as $\text{E}_{\text{CN}}(\mathcal{G}) := \sum\limits_{\delta \in \text{CN-spec}(\mathcal{G})} |\delta|$, where CN-spec$(\mathcal{G})$ is the set of eigenvalues of $CN(\mathcal{G})$ with multiplicities. A graph $\mathcal{G}$ is called CN-integral if CN-spec$(\mathcal{G})$ contains only integers. Also, $\mathcal{G}$ is called CN-hyperenergetic and CN-borderenergetic (see [2]) if $\text{E}_{\text{CN}}(\mathcal{G}) > \text{E}_{\text{CN}}(K_{|V(\mathcal{G})|})$ and $\text{E}_{\text{CN}}(\mathcal{G}) = \text{E}_{\text{CN}}(K_{|V(\mathcal{G})|})$ respectively. In [15, 20] Fasfous et al. and Nath et al. discussed various aspects of CN-spec(Γ_G) and $\text{E}_{\text{CN}}(\Gamma_G)$ for several classes of finite non-abelian groups. Fasfous and Nath [13] also considered CN-spectrum and CN-energy of commuting graph of finite non-commutative rings in their works.

Jannat et al. [17], introduced the notions of common neighborhood Laplacian spectrum (CNL-spectrum), common neighborhood signless Laplacian spectrum (CNSL-spectrum) and energies corresponding to those spectra viz. common neighborhood Laplacian energy (CNL-energy) and common neighborhood signless Laplacian energy (CNSL-energy). The common neighborhood Laplacian matrix (CNL-matrix) and the common neighborhood signless Laplacian matrix (CNSL-matrix) of $\mathcal{G}$, denoted by CNL($\mathcal{G}$) and CNSL($\mathcal{G}$), respectively, are given by

$$\text{CNL}(\mathcal{G}) := \text{CNRS}(\mathcal{G}) - \text{CN}(\mathcal{G}) \text{ and } \text{CNSL}(\mathcal{G}) := \text{CNRS}(\mathcal{G}) + \text{CN}(\mathcal{G}),$$

where CNRS($\mathcal{G}$) is a matrix of size $|V(\mathcal{G})|$ whose (i, j)-th entry is given by

$$\text{CNRS}(\mathcal{G})_{i,j} = \begin{cases} \sum_{k=1}^{|V(\mathcal{G})|} \text{CN}(\mathcal{G})_{i,k}, & \text{if } i = j \text{ and } i = 1, 2, \ldots, |V(\mathcal{G})| \\ 0, & \text{if } i \neq j, \end{cases}$$

The set of eigenvalues of CNL($\mathcal{G}$) and CNSL($\mathcal{G}$) with multiplicities are called CN-Laplacian spectrum (denoted by CNL-spec($\mathcal{G}$)) and CN-signless Laplacian spectrum (denoted by CNL-spec($\mathcal{G}$)) of $\mathcal{G}$, respectively. Let CNL-spec($\mathcal{G}$) $= \{\alpha_1^{a_1}, \alpha_2^{a_2}, \ldots, \alpha_k^{a_k}\}$ and CNSL-spec($\mathcal{G}$) $= \{\beta_1^{b_1}, \beta_2^{b_2}, \ldots, \beta_\ell^{b_\ell}\}$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are the distinct eigenvalues of CNL($\mathcal{G}$) with corresponding multiplicities $a_1, a_2, \ldots, a_k$ and $\beta_1, \beta_2, \ldots, \beta_\ell$ are the distinct eigenvalues of CNSL($\mathcal{G}$) with corresponding multiplicities $b_1, b_2, \ldots, b_\ell$. A graph $\mathcal{G}$ is called CNL (CNSL)-integral if CNL (CNSL)-spectrum contains only integers. The CNL-energy and CNSL-energy of $\mathcal{G}$, denoted by $LE_{CN}(\mathcal{G})$ and $LE^+_{CN}(\mathcal{G})$ respectively, are defined as

$$LE_{CN}(\mathcal{G}) := \sum_{i=1}^{k} a_i \left|\alpha_i - \Delta_{\mathcal{G}}\right| \tag{1.1}$$

and

$$LE^+_{CN}(\mathcal{G}) := \sum_{i=1}^{\ell} b_i \left|\beta_i - \Delta_{\mathcal{G}}\right|, \tag{1.2}$$

where $\Delta_{\mathcal{G}} = \frac{\text{tr(CNRS}(\mathcal{G}))}{|V(\mathcal{G})|}$. It was shown, in [17], that

$$LE_{CN}(K_n) = LE^+_{CN}(K_n) = 2(n-1)(n-2). \tag{1.3}$$

A graph $\mathcal{G}$ is called CNL-hyperenergetic and CNSL-hyperenergetic if $LE_{CN}(\mathcal{G}) > LE_{CN}(K_{|V(\mathcal{G})|})$ and $LE^+_{CN}(\mathcal{G}) > LE^+_{CN}(K_{|V(\mathcal{G})|})$ respectively. Various aspects of CNL-spectrum, CNL-energy, CNSL-spectrum and CNSL-energy of graphs, including their relations with other well-known graph energies and Zagreb indices, were discussed in [17].

In this paper, we compute CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of several classes of finite AC-groups including QD_{2^n} (quasi dihedral group), $PSL(2, 2^k)$ (projective special linear group), $GL(2, q)$ (general linear group where $q > 2$ is a prime power), $A(n, v)$, $A(n, p)$ (Hanaki groups), D_{2m} (dihedral group), groups whose central quotient is isomorphic to $Sz(2)$ or $\mathbb{Z}_p \times \mathbb{Z}_p$ or D_{2m} along with some other groups. It follows that the commuting graphs of all the groups considered in our paper are CNL-integral and CNSL-integral. Additionally, we determine when commuting graphs of these groups are CNL-hyperenergetic and CNSL-hyperenergetic. Recall that a group G is called an AC-group if $C_G(x) := \{y \in G : xy = yx\}$ is abelian for all $x \in G \setminus Z(G)$.

2 CNL (CNSL)-spectrum and CNL (CNSL)-energy

In this section we have computed the CNL (CNSL)-spectrum and CNL (CNSL)-energy of commuting graphs of various groups mentioned above. We start this section with the following result that will be needed for our computations.

Theorem 2.1. *[17] Let $\mathcal{G} = l_1K_{m_1} \cup l_2K_{m_2} \cup l_3K_{m_3}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of the complete graphs K_{m_i} on m_i vertices for $i = 1, 2, 3$. Then*
CNL-spec($\mathcal{G}$) $= \big\{0^{l_1+l_2+l_3}, (m_1(m_1-2))^{l_1(m_1-1)},$
$(m_2(m_2-2))^{l_2(m_2-1)}, (m_3(m_3-2))^{l_3(m_3-1)}\big\}$ *and*
CNSL-spec($\mathcal{G}$) $= \big\{(2(m_1-1)(m_1-2))^{l_1}, ((m_1-2)^2)^{l_1(m_1-1)}, (2(m_2-1)(m_2-2))^{l_2},$
$((m_2-2)^2)^{l_2(m_2-1)}, (2(m_3-1)(m_3-2))^{l_3}, ((m_3-2)^2)^{l_3(m_3-1)}\big\}.$

2.1 Some families of AC-group

Here we compute the CNL (CNSL)-spectrum and CNL (CNSL)-energy of commuting graphs of the quasihedral groups $QD_{2^n} = \{a, b : a^{2^{n-1}} = b^2 = 1, bab^{-1} = a^{2^{n-2}}\}$ for $n \geq 4$, projective special linear groups $PSL(2, 2^k)$ for $k \geq 2$, general linear groups $GL(2, q)$ for any prime power $q > 2$ and the Hanaki groups $A(n, v)$ and $A(n, p)$. We begin with the commuting graph of QD_{2^n}.

Proposition 2.2. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of the quasihedral groups QD_{2^n}, where $n \geq 4$, are given by*

$$\text{CNL-spec}(\Gamma_{QD_{2^n}}) = \{0^{2^{n-1}+1}, ((2^{n-1}-2)(2^{n-1}-4))^{(2^{n-1}-3)}\},$$

$$\text{CNSL-spec}(\Gamma_{QD_{2^n}}) = \{0^{2^{n-1}}, (2(2^{n-1}-3)(2^{n-1}-4))^1, ((2^{n-1}-4)^2)^{(2^{n-1}-3)}\},$$

$$LE_{CN}(\Gamma_{QD_{2^n}}) = \tfrac{(2^n-8)(2^n-6)(2^n-4)(2^n+2)}{8(2^n-2)} \text{ and } LE^+_{CN}(\Gamma_{QD_{2^n}}) = \tfrac{2^{n-3}(2^n-8)(2^n-6)(2^n-4)}{2^n-2}.$$

Proof. By [9, Propositon 2.1], we have $\Gamma_{QD_{2^n}} = K_{2^{n-1}-2} \sqcup 2^{n-2}K_2$. Therefore, by Theorem 2.1, we get

$$\text{CNL-spec}(\Gamma_{QD_{2^n}}) = \{0^{2^{n-2}+1}, ((2^{n-1}-2)(2^{n-1}-4))^{(2^{n-1}-3)}, (2(2-2))^{2^{n-2}(2-1)}\}$$

and

$$\text{CNSL-spec}(\Gamma_{QD_{2^n}}) = \{(2(2^{n-1}-3)(2^{n-1}-4))^1, ((2^{n-1}-4)^2)^{(2^{n-1}-3)}, (2(2-1)(2-2))^{(2^{n-2})}, ((2-2)^2)^{(2^{n-2})(2-1)}\}.$$

Thus, we get the required CNL-spec($\Gamma_{QD_{2^n}}$) and CNSL-spec($\Gamma_{QD_{2^n}}$) on simplification.

Here $|V(\Gamma_{QD_{2^n}})| = 2^n - 2$ and $\text{tr}(\text{CNRS}(\Gamma_{QD_{2^n}})) = \frac{1}{8}(2^n-8)(2^n-6)(2^n-4)$. Therefore, $\Delta_{\Gamma_{QD_{2^n}}} = \frac{(2^n-8)(2^n-6)(2^n-4)}{8(2^n-2)}$. In order to compute CNL-energy of $\Gamma_{QD_{2^n}}$, we first determine the quantities $|\alpha - \Delta_{\Gamma_{QD_{2^n}}}|$, where $\alpha \in$ CNL-spec($\Gamma_{QD_{2^n}}$), so that (1.1) can be used. We have

$$L_1 := |0 - \Delta_{\Gamma_{QD_{2^n}}}| = \frac{(2^n-8)(2^n-6)(2^n-4)}{8(2^n-2)}$$

and $L_2 := |\left(2^{n-1}-2\right)\left(2^{n-1}-4\right) - \Delta_{\Gamma_{QD_{2^n}}}| = 2^n(2^{n-3}-1) - 1 + \frac{6}{2^n-2}$. Hence, by (1.1), we get

$$LE_{CN}(\Gamma_{QD_{2^n}}) = (2^{n-1}+1)L_1 + (2^{n-1}-3)L_2 = \frac{(2^n-8)(2^n-6)(2^n-4)(2^n+2)}{8(2^n-2)}.$$

In order to compute CNSL-energy of $\Gamma_{QD_{2^n}}$, we first determine the quantities $|\beta - \Delta_{\Gamma_{QD_{2^n}}}|$, where $\beta \in$ CNSL-spec($\Gamma_{QD_{2^n}}$), so that (1.2) can be used. While computing CNL-energy, we have already seen that

$$|0 - \Delta_{\Gamma_{QD_{2^n}}}| = \frac{(2^n-8)(2^n-6)(2^n-4)}{8(2^n-2)}.$$

For our convenience, we denote this quantity by B_1. We have

$$B_2 := \left|2\left(2^{n-1}-3\right)\left(2^{n-1}-4\right) - \Delta_{\Gamma_{QD_{2^n}}}\right| = 15 + 2^n(3 \times 2^{n-3} - 5) + \frac{6}{2^n-2}$$

and $B_3 := \left|\left(2^{n-1}-4\right)^2 - \Delta_{\Gamma_{QD_{2^n}}}\right| = 7 + 2^{n+1}(2^{n-4}-1) + \frac{6}{2^n-2}$. Hence, by (1.2), we get

$$LE^+_{CN}(\Gamma_{QD_{2^n}}) = 2^{n-1}B_1 + 1 \times B_2 + (2^{n-1}-3)B_3 = \frac{2^{n-3}(2^n-8)(2^n-6)(2^n-4)}{2^n-2}.$$

This completes the proof. □

Proposition 2.3. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of projective special linear groups $PSL(2, 2^k)$, where $k \geq 2$, are given by*

$$\text{CNL-spec}(\Gamma_{PSL(2,2^k)}) = \Big\{0^{2^k+2^{2k}+1}, ((2^k-1)(2^k-3))^{(2^k+1)(2^k-2)},$$

$$((2^k-2)(2^k-4))^{2^{k-1}(2^k+1)(2^k-3)}, (2^k(2^k-2))^{2^{k-1}(2^k-1)^2}\Big\},$$

CNSL-spec$(\Gamma_{PSL(2,2^k)})$

$$= \Big\{(2(2^k-2)(2^k-3))^{2^k+1}, ((2^k-3)^2)^{(2^k+1)(2^k-2)}, (2(2^k-3)(2^k-4))^{2^{k-1}(2^k+1)},$$
$$((2^k-4)^2)^{2^{k-1}(2^k+1)(2^k-3)}, (2(2^k-1)(2^k-2))^{2^{k-1}(2^k-1)}, \; ((2^k-2)^2)^{2^{k-1}(2^k-1)^2}\Big\},$$

$LE_{CN}(\Gamma_{PSL(2,2^k)}) = \frac{(2^k-2)(19\times 2^k-2^{3k+2}+3\times 4^{k+1}-7\times 16^k-5\times 32^k+3\times 64^k+6)}{8^k-2^k-1}$ *and*

$$LE^+_{CN}(\Gamma_{PSL(2,2^k)}) = \begin{cases} \frac{9260}{59}, & \textit{for } k=2 \\ \frac{-5\times 2^{k+2}+7\times 2^{3k+1}+2^{4k+5}-9\times 4^k+32^k-13\times 64^k+3\times 128^k-12}{8^k-2^k-1}, & \textit{for } k\geq 3. \end{cases}$$

Proof. By [9, Proposition 2.2], we have

$$\Gamma_{PSL(2,2^k)} = (2^k+1)K_{2^k-1} \sqcup 2^{k-1}(2^k+1)K_{2^k-2} \sqcup 2^{k-1}(2^k-1)K_{2^k}.$$

Therefore, by Theorem 2.1, we get
CNL-spec$(\Gamma_{PSL(2,2^k)}) = \Big\{0^{2^k+1}, ((2^k-1)(2^k-1-2))^{(2^k+1)(2^k-1-1)}, 0^{2^{k-1}(2^k+1)},$
$((2^k-2)(2^k-2-2))^{2^{k-1}(2^k+1)(2^k-2-1)}, 0^{2^{k-1}(2^k-1)}, (2^k(2^k-2))^{2^{k-1}(2^k-1)(2^k-1)}\Big\}$ and
CNSL-spec$(\Gamma_{PSL(2,2^k)}) = \Big\{(2(2^k-1-1)(2^k-1-2))^{2^k+1}, ((2^k-1-2)^2)^{(2^k+1)(2^k-1-1)},$
$(2(2^k-2-1)(2^k-2-2))^{2^{k-1}(2^k+1)}, ((2^k-2-2)^2)^{2^{k-1}(2^k+1)(2^k-2-1)},$
$(2(2^k-1)(2^k-2))^{2^{k-1}(2^k-1)}, ((2^k-2)^2)^{2^{k-1}(2^k-1)(2^k-1)}\Big\}$. After simplification, we get the required CNL-spectrum and CNSL-spectrum.

Here $|V(\Gamma_{PSL(2,2^k)})| = 8^k-2^k-1$ and tr(CNRS$(\Gamma_{PSL(2,2^k)})) = (2^k-2)\,(5\times 2^k-3\times 8^k +16^k+3)$. So, $\Delta_{\Gamma_{PSL(2,2^k)}} = \frac{(2^k-2)(5\times 2^k-3\times 8^k+16^k+3)}{8^k-2^k-1}$. We have

$$L_1 := \left|0-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{(2^k-2)\,(5\times 2^k-3\times 8^k+16^k+3)}{8^k-2^k-1},$$

$$L_2 := \left|(2^k-1)\,(2^k-3)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{3+8\times 2^k+4^k(4^k-4\times 2^k-2)}{8^k-2^k-1},$$

$$L_3 := \left|(2^k-2)\,(2^k-4)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{2-5\times 2^k+2^k(2^{2k}(2^k-1))}{8^k-2^k-1}$$

and $L_4 := \left|2^k(2^k-2)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{6+9\times 2^k+4^k(2\times 4^k-4)+8^k(2\times 2^k-7)}{8^k-2^k-1}$. Therefore, by (1.1), we get

$$LE_{CN}(\Gamma_{PSL(2,2^k)}) = (2^k+2^{2k}+1)L_1+(2^k+1)(2^k-2)L_2+2^{k-1}(2^k+1)(2^k-3)L_3 + 2^{k-1}(2^k-1)^2L_4.$$

Hence, the expression for $LE_{CN}(\Gamma_{PSL(2,2^k)})$ is obtained.

Again

$$B_1 := \left|2\,(2^k-2)\,(2^k-3)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{5\times 2^k+2^{3k+2}+3\times 4^k-5\times 16^k+32^k-6}{8^k-2^k-1},$$

$$B_2 := \left|(2^k-3)^2-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = -\frac{-3+4\times 2^k-2^{3k}+2\times 2^{3k}-(2^k-1)2^{3k}}{8^k-2^k-1},$$

$$B_3 := \left|2\,(2^k-3)\,(2^k-4)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \begin{cases} \frac{174}{59}, & \text{for } k=2 \\ \frac{-3\times 2^k+2^{3k+4}+7\times 4^k-9\times 16^k+32^k-18}{8^k-2^k-1}, & \text{for } k\geq 3, \end{cases}$$

$$B_4 := \left|(2^k-4)^2-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = -\frac{-2^k+2^{2k+1}+9\times 8^k-3\times 16^k-10}{8^k-2^k-1},$$

$$B_5 := \left|2\left(2^k-1\right)\left(2^k-2\right)-\Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{2+9\times 2^k+2^{2k}((2^k-1)2^k\times 2^k-4\times 2^k-1)}{8^k-2^k-1}$$

and $B_6 := \left|(2^k-2)^2 - \Delta_{\Gamma_{PSL(2,2^k)}}\right| = \frac{2+7\times 2^k+((2^k-1)-3)8^k+2^{3k}-2^{2k+1}}{8^k-2^k-1}$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_{PSL(2,2^k)}) = (2^k+1)B_1 + (2^k+1)(2^k-2)B_2 + 2^{k-1}(2^k+1)B_3 + 2^{k-1}(2^k+1)(2^k-3)B_4 + 2^{k-1}(2^k-1)B_5 + 2^{k-1}(2^k-1)^2B_6.$$

Hence, the expression for $LE^+_{CN}(\Gamma_{PSL(2,2^k)})$ is obtained. □

Proposition 2.4. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of general linear groups $GL(2,q)$, where $q = p^n > 2$ and p is a prime, are given by*

$$\text{CNL-spec}(\Gamma_{GL(2,q)}) = \Big\{0^{q^2+q+1}, ((q^2-3q+2)(q^2-3q))^{\frac{1}{2}(q^2+q)(q^2-3q+1)}, ((q^2-q)(q^2-q-2))^{\frac{1}{2}(q^2-q)(q^2-q-1)}, ((q^2-2q+1)(q^2-2q-1))^{(q+1)(q^2-2q)}\Big\},$$

$$\text{CNSL-spec}(\Gamma_{GL(2,q)}) = \Big\{(2(q^2-3q+1)(q^2-3q))^{\frac{1}{2}(q^2+q)}, ((q^2-3q)^2)^{\frac{1}{2}(q^2+q)(q^2-3q+1)}, (2(q^2-q-1)(q^2-q-2))^{\frac{1}{2}(q^2-q)}, ((q^2-q-2)^2)^{\frac{1}{2}(q^2-q)(q^2-q-1)}, (2(q^2-2q)(q^2-2q-1))^{q+1}, ((q^2-2q-1)^2)^{(q+1)(q^2-2q)}\Big\},$$

$$LE_{CN}(\Gamma_{GL(2,q)}) = \frac{(q-2)(q-1)q^4(q+1)(2q-3)((q-1)q-1)}{q^3-q-1} \text{ and}$$

$$LE^+_{CN}(\Gamma_{GL(2,q)}) = \begin{cases} \frac{(q-2)q(q+1)(q(q(q(q(2q((q-4)q+6)-7)-7)-1)+11)+2)}{q^3-q-1}, & \text{if } q \le 5 \\ \frac{q(q+1)\left((q(q(q(q(2q-11)+20)-16)+7)+8)q^3-12q-4\right)}{q^3-q-1}, & \text{if } q \ge 6. \end{cases}$$

Proof. By [9, Proposition 2.3], we have

$$\Gamma_{GL(2,q)} = \frac{q(q+1)}{2}K_{q^2-3q+2} \sqcup \frac{q(q-1)}{2}K_{q^2-q} \sqcup (q+1)K_{q^2-2q+1}.$$

Therefore, by Theorem 2.1, we get

$$\text{CNL-spec}(\Gamma_{GL(2,q)}) = \Big\{0^{\frac{q(q+1)}{2}}, ((q^2-3q+2)(q^2-3q+2-2))^{(\frac{q(q+1)}{2})(q^2-3q+2-1)}, 0^{\frac{q(q-1)}{2}}, ((q^2-q)(q^2-q-2))^{(\frac{q(q-1)}{2})(q^2-q-1)}, 0^{q+1}, ((q^2-2q+1)(q^2-2q+1-2))^{(q+1)(q^2-2q+1-1)}\Big\} \text{ and}$$

$$\text{CNSL-spec}(\Gamma_{GL(2,q)}) = \Big\{(2(q^2-3q+2-1)(q^2-3q+2-2))^{\frac{q(q+1)}{2}}, ((q^2-3q+2-2)^2)^{(\frac{q(q+1)}{2})(q^2-3q+2-1)}, (2(q^2-q-1)(q^2-q-2))^{\frac{q(q-1)}{2}}, ((q^2-q-2)^2)^{(\frac{q(q-1)}{2})(q^2-q-1)}, (2(q^2-2q+1-1)(q^2-2q+1-2))^{q+1}, ((q^2-2q+1-2)^2)^{(q+1)(q^2-2q+1-1)}\Big\}.$$

Thus we get CNL-spec$(\Gamma_{GL(2,q)})$ and CNSL-spec$(\Gamma_{GL(2,q)})$ on simplification.

Here $|V(\Gamma_{GL(2,q)})| = (q-1)(q^3-q-1)$ and $\text{tr}(\text{CNRS}(\Gamma_{GL(2,q)})) = (q-2)(q-1)q(q+1)\left((q-2)(q-1)q^2+1\right)$. So, $\Delta_{\Gamma_{GL(2,q)}} = \frac{(q-2)q(q+1)\left((q-2)(q-1)q^2+1\right)}{q^3-q-1}$. We have

$$L_1 := \left|0 - \Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)q(q+1)((q-2)(q-1)q^2+1)}{q^3-q-1},$$

$$L_2 := \left|(q^2-3q+2)(q^2-3q) - \Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)q\left((q(2q-3)-1)q^2+4\right)}{q^3-q-1},$$

$$L_3 := \left|(q^2-q)(q^2-q-2) - \Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)q^3(q+1)(2q-3)}{q^3-q-1},$$

and $L_4 := \left|(q^2-2q+1)(q^2-2q-1) - \Delta_{\Gamma_{GL(2,q)}}\right| = \frac{q(-3+q(3+(-2+q)q))-1}{q^3-q-1}$. Therefore, by (1.1), we get

$$LE_{CN}(\Gamma_{GL(2,q)}) = (q^2+q+1)L_1 + \frac{q(q+1)}{2}(q^2-3q+1)L_2 + \frac{q(q-1)}{2}(q^2-q-1)L_3 + (q+1)(q^2-2q)L_4.$$

Hence, the expression for $LE_{CN}(\Gamma_{GL(2,q)})$ is obtained.

Again

$$B_1 := \left|2\left(q^2-3q+1\right)\left(q^2-3q\right)-\Delta_{\Gamma_{GL(2,q)}}\right| = \begin{cases} -\frac{q\left(q\left((q-5)(q-3)q^3-5q-13\right)+8\right)}{q^3-q-1}, & \text{for } q \le 5 \\ \frac{q\left(q\left((q-5)(q-3)q^3-5q-13\right)+8\right)}{q^3-q-1}, & \text{for } q \ge 6, \end{cases}$$

$$B_2 := \left|\left(q^2-3q\right)^2-\Delta_{\Gamma_{GL(2,q)}}\right| = -\frac{q\left(-2q^5+5q^4+q^3-8q+2\right)}{q^3-q-1},$$

$$B_3 := \left|2\left(q^2-q-1\right)\left(q^2-q-2\right)-\Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)(q+1)\left(q^5+q^4-6q^3+3q+2\right)}{q^3-q-1},$$

$$B_4 := \left|\left(q^2-q-2\right)^2-\Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)(q+1)\left(2q^4-5q^3+2q+2\right)}{q^3-q-1}$$

$$B_5 := \left|2\left(q^2-2q\right)\left(q^2-2q-1\right)-\Delta_{\Gamma_{GL(2,q)}}\right| = \frac{(q-2)q\left((q-3)(q+1)q^3+5q+1\right)}{q^3-q-1}$$

and $B_6 := \left|\left(q^2-2q-1\right)^2-\Delta_{\Gamma_{GL(2,q)}}\right| = -\frac{q(q(q(q(3-2q)+6)-5)-3)-1}{q^3-q-1}$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_{GL(2,q)}) = \frac{1}{2}(q^2+1)B_1 + \frac{1}{2}(q^2+1)\left(q^2-3q+1\right)B_2 + \frac{1}{2}(q^2-q)B_3 + \frac{1}{2}(q^2-q)(q^2-q-1)B_4 + (q+1)B_5 + (q+1)(q^2-2q)B_6.$$

Hence, the expression for $LE^+_{CN}(\Gamma_{GL(2,q)})$ is obtained. □

Proposition 2.5. *Let $F = GF(2^n)$ (where $n \geq 2$) and ν be the Frobenius automorphism of F, that is $\nu(x) = x^2$, for all $x \in F$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the groups*

$$A(n,\nu) = \left\{U(a,b) = \begin{pmatrix}1 & 0 & 0\\ a & 1 & 0\\ b & \nu(a) & 1\end{pmatrix} : a,b \in F\right\}$$

under matrix multiplication are given by

$$\text{CNL-spec}(\Gamma_{A(n,\nu)}) = \left\{0^{(2^n-1)}, (2^n(2^n-2))^{(2^n-1)^2}\right\},$$

$$\text{CNSL-spec}(\Gamma_{A(n,\nu)}) = \left\{(2(2^n-1)(2^n-2))^{(2^n-1)}, ((2^n-2)^2)^{(2^n-1)^2}\right\}$$

and $LE_{CN}(\Gamma_{A(n,\nu)}) = LE^+_{CN}(\Gamma_{A(n,\nu)}) = 2\left(2^n-2\right)\left(2^n-1\right)^2$.

Proof. By [9, Proposition 2.4], we have $\Gamma_{A(n,\nu)} = (2^n-1)K_{2^n}$. Therefore, by Theorem 2.1, we get CNL-spec$(\Gamma_{A(n,\nu)})$ and CNSL-spec$(\Gamma_{A(n,\nu)})$.

Here $|V(\Gamma_{A(n,\nu)})| = 4^n - 2^n$ and $\text{tr}(\text{CNRS}(\Gamma_{A(n,\nu)})) = 2^n\left(2^n-2\right)\left(2^n-1\right)^2$. So, $\Delta_{\Gamma_{A(n,\nu)}} = 4^n - 3\times 2^n + 2$. We have

$L_1 := \left|0-\Delta_{\Gamma_{A(n,\nu)}}\right| = 4^n - 3\times 2^n + 2$ and $L_2 := \left|2^n(2^n-2)-\Delta_{\Gamma_{A(n,\nu)}}\right| = 2^n-2$. Hence, by (1.1), we get

$$LE_{CN}(\Gamma_{A(n,\nu)}) = (2^n-1)L_1 + (2^n-1)^2L_2 = 2(2^n-2)(2^n-1)^2.$$

Again,

$B_1 := \left|2(2^n-1)(2^n-2)-\Delta_{\Gamma_{A(n,\nu)}}\right| = 4^n - 3\times 2^n + 2$ and $B_2 := \left|(2^n-2)^2-\Delta_{\Gamma_{A(n,\nu)}}\right| = 2^n-2$. Hence, by (1.2), we get

$LE^+_{CN}(\Gamma_{A(n,\nu)}) = (2^n-1)B_1 + (2^n-1)^2B_2 = 2\left(2^n-2\right)\left(2^n-1\right)^2$. □

Proposition 2.6. *Let $F = GF(p^n)$, where p is a prime. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of commuting graphs of the groups*

$$A(n,p) = \left\{ V(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \ : \ a,b,c \in F \right\}$$

under matrix multiplication $V(a,b,c)V(a',b',c') = V(a+a', b+b'+ca', c+c')$ are given by

$$\text{CNL-spec}(\Gamma_{A(n,p)}) = \left\{0^{p^n+1}, ((p^{2n}-p^n)(p^{2n}-p^n-2))^{(p^n+1)(p^{2n}-p^n-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{A(n,p)}) = \left\{(2(p^{2n}-p^n-1)(p^{2n}-p^n-2))^{(p^n+1)}, ((p^{2n}-p^n-2)^2)^{(p^n+1)(p^{2n}-p^n-1)}\right\}$$

and $LE_{CN}(\Gamma_{A(n,p)}) = LE^+_{CN}(\Gamma_{A(n,p)}) = 2\left(p^n+1\right)^2\left(-3p^{2n}+p^{3n}+p^n+2\right)$.

Proof. By [9, Proposition 2.5], $\Gamma_{A(n,p)} = (p^n+1)K_{p^{2n}-p^n}$. Therefore, by Theorem 2.1, we get CNL-spec$(\Gamma_{A(n,p)})$ and CNSL-spec$(\Gamma_{A(n,p)})$.

Here $|V(\Gamma_{A(n,p)})| = p^n\left(p^{2n}-1\right)$ and $\text{tr}(\text{CNRS}(\Gamma_{A(n,p)})) = p^n\left(p^n-2\right)\left(p^n-1\right) \times \left(p^n+1\right)^2 \left(p^n\left(p^n-1\right)-1\right)$. So, $\Delta_{\Gamma_{A(n,p)}} = \left(p^n-2\right)\left(p^n+1\right)\left(p^n\left(p^n-1\right)-1\right)$. We have

$$L_1 := \left|0 - \Delta_{\Gamma_{A(n,p)}}\right| = \left(p^n-2\right)\left(p^n+1\right)\left(p^n\left(p^n-1\right)-1\right),$$

and $L_2 := \left|(p^{2n}-p^n)(p^{2n}-p^n-2) - \Delta_{\Gamma_{A(n,p)}}\right| = (p^n-2)(p^n+1)$. Hence, by (1.1), we get

$$\begin{aligned} LE_{CN}(\Gamma_{A(n,p)}) &= (p^n+1)L_1 + (p^{2n}-p^n-1)(p^n+1)L_2 \\ &= 2(p^n+1)^2(-3p^{2n}+p^{3n}+p^n+2). \end{aligned}$$

Again

$$B_1 := \left|2(p^{2n}-p^n-1)(p^{2n}-p^n-2) - \Delta_{\Gamma_{A(n,p)}}\right| = \left(p^n-2\right)\left(p^n+1\right)\left(p^n\left(p^n-1\right)-1\right)$$

and $B_2 := \left|(p^{2n}-p^n-2)^2 - \Delta_{\Gamma_{A(n,p)}}\right| = p^{2n}-p^n-2$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_{A(n,p)}) = (p^n+1)B_1 + \left(p^{2n}-p^n-1\right)\left(p^n+1\right)B_2.$$

Hence, the expression for $LE^+_{CN}(\Gamma_{A(n,p)})$ is obtained. □

2.2 Groups whose central quotient is isomorphic to $Sz(2)$, D_{2m} or $\mathbb{Z}_p \times \mathbb{Z}_p$

Let us begin with the groups G such that $G/Z(G)$ is isomorphic to the Suzuki group $Sz(2) = \langle a, b \ : \ a^5 = b^4 = 1, \ b^{-1}ab = a^2 \rangle$.

Theorem 2.7. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong Sz(2)$ and $|Z(G)| = z$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of Γ_G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^6, (4z(4z-2))^{(4z-1)}, (3z(3z-2))^{5(3z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2(4z-1)(4z-2))^1, ((4z-2)^2)^{(4z-1)}, (2(3z-1)(3z-2))^5, ((3z-2)^2)^{5(3z-1)}\right\},$$

$$LE_{CN}(\Gamma_G) = \begin{cases} \frac{648}{19}, & \text{for } z = 1 \\ \frac{2}{19}(4z-1)(z(105z+31)-38), & \text{for } z \geq 2 \end{cases}$$

and $LE^+_{CN}(\Gamma_G) = \frac{10}{19}(3z-1)(z(28z+45)-38)$.

Proof. By [9, Theorem 2.2], we have $\Gamma_G = K_{4z} \sqcup 5K_{3z}$. Therefore, by Theorem 2.1, we get

$$\text{CNL-spec}(\Gamma_G) = \left\{0^1, (4z(4z-2))^{(4z-1)}, 0^5, (3z(3z-2))^{5(3z-1)}\right\}$$

and $\text{CNSL-spec}(\Gamma_G) = \{(2(4z-1)(4z-2))^1, ((4z-2)^2)^{(4z-1)}, (2(3z-1)(3z-2))^5, ((3z-2)^2)^{5(3z-1)}\}$. Hence, we get the required CNL-spec(Γ_G) and CNSL-spec(Γ_G) on simplification.

Here $|V(\Gamma_G)| = 19z$ and $\text{tr}(\text{CNRS}(\Gamma_G)) = 199z^3 - 183z^2 + 38z$. So, $\Delta_{\Gamma_G} = \frac{199z^2-183z+38}{19}$. We have

$$L_1 := |0 - \Delta_{\Gamma_G}| = \left|-\frac{199z^2 - 183z + 38}{19}\right| = \frac{199z^2 - 183z + 38}{19},$$

$$L_2 := |4z(4z-2) - \Delta_{\Gamma_G}| = \frac{105z^2 + 31z - 38}{19}$$

and $L_3 := |3z(3z-2) - \Delta_{\Gamma_G}| = \left|\frac{z(69-28z)-38}{19}\right| = \begin{cases} \frac{3}{19}, & \text{for } z = 1 \\ -\frac{z(69-28z)-38}{19}, & \text{for } z \geq 2. \end{cases}$
Therefore, by (1.1), we get $LE_{CN}(\Gamma_G) = 6L_1 + (4z-1)L_2 + 5(3z-1)L_3$. Hence, the expression for $LE_{CN}(\Gamma_G)$ is obtained.

Again

$$B_1 := |2(4z-1)(4z-2) - \Delta_{\Gamma_G}| = \frac{409z^2 - 273z + 38}{19},$$

$$B_2 := \left|(4z-2)^2 - \Delta_{\Gamma_G}\right| = \frac{105z^2 - 121z + 38}{19},$$

$$B_3 := |2(3z-1)(3z-2) - \Delta_{\Gamma_G}| = \frac{143z^2 - 159z + 38}{19}$$

and $B_4 := \left|(3z-2)^2 - \Delta_{\Gamma_G}\right| = \frac{28z^2+45z-38}{19}$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_G) = 1 \times B_1 + (4z-1)B_2 + 5 \times B_3 + 5(3z-1)B_4.$$

Hence, the expression for $LE^+_{CN}(\Gamma_G)$ is obtained. □

Theorem 2.8. *Let G be a finite non-abelian group such that $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime and $|Z(G)| \geq 2$. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^{(p+1)}, (((p-1)z)((p-1)z-2))^{(p+1)((p-1)z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2((p-1)z-1)((p-1)z-2))^{(p+1)}, (((p-1)z-2)^2)^{(p+1)((p-1)z-1)}\right\} \text{ and}$$

$$LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = 2(p+1)((p-1)z-2)((p-1)z-1),$$

where $z = |Z(G)|$.

Proof. From [10, Theorem 2.1], we have $\Gamma_G = (p+1)K_{(p-1)z}$. Therefore, by Theorem 2.1, we get CNL-spec(Γ_G) and CNSL-spec(Γ_G).

Here $|V(\Gamma_G)| = (p^2-1)\, z$ and $\text{tr}(\text{CNRS}(\Gamma_G)) = (p-1)(p+1)z((p-1)z-2)((p-1)z-1)$. So, $\Delta_{\Gamma_G} = ((p-1)z-2)((p-1)z-1)$. We have

$$L_1 := |0 - \Delta_{\Gamma_G}| = ((p-1)z-2)((p-1)z-1)$$

and $L_2 := |((p-1)z)((p-1)z-2) - \Delta_{\Gamma_G}| = (p-1)z - 2$. Therefore, by (1.1), we get

$$LE_{CN}(\Gamma_G) = (p+1)L_1 + (p+1)((p-1)z-1)L_2.$$

Hence, the expression for $LE_{CN}(\Gamma_G)$ is obtained.

Again

$$B_1 := |2((p-1)z-1)((p-1)z-2) - \Delta_{\Gamma_G}| = ((p-1)z-2)((p-1)z-1)$$

and $B_2 := \left|((p-1)z-2)^2 - \Delta_{\Gamma_G}\right| = pz - z - 2$. Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_G) = (p+1)B_1 + (p+1)((p-1)z-1)B_2.$$

Hence, the expression for $LE^+_{CN}(\Gamma_G)$ is obtained. □

Corollary 2.9. *Let G be a non-abelian group of order p^3, where p is a prime. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^{(p+1)}, (p(p-1)(p(p-1)-2))^{(p+1)((p-1)p-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2((p-1)p-1)((p-1)p-2))^{(p+1)}, (((p-1)p-2)^2)^{(p+1)((p-1)p-1)}\right\} \textit{ and}$$

$$LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = 2(p+1)((p-1)p-2)((p-1)p-1).$$

Proof. We know that for a non-abelian group G of order p^3, $|Z(G)| = p$ and $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from the Theorem 2.8. □

Corollary 2.10. *Let G be a finite 4-centralizer group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^3, ((z(z-2))^{3((p-1)z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2(z-1)(z-2))^3, ((z-2)^2)^{3(z-1)}\right\} \textit{ and}$$

$$LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = \begin{cases} 0, & \textit{if } p = 2, z = 1 \\ 6(z-2)(z-1), & \textit{otherwise;} \end{cases}$$

where $z = |Z(G)|$.

Proof. Here $\frac{G}{Z(G)} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence the results follows from the Theorem 2.8. □

Corollary 2.11. *Let G be a finite 5-centralizer group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^4, (2z(2z-2))^{4(2z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2(2z-1)(2z-2))^4, ((2z-2)^2)^{4(z-1)}\right\}$$

and $LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = 8(2z-2)(2z-1)$, *where* $z = |Z(G)|$.

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Hence the result follows from Theorem 2.8. □

Corollary 2.12. *Let G be a finite $(p+2)$-centralizer p-group. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^{(p+1)}, (((p-1))((p-1)|Z(G)|-2))^{(p+1)((p-1)|Z(G)|-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2((p-1)|Z(G)|-1)((p-1)|Z(G)|-2))^{(p+1)}, \right.$$
$$\left. (((p-1)|Z(G)|-2)^2)^{(p+1)((p-1)|Z(G)|-1)}\right\} \textit{ and}$$

$$LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G) = 2(p+1)((p-1)|Z(G)|-2)((p-1)|Z(G)|-1).$$

Proof. We have $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence the result follows from Theorem 2.8. □

Theorem 2.13. *If G is a finite group such that $\frac{G}{Z(G)} \cong D_{2m}$, $m \geq 3$, then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graph of G are given by*

$$\text{CNL-spec}(\Gamma_G) = \left\{0^{m+1}, ((m-1)z((m-1)z-2))^{((m-1)z-1)}, (z(z-2))^{m(z-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_G) = \left\{(2((m-1)z-1)((m-1)z-2))^1, (((m-1)z-2)^2)^{((m-1)z-1)}, \right.$$
$$\left. (2(z-1)(z-2))^m, ((z-2)^2)^{m(z-1)}\right\},$$

$$LE_{CN}(\Gamma_G) = \frac{2((m-1)z-1)(m(z((m-2)mz-m+3)-4)+z+2)}{2m-1}$$

$$\textit{and } LE^+_{CN}(\Gamma_G) = \begin{cases} \frac{-2m^4+6m^3+3m^2-13m+7}{1-2m}, & \textit{for } m = 3, 4 \,\&\, z = 1 \\ \frac{2(m-2)(m-1)mz^2(mz-3)}{2m-1}, & \textit{otherwise;} \end{cases}$$

where $z = |Z(G)|$.

Proof. From [10, Theorem 2.5], we have $\Gamma(G) = K_{(m-1)z} \sqcup mK_z$. Therefore, by Theorem 2.1, we get CNL-spec(Γ_G) and CNSL-spec(Γ_G).

Here $|V(\Gamma_G)| = (2m-1)z$ and tr(CNRS$(\Gamma_G)) = m(z-2)(z-1)z + (m-1)z((m-1)z - 2)((m-1)z-1)$. So, $\Delta_{\Gamma_G} = \frac{(m((m-3)m+4)-1)z^2-3((m-1)m+1)z+4m-2}{2m-1}$. We have

$$L_1 := |0 - \Delta_{\Gamma_G}| = -\frac{(2-4m)+3z(1-m)+z^2(1-4m)+m^2z^2(3-\frac{m}{2})+m^2z(3-\frac{mz}{2})}{2m-1},$$

$$L_2 := |((m-1)z)((m-1)z-2) - \Delta_{\Gamma_G}| = \frac{m(z((m-2)mz-m+3)-4)+z+2}{2m-1}$$

$$\text{and } L_3 := |z(z-2) - \Delta_{\Gamma_G}| = -\frac{m(-z((m-2)(m-1)z-3m+7)-4)+5z+2}{2m-1}.$$

Therefore, by (1.1), we get

$$LE_{CN}(\Gamma_G) = (m+1)L_1 + ((m-1)z-1)L_2 + m(z-1)L_3.$$

Hence, the expression for $LE_{CN}(\Gamma_G)$ is obtained.

Again

$$\begin{aligned} B_1 &:= |2((m-1)z-1)((m-1)z-2) - \Delta_{\Gamma_G}| \\ &= \frac{((m-1)m(3m-4)-1)z^2 - 3(m(3m-5)+1)z + 4m - 2}{2m-1}, \end{aligned}$$

$$B_2 := \left|((m-1)z-2)^2 - \Delta_{\Gamma_G}\right| = \frac{m(z((m-2)mz-5m+9)+4)-z-2}{2m-1},$$

$$B_3 := |2(z-1)(z-2) - \Delta_{\Gamma_G}| = -\frac{-\left(\left((m-3)m^2+1\right)z^2\right)+3((m-5)m+3)z+4m-2}{2m-1}$$

$$\text{and } B_4 := \left|(z-2)^2 - \Delta_{\Gamma_G}\right| = \begin{cases} \frac{m(4-z((m-2)(m-1)z-3m+11))+7z-2}{2m-1}, & \text{for } m = 3,4, z = 1; \\ -\frac{m(4-z((m-2)(m-1)z-3m+11))+7z-2}{2m-1}, & \text{otherwise.} \end{cases}$$

Therefore, by (1.2), we get

$$LE^+_{CN}(\Gamma_G) = 1 \times B_1 + ((m-1)z-1)\,B_2 + m\,B_3 + m(z-1)\,B_4.$$

Hence, the expression for $LE^+_{CN}(\Gamma_G)$ is obtained. □

Corollary 2.14. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the metacyclic groups* $\mathcal{M}_{2mn} = \langle x, y : x^m = y^{2n} = 1,\ yxy^{-1} = x^{-1}\rangle$, *where* $m \geq 3$, $n \geq 2$, *are given by*

CNL-spec$(\Gamma_{\mathcal{M}_{2mn}}) =$

$$\begin{cases} \left\{0^{m+1}, ((m-1)n((m-1)n-2))^{((m-1)n-1)}, (n(n-2))^{m(n-1)}\right\}, & \textit{if } 2 \nmid m \\ \left\{0^{\frac{m}{2}+1}, ((\frac{m}{2}-1)2n((\frac{m}{2}-1)2n-2))^{((\frac{m}{2}-1)2n-1)}, (2n(2n-2))^{\frac{m}{2}(2n-1)}\right\}, & \textit{if } 2 \mid m, \end{cases}$$

CNSL-spec$(\Gamma_{\mathcal{M}_{2mn}}) =$

$$\begin{cases} \{(2((m-1)n-1)((m-1)n-2))^1, (((m-1)n-2)^2)^{((m-1)n-1)}, \\ \qquad (2(n-1)(n-2))^m, ((n-2)^2)^{m(n-1)}\}, & \textit{if } 2 \nmid m \\ \{(2((\frac{m}{2}-1)2n-1)((\frac{m}{2}-1)2n-2))^1, (((\frac{m}{2}-1)2n-2)^2)^{((\frac{m}{2}-1)2n-1)}, \\ \qquad (2(2n-1)(2n-2))^{\frac{m}{2}}, ((2n-2)^2)^{\frac{m}{2}(2n-1)}\}, & \textit{if } 2 \mid m, \end{cases}$$

$$LE_{CN}(\Gamma_{\mathcal{M}_{2mn}}) = \begin{cases} \frac{2((m-1)n-1)(m(n((m-2)mn-m+3)-4)+n+2)}{2m-1}, & \textit{if } 2 \nmid m \\ \frac{((m-2)n-1)(m(n((m-4)mn-m+6)-4)+4(n+1))}{m-1}, & \textit{if } 2 \mid m \end{cases}$$

$$\textit{and } LE^+_{CN}(\Gamma_{\mathcal{M}_{2mn}}) = \begin{cases} \frac{2(m-2)(m-1)mn^2(mn-3)}{2m-1}, & \textit{if } 2 \nmid m \\ \frac{(m-4)(m-2)mn^2(mn-3)}{m-1}, & \textit{if } 2 \mid m. \end{cases}$$

Proof. We have $Z(\mathcal{M}_{2mn}) = \begin{cases} \langle y^2\rangle, & \text{if } 2\nmid m\\ \langle y^2\rangle \cup x^{\frac{m}{2}}\langle y^2\rangle, & \text{if } 2\mid m\end{cases}$
and so $|Z(\mathcal{M}_{2mn})| = n$ and $2n$ according as $2\nmid m$ and $2\mid m$. Therefore,

$$\frac{\mathcal{M}_{2mn}}{Z(\mathcal{M}_{2mn})} \cong \begin{cases} D_{2m}, & \text{if } 2\nmid m\\ D_m, & \text{if } 2\mid m.\end{cases}$$

Hence, the result follows from Theorem 2.13. □

Corollary 2.15. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the dihedral groups, $D_{2m} = \langle x, y : x^{2m} = y^2 = 1, yxy^{-1} = x^{-1}\rangle$, $m \geq 3$, are given by*

$$\text{CNL-spec}(\Gamma_{D_{2m}}) = \begin{cases} \left\{0^{m+1}, ((m-1)(m-3))^{(m-2)}\right\}, & \text{if } 2\nmid m\\ \left\{0^{(m+1)}, ((m-2)(m-4))^{(m-3)}\right\}, & \text{if } 2\mid m,\end{cases}$$

$$\text{CNSL-spec}(\Gamma_{D_{2m}}) = \begin{cases} \left\{0^{m}, (2(m-2)(m-3))^1, ((m-3)^2)^{(m-2)}\right\}, & \text{if } 2\nmid m\\ \left\{0^{m}, (2(m-3)(m-4))^1, ((m-4)^2)^{(m-3)}\right\}, & \text{if } 2\mid m,\end{cases}$$

$$LE_{CN}(\Gamma_{D_{2m}}) = \begin{cases} \frac{2(m-3)(m-2)(m-1)(m+1)}{2m-1}, & \text{if } 2\nmid m\\ \frac{(m-4)(m-3)(m-2)(m+1)}{m-1}, & \text{if } 2\mid m\end{cases}$$

and $LE^+_{CN}(\Gamma_{D_{2m}}) = \begin{cases} \frac{2(m-3)(m-2)(m-1)m}{2m-1}, & \text{if } 2\nmid m\\ \frac{(m-4)(m-3)(m-2)m}{m-1}, & \text{if } 2\mid m.\end{cases}$

Proof. We know that $|Z(D_{2m})| = \begin{cases} 1, & \text{if } 2\nmid m\\ 2, & \text{if } 2\mid m\end{cases}$ and $\frac{D_{2m}}{Z(D_{2m})} = \begin{cases} D_{2m}, & \text{if } 2\nmid m\\ D_m, & \text{if } 2\mid m.\end{cases}$ Therefore, by using Theorem 2.13 we get the required result. □

Corollary 2.16. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the groups $U_{6n} = \langle x, y : x^{2n} = y^3 = 1, x^{-1}yx = y^{-1}\rangle$ are given by*

$$\text{CNL-spec}(\Gamma_{U_{6n}}) = \left\{0^4, (2n(2n-2))^{2n-1}, (n(n-2))^{3(n-1)}\right\},$$

$$\text{CNSL-spec}(\Gamma_{U_{6n}}) = \left\{((2n-1)(2n-2))^1, ((2n-2)^2)^{2n-1}, (2(n-1)(n-2))^3, ((n-2)^2)^{3(n-1)}\right\},$$

$$LE_{CN}(\Gamma_{U_{6n}}) = \tfrac{2}{5}(n-1)(2n-1)(9n+10)$$

and $LE^+_{CN}(\Gamma_{U_{6n}}) = \begin{cases} \frac{6}{5}(n-1)(n+10) = 0, & \text{for } n = 1\\ \frac{36}{5}(n-1)n^2, & \text{for } n \geq 2.\end{cases}$

Proof. We have $Z(U_{6n}) = \langle x^2\rangle$ and $\frac{U_{6n}}{Z(U_{6n})} \cong D_6$ with $|Z(U_{6n})| = n$. Therefore we get the required result by putting $m = 3$ and $z = n$ in Theorem 2.13. □

Corollary 2.17. *The CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of the commuting graphs of the dicyclic groups, $Q_{4n} = \langle x, y : y^{2n} = 1, x^2 = y^n, xyx^{-1} = y^{-1}\rangle$, $n \geq 2$, are given by*

$$\text{CNL-spec}(\Gamma_{Q_{4n}}) = \left\{0^{2n+1}, ((2n-2)(2n-4))^{2n-3}\right\},$$

$$\text{CNSL-spec}(\Gamma_{Q_{4n}}) = \left\{0^{2n}, (2(2n-3)(2n-4))^1, ((2n-4)^2)^{2n-3}\right\},$$

$$LE_{CN}(\Gamma_{Q_{4n}}) = \frac{4(n-2)(n-1)(2n-3)(2n+1)}{2n-1} \text{ and } LE^+_{CN}(\Gamma_{Q_{4n}}) = \frac{8(n-2)(n-1)n(2n-3)}{2n-1}.$$

Proof. We have $Z(Q_{4n}) = \{1, x^n\}$ and $\frac{Q_{4n}}{Z(Q_{4n})} \cong D_{2n}$. Therefore, by using Theorem 2.13, we get the required result. □

We conclude this section with the following two results for finite non-abelian AC-group in general.

Theorem 2.18. *Let G be a finite non-abelian AC-group and $X_1, X_2, \ldots, X_n$ be the distinct centralizers of non-central elements of G. Then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of Γ_G are given by*
$\text{CNL-spec}(\Gamma_G) = \{0^n, ((|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2))^{(|X_i|-|Z(G)|-1)}$, *where* $1 \leq i \leq n\}$,

CNSL-spec$(\Gamma_G)=\{(2(|X_i|-|Z(G)|-1)(|X_i|-|Z(G)|-2))^1, ((|X_i|-|Z(G)|-2)^2)^{(|X_i|-|Z(G)|-1)}$, *where* $1 \leq i \leq n\}$,
$LE_{CN}(\Gamma_G) = nL_0 + \sum_{i=1}^{n}(|X_i| - |Z(G)| - 1)L_{X_i}$, *where* $L_0 = |0 - \Delta_{\Gamma_G}|$ *and* $L_{X_i} = |(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2) - \Delta_{\Gamma_G}|$ *and*
$LE^{+}_{CN}(\Gamma_G) = \sum_{i=1}^{n} B_{X_i} + \sum_{i=1}^{n}(|X_i| - |Z(G)| - 1)B'_{X_i}$, *where* $B_{X_i} = |2(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)| - 2) - \Delta_{\Gamma_G}|$ *and* $B'_{X_i} = |(|X_i| - |Z(G)| - 2)^2 - \Delta_{\Gamma_G}|$.

Proof. By [9, Lemma 2.1], we have $\Gamma_G = \sqcup_{i=1}^{n} K_{|X_i|-|Z(G)|}$. Here $|V(\Gamma_G)| = \sum_{i=1}^{n} |X_i| - |Z(G)|$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma_G)) = \sum_{i=1}^{n}(|X_i| - |Z(G)| - 1)(|X_i| - |Z(G)|)(|X_i| - |Z(G)| - 2)$. Therefore, $\Delta_{\Gamma_G} = \frac{\sum_{i=1}^{n}(|X_i|-|Z(G)|-1)(|X_i|-|Z(G)|)(|X_i|-|Z(G)|-2)}{\sum_{i=1}^{n}|X_i|-|Z(G)|}$. Hence, the result follows from the Theorem 2.1. □

Theorem 2.19. *Let G be a finite non-abelian AC-group and $X_1, X_2, \ldots, X_n$ be the distinct centralizers of non-central elements of G. If A is a finite abelian group then the CNL-spectrum, CNSL-spectrum, CNL-energy and CNSL-energy of $\Gamma_{G\times A}$ are given by*
CNL-spec$(\Gamma_{G\times A}) = \{0^n, (|A|(|X_i| - |Z(G)|)(|A|(|X_i| - |Z(G)|) - 2))^{(|A|(|X_i|-|Z(G)|)-1)}$, *where* $1 \leq i \leq n\}$,
CNSL-spec$(\Gamma_{G\times A}) = \{(2(|A|(|X_i| - |Z(G)|) - 1)(|A|(|X_i| - |Z(G)|) - 2))^1, ((|A|(|X_i| - |Z(G)|) - 2)^2)^{(|A|(|X_i|-|Z(G)|)-1)}$, *where* $1 \leq i \leq n\}$,
$LE_{CN}(\Gamma_G) = nL_0 + \sum_{i=1}^{n}(|A|(|X_i| - |Z(G)|) - 1)L_{X_i}$, *where* $L_0 = |-\Delta_{\Gamma_{G\times A}}|$ *and* $L_{X_i} = ||A|(|X_i| - |Z(G)|)(|A|(|X_i| - |Z(G)|) - 2) - \Delta_{\Gamma_{G\times A}}|$ *and*
$LE^{+}_{CN}(\Gamma_G) = \sum_{i=1}^{n} S_{X_i} + \sum_{i=1}^{n}(|A|(|X_i| - |Z(G)|) - 1)S'_{X_i}$, *where* $S_{X_i} = |2(|A|(|X_i| - |Z(G)|) - 1)(|A|(|X_i| - |Z(G)|) - 2) - \Delta_{\Gamma_{G\times A}}|$ *and* $S'_{X_i} = |(|A|(|X_i| - |Z(G)|) - 2)^2 - \Delta_{\Gamma_{G\times A}}|$.

Proof. We have $Z(G \times A) = Z(G) \times A$ and $X_1 \times A, X_2 \times A, \ldots, X_n \times A$ are the distinct centralizers of non-central elements of $G \times A$. Since G is an AC-group, $G \times A$ is also an AC-group. Therefore,

$$\Gamma_{G\times A} = \sqcup_{i=1}^{n} K_{|X_i\times A|-|Z(G)\times A|} = \sqcup_{i=1}^{n} K_{|A|(|X_i|-|Z(G)|)}.$$

We have, $|V(\Gamma_{G\times A})| = \sum_{i=1}^{n} |A|(|X_i|-|Z(G)|)$ and $\mathrm{tr}(\mathrm{CNRS}(\Gamma_{G\times A})) = \sum_{i=1}^{n}(|A|(|X_i|-|Z(G)|)-1)(|A|(|X_i| - |Z(G)|)(|A|(|X_i| - |Z(G)|) - 2))$. Therefore $\Delta_{\Gamma_{G\times A}} = \frac{\sum_{i=1}^{n}(|A|(|X_i|-|Z(G)|)-1)(|A|(|X_i|-|Z(G)|)(|A|(|X_i|-|Z(G)|)-2))}{\sum_{i=1}^{n}|A|(|X_i|-|Z(G)|)}$. Hence the result follows from the Theorem 2.1. □

3 CNL (CNSL)-integral and CNL (CNSL)-hyperenergetic graphs

We begin this section with the following theorem which follows from the results obtained in Section 2.

Theorem 3.1. *Let G be a finite non-abelian group.*

(a) *If G is isomorphic to QD_{2^n} (where $n \geq 4$), $PSL(2, 2^k)$ (where $k \geq 2$), $GL(2, q)$ (where $q > 2$ is a prime power), $A(n, \nu)$ and $A(n, p)$ then Γ_G is CNL (CNSL)-integral.*

(b) *If $\frac{G}{Z(G)}$ is isomorphic to $Sz(2)$, D_{2m} and $\mathbb{Z}_p \times \mathbb{Z}_p$, then Γ_G is CNL (CNSL)-integral.*

(c) *If G is a 4, 5-centralizer finite group or a $(p+2)$-centralizer finite p-group then Γ_G is CNL (CNSL)-integral.*

(d) *If G is isomorphic to $\mathcal{M}_{2mn}$, D_{2m}, U_{6n} and Q_{4n} then Γ_G is CNL (CNSL)-integral.*

(e) *If G is an AC-group then Γ_G is CNL (CNSL)-integral.*

Theorem 3.2. *The commuting graph of the quasihedral group QD_{2^n} ($n \geq 4$) is*

(a) *CNL (CNSL)-hyperenergetic if $n \geq 5$.*

(b) *not CNL (CNSL)-hyperenergetic if $n = 4$.*

Proof. We have $|V(\Gamma_{QD_{2^n}})| = 2^n - 2$. Using (1.3) and Proposition 2.2, we get

$$LE_{CN}(K_{|V(\Gamma_{QD_{2^n}})|} - LE_{CN}(\Gamma_{QD_{2^n}}) = -\frac{2^{(-3+n)}(-4+2^n)(100-7\times 2^{2+n}+4^n)}{-2+2^n}.$$

Note that $LE_{CN}(K_{|V(\Gamma_{QD_{2^n}})|} - LE_{CN}(\Gamma_{QD_{2^n}}) < 0$ or > 0 according as $n \geq 5$ or $n = 4$. Also,

$$LE^+_{CN}(K_{|V(\Gamma_{QD_{2^n}})|} - LE_{CN}(\Gamma_{QD_{2^n}}) = -\frac{(-4+2^n)^2(24-13\times 2^{1+n}+4^n)}{8(-2+2^n))}.$$

Note that $LE^+_{CN}(K_{|V(\Gamma_{QD_{2^n}})|} - LE_{CN}(\Gamma_{QD_{2^n}}) > 0$ or < 0 according as $n \geq 5$ or $n = 4$. Hence, the result follows. □

Proposition 3.3. *The commuting graph of the projective special linear group $PSL(2, 2^k)$, $k \geq 2$, is not CNL (CNSL)-hyperenergetic.*

Proof. We have $|V(\Gamma_{PSL(2,2^k)})| = -2^k + 2^{3k} - 1$. Using (1.3) and Proposition 2.3, we get

$$\begin{aligned} &LE_{CN}(K_{|V(\Gamma_{PSL(2,2^k)})|}) - LE_{CN}(\Gamma_{PSL(2,2^k)}) \\ &= \frac{2^k\left(7\times 2^{3k+1} - 7\times 2^k + 2\times 2^{8k} - 2^{5k} - 9\times 2^{6k} + 3\times 2^{4k}\right)}{8^k - 2^k - 1} > 0. \end{aligned}$$

Therefore, $\Gamma_{PSL(2,2^k)}$ is not CNL-hyperenergetic.

For $k \geq 3$, we have

$$\begin{aligned} &LE^+_{CN}(K_{|V(\Gamma_{PSL(2,2^k)})|}) - LE^+_{CN}(\Gamma_{PSL(2,2^k)}) \\ &= \frac{2^k\left(-3\times 2^k + 3\times 2^{2k+1} + 2^{8k+1} - 8^{k+1} + 5\times 16^k + 32^k - 9\times 64^k - 2\right)}{-2^k + 8^k - 1} > 0. \end{aligned}$$

Also, for $k = 2$, $LE^+_{CN}(K_{|V(\Gamma_{PSL(2,2^k)})|}) - LE^+_{CN}(\Gamma_{PSL(2,2^k)}) = \frac{380848}{59} > 0$. Therefore, $\Gamma_{PSL(2,2^k)}$ is not CNSL-hyperenergetic. □

Theorem 3.4. *The commuting graph of the general linear group $GL(2, q)$, where $q = p^n > 2$ and p is prime, is not CNL (CNSL)-hyperenergetic.*

Proof. We have $|V(\Gamma_{GL(2,q)})| = q^4 - q^3 - q^2 + 1$. Using (1.3) and Proposition 2.4, we get

$$\begin{aligned} &LE_{CN}(K_{|V(\Gamma_{GL(2,q)})|}) - LE_{CN}(\Gamma_{GL(2,q)}) \\ &= \frac{q^2(q+1)((q-1)q-1)\left(q^2(q-1)(5q-8+2q^2(q-2))+2\right)}{q^3-q-1} > 0. \end{aligned}$$

Therefore, $\Gamma_{GL(2,q)}$ is not CNL-hyperenergetic.

Also

$$\begin{aligned} &LE^+_{CN}(K_{|V(\Gamma_{GL(2,q)})|}) - LE^+_{CN}(\Gamma_{GL(2,q)}) \\ &= \begin{cases} \frac{2q^{11}-6q^{10}+6q^9-10q^8+9q^7+24q^6-22q^5-28q^4+3q^3+22q^2+4q}{q^3-q-1} > 0, & \text{if } q \leq 5 \\ \frac{2q^{11}-6q^{10}+5q^9-3q^8+2q^7+9q^6-17q^5-10q^4+8q^3+14q^2+4q}{q^3-q-1} > 0, & \text{if } q \geq 6. \end{cases} \end{aligned}$$

Therefore $\Gamma_{GL(2,q)}$ is not CNSL-hyperenergetic. □

Theorem 3.5. *Let G be a finite group such that $\frac{G}{Z(G)} \cong Sz(2)$. Then the commuting graph of G is*

(a) *CNL-hyperenergetic if $|Z(G)| \geq 17$.*

(b) *not CNL-hyperenergetic if $1 \leq |Z(G)| \leq 16$.*

(c) *CNSL-hyperenergetic if $|Z(G)| \geq 16$.*

(d) *not CNSL-hyperenergetic if $1 \leq |Z(G)| \leq 15$.*

Proof. We have $|V(\Gamma_G)| = 3 \times 5z + 4z = 19z$, where $z = |Z(G)|$. Using (1.3) and Theorem 2.7, for $z \geq 2$, we get

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = -\frac{120}{19}z(z(7z - 114) + 15).$$

Note that $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) < 0$ or > 0 according as $z \geq 17$ or $1 \leq z \leq 16$. Therefore, Γ_G is CNL-hyperenergetic and not CNL-hyperenergetic according as $z \geq 17$ and $1 \leq z \leq 16$.

Again

$$LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) = -\frac{8}{19}(3z(z(35z - 527) + 24) + 38).$$

Note that $LE^+_{CN}(K_{|V(\Gamma_G)|}) - LE^+_{CN}(\Gamma_G) < 0$ or > 0 according as $z \geq 16$ or $1 \leq z \leq 15$. Therefore, Γ_G is CNSL hyperenergetic and not CNSL hyperenergetic according as $z \geq 16$ and $1 \leq z \leq 15$. □

Theorem 3.6. *Let G be a finite group with $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Then the commuting graph of G is not CNL (CNSL)-hyperenergetic.*

Proof. We have $|V(\Gamma_G)| = (p^2 - 1)z$, where $z = |Z(G)|$. By (1.3) and Theorem 2.8, we get

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = 2p((p-1)^2(p+1)z^2 - 2) > 0$$

Hence, Γ_G is not CNL-hyperenergetic. Again, $LE_{CN}(\Gamma_G) = LE^+_{CN}(\Gamma_G)$. Therefore, Γ_G is also not CNSL-hyperenergetic. □

As a corollary of Theorem 3.6 we get the following result.

Corollary 3.7. *The commuting graph of G is not CNL (CNSL)-hyperenergetic if G is a finite*

(a) *non-abelian group of order p^3, where p is a prime.*

(b) *4-centralizer group.*

(c) *5-centralizer group.*

(d) *$(p+2)$-centralizer p-group.*

(e) *non-abelian group such that the maximal size of the set of pairwise non-commuting elements of G is 3 or 4.*

Proof. Parts (a)-(d) are follow from Theorem 3.6. Under the hypothesis in part (e), G is either 4-centralizer or 5-centralizer finite group. Hence, the result follows from parts (b) and (c). □

Theorem 3.8. *Let G be a finite group with $\frac{G}{Z(G)} \cong D_{2m}$, where $m \geq 2$ and $|Z(G)| = z$. Then the commuting graph of G is*

(a) *not CNL-hyperenergetic if $m = 3$ and $z \leq 6$; $m = 4$ and $z = 1, 2, 3$; $m = 5$ and $z = 1, 2$; and $m = 6, 7, 8, 9, 10$ and $z = 1$. Otherwise, it is CNL-hyperenergetic.*

(b) *not CNSL-hyperenergetic if $m = 3$ and $1 \leq z \leq 7$; $m = 4$ and $z = 1, 2, 3$; $m = 5, 6$ and $z = 1, 2$; and $m = 7, 8, 9, 10, 11$ and $z = 1$. Otherwise, it is CNSL-hyperenergetic.*

Proof. We have $|V(\Gamma_G)| = 2mz - z$, where $z = |Z(G)|$.

By (1.3) and Theorem 2.13, we get

$$LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) = -\frac{2(m-1)mz(z(m^2z - 2m(z+5)+8)+9)}{2m-1}.$$

It can be seen that $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) > 0$ if if $m = 3$ and $z \leq 6$; $m = 4$ and $z = 1, 2, 3$; $m = 5$ and $z = 1, 2$; and $m = 6, 7, 8, 9, 10$ and $z = 1$. Otherwise $LE_{CN}(K_{|V(\Gamma_G)|}) - LE_{CN}(\Gamma_G) < 0$. Hence, Γ_G is not CNL-hyperenergetic if $m = 3$ and $z \leq 6$; $m = 4$ and $z = 1, 2, 3$; $m = 5$ and $z = 1, 2$; and $m = 6, 7, 8, 9, 10$ and $z = 1$. Otherwise it is CNL-hyperenergetic.

Now we determine whether Γ_G is CNSL-hyperenergetic by considering the following cases.

Case 1. $m = 3$, $z = 1$ and $m = 4$, $z = 1$

By (1.3) and Theorem 2.13, we get

$$LE^+_{CN}(K_{|V(\Gamma_G)|} - LE^+_{CN}(\Gamma_G) = \begin{cases} 24 > 0, & \text{for } m = 3, z = 1 \\ \frac{372}{7} > 0, & \text{for } m = 4, z = 1. \end{cases}$$

Therefore, Γ_G is not CNSL-hyperenergetic.

Case 2. $m = 3$ and $z \geq 2$; $m = 4$ and $z \geq 2$; and $m \geq 5$ and $z \geq 1$

By (1.3) and Theorem 2.13, we get

$$LE^+_{CN}(K_{|V(\Gamma_G)|} - LE^+_{CN}(\Gamma_G) = \frac{2(2m-1)((2m-1)z-2)((2m-1)z-1) - 2(m-2)(m-1)mz^2(mz-3)}{2m-1}.$$

It can be seen that $LE^+_{CN}(K_{|V(\Gamma_G)|} - LE^+_{CN}(\Gamma_G) > 0$ if $m = 3$ and $2 \leq z \leq 7$; $m = 4$ and $z = 2, 3$; $m = 5, 6$ and $z = 1, 2$; and $m = 7, 8, 9, 10, 11$ and $z = 1$. Otherwise $LE^+_{CN}(K_{|V(\Gamma_G)|} - LE^+_{CN}(\Gamma_G) < 0$. Hence, Γ_G is not CNSL-hyperenergetic if $m = 3$ and $2 \leq z \leq 7$; $m = 4$ and $z = 2, 3$; $m = 5, 6$ and $z = 1, 2$; and $m = 7, 8, 9, 10, 11$ and $z = 1$. Otherwise it is CNSL-hyperenergetic. Hence, the result follows. □

As a consequences of Theorem 3.8 we get the following results.

Corollary 3.9. *Suppose that G is isomorphic to the metacyclic group $\mathcal{M}_{2mn}$.*

(a) *If m is even then*

  (i) *Γ_G is not CNL-hyperenergetic whenever $m = 4$ and $n \geq 2$; $m = 6$ and $n = 2, 3$. Otherwise, Γ_G is CNL-hyperenergetic.*

  (ii) *Γ_G is not CNSL-hyperenergetic whenever $m = 4$ and $n \geq 2$; $m = 6$ and $n = 2, 3$. Otherwise, Γ_G is CNSL-hyperenergetic.*

(b) *If m is odd then*

  (i) *Γ_G is not CNL-hyperenergetic whenever $m = 3$ and $n \leq 6$; $m = 5$ and $n = 2$. Otherwise, Γ_G is CNL-hyperenergetic.*

  (ii) *Γ_G is not CNSL-hyperenergetic whenever $m = 3$ and $n \leq 7$. Otherwise, Γ_G is CNSL-hyperenergetic.*

Corollary 3.10. *Suppose that G is isomorphic to the dihedral group D_{2m}.*

(a) *If m is even then*

  (i) *Γ_G is not CNL-hyperenergetic whenever $4 \leq m \leq 10$. Otherwise, Γ_G is CNL-hyperenergetic.*

  (ii) *Γ_G is not CNSL-hyperenergetic whenever $4 \leq m \leq 12$. Otherwise, Γ_G is CNSL-hyperenergetic.*

(b) *If m is odd then*

(i) Γ_G *is not CNL-hyperenergetic whenever* $3 \leq m \leq 9$*. Otherwise,* Γ_G *is CNL- hyperenergetic.*

(ii) Γ_G *is not CNSL-hyperenergetic whenever* $3 \leq m \leq 11$*. Otherwise,* Γ_G *is CNSL-hyperenergetic.*

Corollary 3.11. *Let G be a finite non-abelian group.*

(a) *If G is isomorphic to the generalized quartanion group of order* $4n$*,* Q_{4n} *then*

(i) Γ_G *is not CNL-hyperenergetic when* $n = 2, 3, 4, 5$*. Otherwise,* Γ_G *is CNL- hyperenergetic.*

(ii) Γ_G *is not CNSL-hyperenergetic when* $n = 2, 3, 4, 5, 6$*. Otherwise,* Γ_G *is CNSL-hyperenergetic.*

(b) *If G is isomorphic to* U_{6n} *then*

(i) Γ_G *is not CNL-hyperenergetic when* $n \leq 6$*. Otherwise,* Γ_G *is CNL-hyperenergetic.*

(ii) Γ_G *is not CNSL-hyperenergetic when* $n \leq 7$*. Otherwise,* Γ_G *is CNSL-hyperenergetic.*

We conclude this section with the following proposition.

Proposition 3.12. *If G is one of the groups considered in Propositions 2.5-2.6 then commuting graph of G is not CNL (CNSL)-hyperenergetic.*

4 Conclusion remarks

We observed that the commuting graphs of the groups discussed above are CNL (CNSL)-integral and this leads us to the following question: “Which finite non-abelian groups are CNL (CNSL)-inetgral?" It is also observed that the commuting graphs of some AC-groups are CNL (CNSL)-hyperenergetic but some are not CNL (CNSL)-hyperenergetic. Therefore, one can try to find general conditions such that the commuting graphs of finite AC-groups are CNL (CNSL)-hyperenergetic.

Similar investigations may be carried for other graphs defined on groups, for example, power graph of finite groups [18]; and various graphs defined on rings and matrices [3, 5].

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