

# On extended beta and hypergeometric functions involving Kummer’s ${}_1F_1$ and Mittag-Leffler functions

R. Kanaga

Communicated by: Ayman Badawi

MSC 2010 Classifications: Primary 33C05, 33C10, 33C15, 33E12; Secondary 33B10, 33B15, 33C20, 30A10.

Keywords and phrases: Carlson notations, confluent hypergeometric function, generalized extended beta function, Gauss hypergeometric function, Mittag-Leffler function, Bessel-Struve kernel function, log-convexity, and Turán-type inequality, Mellin transform, generalized extended beta distribution, generalized extended hypergeometric type functions.

*The authors would like to thank the reviewers and editor for their constructive comments and valuable suggestions that improved the quality of our paper.*

**Corresponding Author: R. Kanaga**

**Abstract** This study establishes generalized beta and hypergeometric function extensions using Kummer’s and Mittag-Leffler’s functions. It gives integral representations, summation formulae, Mellin transforms, and recurrence relations while establishing properties like log-convexity and Turán-type inequalities for the extended beta function. Also, transformation, differentiation, and difference relations are provided for the extended hypergeometric functions. A generating function is also proposed. As a statistical application, a generalized extended beta distribution is defined, with key properties such as variance, coefficient of variation, moment-generating function, characteristic function, cumulative distribution function (CDF), and complementary CDF systematically obtained.

## 1 Introduction

To ensure brevity and consistency, we adopt the following notation, as introduced by Carlson [7] and elaborated in [20]:

$$\begin{aligned} \mathbb{C}_> &:= \{w \in \mathbb{C} : \operatorname{Re}(w) > 0\}, \\ \mathbb{C}_{>>} &:= \{z, w \in \mathbb{C} : \operatorname{Re}(z) > \operatorname{Re}(w) > 0\}, \\ \mathbb{C}_{>-} &:= \{w \in \mathbb{C} : \operatorname{Re}(w) > -1\}, \end{aligned}$$

where  $\mathbb{C}$  represents the set of complex numbers. We begin by recalling Euler’s beta function  $B(u, v)$  [10, 33]:

$$B(u, v) := \begin{cases} \int_0^1 t^{u-1}(1-t)^{v-1} dt, & (u, v \in \mathbb{C}_>), \\ \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, & (u, v \in \mathbb{C} \setminus \mathbb{Z}_0^-), \end{cases}$$

where  $\Gamma(w)$  is the gamma function, defined as

$$\Gamma(w) := \int_0^\infty e^{-t}t^{w-1} dt, \quad (w \in \mathbb{C}_>).$$

Throughout this work, the following sets are used.  $\mathbb{R}_0^+$  denotes non-negative real numbers,  $\mathbb{N}$  represents positive integers, and  $\mathbb{Z}_0^-$  represents non-positive integers. The Gauss hypergeometric function  ${}_2F_1(\delta, \gamma; \beta; w)$  and Kummer’s confluent hypergeometric function  ${}_1F_1(\gamma; \beta; w) = \Phi(\gamma; \beta; w)$  are expressed as follows [10, 33, 35]:

$${}_2F_1(\delta, \gamma; \beta; w) = \sum_{n=0}^\infty \frac{(\delta)_n(\gamma)_n}{(\beta)_n} \frac{w^n}{n!}, \quad (|w| < 1; \delta, \gamma \in \mathbb{C}; \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-),$$

$${}_1F_1(\gamma; \beta; w) := \Phi(\gamma; \beta; w) = \sum_{n=0}^{\infty} \frac{(\gamma)_n w^n}{(\beta)_n n!}, \quad (w, \gamma \in \mathbb{C}; \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

Here, the Pochhammer symbol  $(\nu)_n$  for  $\nu \in \mathbb{C}$  is defined as

$$(\nu)_n := \begin{cases} 1, & (n = 0), \\ \nu(\nu + 1) \cdots (\nu + n - 1), & (n \in \mathbb{N}). \end{cases}$$

The integral representations of  ${}_2F_1(\delta, \gamma; \beta; w)$  and  ${}_1F_1(\gamma; \beta; w)$  are given by

$$\begin{aligned} {}_2F_1(\delta, \gamma; \beta; w) &= \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\beta-\gamma-1} (1-wt)^{-\delta} dt, \\ &(\beta, \gamma \in \mathbb{C}_{>>}; |\arg(1-w)| \leq \pi - \epsilon, 0 < \epsilon < \pi), \\ {}_1F_1(\gamma; \beta; w) &= \Phi(\gamma; \beta; w) = \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\beta-\gamma-1} e^{wt} dt, \quad (\beta, \gamma \in \mathbb{C}_{>>}). \end{aligned} \tag{1.1}$$

In 1997, Chaudhry et al. [8] introduced an extended beta function

$$B(u, v; p) = \int_0^1 t^{u-1} (1-t)^{v-1} \exp\left(-\frac{p}{t(1-t)}\right) dt, \quad (p \in \mathbb{C}_{>}).$$

For  $p = 0$ ,  $B(u, v; p)$  is reduced to classical beta function. This extended beta function was used to generalize the hypergeometric and confluent hypergeometric functions [9]

$$\begin{aligned} F_p(\delta, \gamma; \beta; w) &= \sum_{n=0}^{\infty} \frac{B(\gamma + n, \beta - \gamma; p)}{B(\gamma, \beta - \gamma)} \frac{(\delta)_n w^n}{n!}, \quad (p \geq 0, |w| < 1; \beta, \gamma \in \mathbb{C}_{>>}), \\ \Phi_p(\gamma; \beta; w) &= \sum_{n=0}^{\infty} \frac{B(\gamma + n, \beta - \gamma; p)}{B(\gamma, \beta - \gamma)} \frac{w^n}{n!}, \quad (p \geq 0; \beta, \gamma \in \mathbb{C}_{>>}). \end{aligned}$$

Later, Choi et al. [10] proposed another extended beta function

$$\begin{aligned} B(u, v; p, q) &= \int_0^1 t^{u-1} (1-t)^{v-1} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right) dt, \\ &(\min\{\operatorname{Re}(u), \operatorname{Re}(v)\} > 0; \min\{\operatorname{Re}(p), \operatorname{Re}(q)\} > 0). \end{aligned} \tag{1.2}$$

This function was applied to extend hypergeometric functions

$$\begin{aligned} F_{p,q}(\delta, \gamma; \beta; w) &= \sum_{n=0}^{\infty} \frac{B(\gamma + n, \beta - \gamma; p, q)}{B(\gamma, \beta - \gamma)} \frac{(\delta)_n w^n}{n!}, \quad (p, q \in \mathbb{C}_{>}, |w| < 1; \beta, \gamma \in \mathbb{C}_{>>}), \\ \Phi_{p,q}(\gamma; \beta; w) &= \sum_{n=0}^{\infty} \frac{B(\gamma + n, \beta - \gamma; p, q)}{B(\gamma, \beta - \gamma)} \frac{w^n}{n!}, \quad (p, q \in \mathbb{C}_{>}; \beta, \gamma \in \mathbb{C}_{>>}). \end{aligned}$$

Recently, an extension of Gauss hypergeometric and confluent hypergeometric functions is studied in [34] using the Bessel-Struve kernel function. Also, several extensions are discussed in [1, 2, 3, 4, 5, 11, 12, 13, 14, 15, 16, 17, 18, 19, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 36, 37]. Building on these studies, this paper introduces a new generalized extended beta function in the next section and systematically develops its integral representations, Mellin transforms, summation formulae, and log-convexity properties. Here,

**Definition 1.1.** The Mellin transform [6, p. 343] of a function  $f$ , defined for  $x > 0$ , is given by

$$\mathcal{M}\{f(x)\}(s) = \int_0^\infty x^{s-1} f(x) dx,$$

where  $s \in \mathbb{C}$  and the integral converges. The variable  $s$  is typically complex, and the transform exists in a vertical strip of the complex plane where the integral converges.

In addition, a generalized extended beta distribution and extensions of hypergeometric functions are proposed, and their essential properties are analyzed.

## 2 Extended beta function and its properties

**Definition 2.1** (Extended beta function). For  $\min\{\operatorname{Re}(u), \operatorname{Re}(v)\} > 0$ ,  $\min\{\operatorname{Re}(p), \operatorname{Re}(q)\} > 0$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ , the extended Beta function  $B_{\alpha;\eta;\xi}(u, v; p; q)$  is defined as

$$B_{\alpha;\eta;\xi}(u, v; p; q) = \int_0^1 t^{u-1}(1-t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \tag{2.1}$$

Here,  $E_\alpha(w)$  denotes the Mittag-Leffler function, defined as (see [38, p.255, Eq. (3.1)])

$$E_\alpha(w) = \sum_{n=0}^\infty \frac{w^n}{\Gamma(\alpha n + 1)} \quad (w \in \mathbb{C}, \alpha \in \mathbb{C}_{>-}),$$

and  $\Phi(\eta; \xi; w)$  represents Kummer’s confluent hypergeometric function  ${}_1F_1(\eta; \xi; w)$ , expressed as

$$\Phi(\eta; \xi; w) := {}_1F_1(\eta; \xi; w) = \frac{\Gamma(\eta)}{\Gamma(\xi)\Gamma(\eta - \xi)} \sum_{m=0}^\infty B(\xi + m, \eta - \xi) \frac{w^m}{m!} \quad (w \in \mathbb{C}, \eta, \xi \in \mathbb{C}_{>-}). \tag{2.2}$$

For  $p = 0 = q$  in (2.1), we get  $E_\alpha(0) = 1$  and  $\Phi(\eta; \xi; 0) = 1$ . As a result, (2.1) reduces to the standard beta function. (see [38])

Also, for  $\xi = \eta$  and  $w = -p/t$ , the function

$$\Phi\left(\eta; \xi; -\frac{p}{t}\right) = e^{-\frac{p}{t}}, \tag{2.3}$$

and for  $\alpha = 1$ ,  $w = -q/(1-t)$ , the function

$$E_\alpha\left(-\frac{q}{1-t}\right) = e^{-\frac{q}{1-t}}. \tag{2.4}$$

Now for  $\alpha = 1$  and  $\xi = \eta$ ; from (2.3) and (2.4), the equation (2.1) reduces to (1.2).

### 2.1 Integral representations of extended beta function

In this section, we establish the integral representation of an extended beta function. To derive these results, we first present the following lemmas.

**Lemma 2.2.** *The following integral formula for the confluent hypergeometric function holds:*

$$\int_0^\infty p^{b-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) dp = t^b \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{\Gamma(\eta)\Gamma(\xi - b)}, \tag{2.5}$$

where  $\eta, \xi, b \in \mathbb{C}$ ,  $p \geq 0$ ,  $\eta, \xi \in \mathbb{C}_{>}$ , and  $0 < t < 1$ .

*Proof.* We know that the integral representation of  $\Phi(\eta; \xi; -w)$  (also see [33, Eq. (9), p. 137]) is given by

$$\Phi(\eta; \xi; -w) = \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi - \eta)} \int_0^1 t^{\eta-1}(1-t)^{\xi-\eta-1} e^{-wt} dt. \tag{2.6}$$

Substituting  $w \rightarrow wp$  and integrating with respect to  $p$  from 0 to  $\infty$ , we get

$$\int_0^\infty p^{b-1} \Phi(\eta; \xi; -wp) dp = \int_0^\infty p^{b-1} \left[ \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi - \eta)} \int_0^1 t^{\eta-1}(1-t)^{\xi-\eta-1} e^{-wpt} dt \right] dp. \tag{2.7}$$

By the uniform convergence of the integrals, we can interchange the order of integration

$$\int_0^\infty p^{b-1} \Phi(\eta; \xi; -wp) dp = \frac{\Gamma(\xi)}{\Gamma(\eta)\Gamma(\xi - \eta)} \int_0^1 t^{\eta-1}(1-t)^{\xi-\eta-1} \left[ \int_0^\infty p^{b-1} e^{-wpt} dp \right] dt. \tag{2.8}$$

Using the formula of the gamma function

$$\Gamma(\beta)\theta^{-\beta} = \int_0^\infty t^{\beta-1} e^{-\theta t} dt, \quad (\theta, \beta \in \mathbb{C}_>),$$

we evaluate the inner integral

$$\int_0^\infty p^{b-1} e^{-wpt} dp = \frac{\Gamma(b)}{(wt)^b}.$$

Substituting this result back, we get

$$\int_0^\infty p^{b-1} \Phi(\eta; \xi; -wp) dp = \frac{\Gamma(b)\Gamma(\xi)}{w^b\Gamma(\eta)\Gamma(\xi - \eta)} \int_0^1 t^{\eta-b-1}(1-t)^{\xi-\eta-1} dt. \tag{2.9}$$

The remaining integral is a beta function, giving

$$\int_0^1 t^{\eta-b-1}(1-t)^{\xi-\eta-1} dt = \frac{\Gamma(\eta - b)\Gamma(\xi - \eta)}{\Gamma(\xi - b)}.$$

Thus, we get

$$\int_0^\infty p^{b-1} \Phi(\eta; \xi; -wp) dp = \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{w^b\Gamma(\eta)\Gamma(\xi - b)}. \tag{2.10}$$

Finally, substituting  $w = \frac{1}{t}$  into the result gives

$$\int_0^\infty p^{b-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) dp = t^b \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{\Gamma(\eta)\Gamma(\xi - b)}. \tag{2.11}$$

This completes the proof. □

**Lemma 2.3.** *The Mittag-Leffler function satisfies the following integral formula*

$$\int_0^\infty q^{c-1} E_\alpha\left(\frac{-q}{1-t}\right) dq = (1-t)^c \frac{\Gamma(c)\Gamma(1-c)}{\Gamma(1-\alpha c)}, \tag{2.12}$$

where  $\alpha, c \in \mathbb{C}$ ,  $p \geq 0$ ,  $\alpha \in \mathbb{C}_>$ , and  $0 < t < 1$ .

*Proof.* We start with the integral formula provided in [38, p. 273, Eq. (3.119)]

$$\int_0^\infty q^{c-1} E_\alpha(-\lambda q) dq = \frac{\Gamma(c)\Gamma(1-c)}{\lambda^c \Gamma(1-\alpha c)}. \tag{2.13}$$

By substituting  $\lambda = \frac{1}{1-t}$ , the desired result in (2.12) follows directly. □

**Theorem 2.4.** *The following integral formula is valid:*

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta;\xi}(u, v; p; q) dp dq = \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{\Gamma(\eta)\Gamma(\xi - b)} \frac{\Gamma(c)\Gamma(1 - c)}{\Gamma(1 - \alpha c)} B(u + b, v + c), \tag{2.14}$$

where  $p, q, b, c \in \mathbb{C}_>$ ,  $(u + b), (v + c) \in \mathbb{C}_>$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Multiplying both sides of (2.1) by  $p^{b-1}q^{c-1}$  and integrating over  $p$  and  $q$  with limits  $0 \leq p, q \leq \infty$ , we get

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta;\xi}(u, v; p; q) dq dp = \int_0^\infty \int_0^\infty p^{b-1} q^{c-1} \tag{2.15}$$

$$\cdot \left\{ \int_0^1 t^{u-1} (1-t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt \right\} dp dq. \tag{2.16}$$

The uniform convergence of the integral allows us to interchange the order of integration

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta;\xi}(u, v; p; q) dp dq = \int_0^1 t^{u-1} (1-t)^{v-1} \left\{ \int_0^\infty p^{b-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) dp \cdot \int_0^\infty q^{c-1} E_\alpha\left(-\frac{q}{1-t}\right) dq \right\} dt.$$

Using equations (2.12) and (2.5), we find

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta;\xi}(u, v; p; q) dp dq = \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{\Gamma(\eta)\Gamma(\xi - b)} \frac{\Gamma(c)\Gamma(1 - c)}{\Gamma(1 - \alpha c)} \left\{ \int_0^1 t^{u+b-1} (1-t)^{v+c-1} dt \right\}. \tag{2.17}$$

Evaluating the integral using the beta function

$$\int_0^\infty \int_0^\infty p^{b-1} q^{c-1} B_{\alpha;\eta;\xi}(u, v; p; q) dp dq = \frac{\Gamma(b)\Gamma(\xi)\Gamma(\eta - b)}{\Gamma(\eta)\Gamma(\xi - b)} \frac{\Gamma(c)\Gamma(1 - c)}{\Gamma(1 - \alpha c)} B(u + b, v + c).$$

This completes the proof. □

**Remark 2.5.** When choosing the specific values  $\alpha = 1$  and  $\xi = \eta$  in equation (2.14), the above theorem is reduced to Theorem 1 in [10].

**Theorem 2.6.** *Various integral representations for  $B_{\alpha;\eta;\xi}(u, v; p; q)$  are given as follows:*

$$B_{\alpha;\eta;\xi}(u, v; p; q) = 2 \int_0^{\frac{\pi}{2}} (\cos^{2u-1} \theta) (\sin^{2v-1} \theta) \Phi(\eta; \xi; -p \sec^2 \theta) E_\alpha(-q \csc^2 \theta) d\theta, \tag{2.18}$$

$$B_{\alpha;\eta;\xi}(u, v; p; q) = \int_0^\infty \phi^{u-1} (1 + \phi)^{-(u+v)} \Phi\left(\eta; \xi; \frac{-p(1 + \phi)}{\phi}\right) E_\alpha(-q(1 + \phi)) d\phi, \tag{2.19}$$

$$B_{\alpha;\eta;\xi}(u, v; p; q) = 2^{1-u-v} \int_{-1}^1 (1 + \phi)^{u-1} (1 - \phi)^{v-1} \Phi\left(\eta; \xi; \frac{-2p}{1 + \phi}\right) E_\alpha\left(\frac{-2q}{1 - \phi}\right) d\phi, \tag{2.20}$$

$$B_{\alpha;\eta;\xi}(u, v; p; q) = (g - f)^{1-u-v} \int_f^g (\phi - f)^{u-1} (g - \phi)^{v-1} \cdot \Phi\left(\eta; \xi; \frac{-p(g - f)}{\phi - f}\right) E_\alpha\left(\frac{-q(g - f)}{g - \phi}\right) d\phi, \tag{2.21}$$

where  $p, q, u, v \in \mathbb{C}_>$  and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Using the transformations  $t = \cos^2 \theta$ ,  $t = \phi/(1 + \phi)$ ,  $t = (1 + \phi)/2$ , and  $t = (\phi - f)/(g - f)$  in equation (2.18), the integral forms (2.18), (2.19), (2.20), and (2.21) can be obtained, respectively.  $\square$

**Remark 2.7.** The corresponding results presented in [10] are obtained by simplifying the integrals in Theorem 2.6 for the specific values  $\alpha = 1$  and  $\xi = \eta$ . Furthermore, by assuming  $p = q$ , the integrals in Theorem 2.6 reduce to the analogous results found in [9] under the conditions  $\alpha = 1$  and  $\xi = \eta$ . Moreover, it can be readily shown that the integrals in Theorem 2.6 lead to well-known formulas for the beta function when  $p = 0 = q$ .

**2.2 Mellin transforms of extended beta function**

**Theorem 2.8.** *The Mellin transform of  $B_{\alpha;\eta;\xi}(u, v; p; q)$  is given by the following formula.*

$$B_{\alpha;\eta;\xi}(u, v; p, q) = \frac{1}{(2\pi i)^2} \int_{\epsilon_1 - i\infty}^{\epsilon_1 + i\infty} \int_{\epsilon_2 - i\infty}^{\epsilon_2 + i\infty} \frac{\Gamma(x)\Gamma(\xi)\Gamma(\eta - x)}{\Gamma(\eta)\Gamma(\xi - x)} \frac{\Gamma(y)\Gamma(1 - y)}{\Gamma(1 - \alpha y)} \cdot \frac{\Gamma(u + x)\Gamma(v + y)}{\Gamma(u + x + v + y)} p^{-x} q^{-y} dx dy, \tag{2.22}$$

where  $p, q \in \mathbb{C}_{>}$ ,  $\epsilon_1 > 0$ ,  $\epsilon_2 > 0$ , and  $\alpha, \eta, \xi \in \mathbb{C}^+$ .

*Proof.* Applying the Mellin transform to both sides of  $B_{\alpha;\eta;\xi}(u, v; p, q)$  results in

$$M\{B_{\alpha;\eta;\xi}(u, v; p, q); p \rightarrow x, q \rightarrow y\} = \int_0^\infty \int_0^\infty p^{x-1} q^{y-1} \cdot \left[ \int_0^1 t^{u-1} (1-t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt \right] dp dq. \tag{2.23}$$

By the uniform convergence of the integral, the order of integration can be interchanged.

$$M\{B_{\alpha;\eta;\xi}(u, v; p, q); p \rightarrow x, q \rightarrow y\} = \int_0^1 t^{u-1} (1-t)^{v-1} \cdot \left[ \int_0^\infty p^{x-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) dp \right] \cdot \left[ \int_0^\infty q^{y-1} E_\alpha\left(-\frac{q}{1-t}\right) dq \right] dt. \tag{2.24}$$

Using the results of known integral transforms, such as (2.12) and (2.5), in equation (2.24), and applying the beta function definition, we get

$$M\{B_{\alpha;\eta;\xi}(u, v; p, q); p \rightarrow x, q \rightarrow y\} = \frac{\Gamma(x)\Gamma(\xi)\Gamma(\eta - x)}{\Gamma(\eta)\Gamma(\xi - x)} \frac{\Gamma(y)\Gamma(1 - y)}{\Gamma(1 - \alpha y)} \frac{\Gamma(u + x)\Gamma(v + y)}{\Gamma(u + x + v + y)}. \tag{2.25}$$

Finally, the inverse Mellin transform on both sides of the expression obtained above proves the formula (2.22).  $\square$

**2.3 Properties of the extended beta function**

**Theorem 2.9.** *For  $B_{\alpha;\eta;\xi}(u, v; p; q)$ , the following relation is valid:*

$$B_{\alpha;\eta;\xi}(u, v; p; q) = B_{\eta;\alpha}(v, u; q; p), \tag{2.26}$$

where  $p, q \in \mathbb{C}_{>}$  and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Using the transformation  $t = 1 - t$  in equation (2.1), we can derive the result in (2.26).  $\square$

**Remark 2.10.** The specific case  $\alpha = 1$  and  $\xi = \eta$  in equation (2.26) leads to a result equivalent to that presented by Choi et al. [10]. Furthermore, substituting  $\alpha = 1$ ,  $\xi = \eta$ , and setting  $p = q$  into equation (2.26) gives a result analogous to the one reported by Chaudhry et al. [9] when  $p = 0 = q$ , the expression in (2.26) reduce the symmetric property of the beta function.

**Theorem 2.11.** For the extended beta function  $B_{\alpha;\eta;\xi}(u, v; p; q)$ , the following recurrence relation holds:

$$B_{\alpha;\eta;\xi}(u, v; p; q) = B_{\alpha;\eta;\xi}(u + 1, v; p; q) + B_{\alpha;\eta;\xi}(u, v + 1; p; q). \tag{2.27}$$

*Proof.* From the definition of  $B_{\alpha;\eta;\xi}(u, v; p; q)$ , we get

$$\begin{aligned} B_{\alpha;\eta;\xi}(u, v; p; q) &= \int_0^1 t^{u-1}(1-t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \int_0^1 t^{u-1}(1-t)^{v-1} [t + (1-t)] \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \int_0^1 t^{u+1-1}(1-t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt \\ &\quad + \int_0^1 t^{u-1}(1-t)^{v+1-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \end{aligned}$$

By recognizing the integrals in the last step as  $B_{\alpha;\eta;\xi}(u + 1, v; p; q)$  and  $B_{\alpha;\eta;\xi}(u, v + 1; p; q)$ , we obtain

$$B_{\alpha;\eta;\xi}(u, v; p; q) = B_{\alpha;\eta;\xi}(u + 1, v; p; q) + B_{\alpha;\eta;\xi}(u, v + 1; p; q), \tag{2.28}$$

which completes the proof. □

**Remark 2.12.** The specific case of  $\alpha = 1$  and  $\xi = \eta$  in equation (2.27) leads to a result that corresponds to one found in [10]. Furthermore, by setting  $\alpha = 1$ ,  $\xi = \eta$ , and  $p = q$  in equation (2.27), an equivalent result to that in [9] is obtained. Also, it can be observed that the identity given in (2.27) reduces to a well-known relation for the beta function when  $p = 0 = q$ .

**Theorem 2.13.** The following summation formula is satisfied by the extended beta function  $B_{\alpha;\eta;\xi}(u, v; p; q)$ :

$$B_{\alpha;\eta;\xi}(u, 1 - v; p; q) = \sum_{k=0}^{\infty} \frac{(v)_k}{k!} B_{\alpha;\eta;\xi}(u + k, 1; p; q), \tag{2.29}$$

where  $p, q \in \mathbb{C}_>$  and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Using the generalized binomial theorem, we get

$$(1 - w)^{-v} = \sum_{k=0}^{\infty} \frac{(v)_k}{k!} w^k, \quad (|w| < 1), \tag{2.30}$$

where  $(\lambda)_k = \Gamma(\lambda + k)/\Gamma(\lambda)$  is the Pochhammer symbol. Substituting this into equation (2.1), we write

$$\begin{aligned} B_{\alpha;\eta;\xi}(u, 1 - v; p; q) &= \int_0^1 t^{u-1}(1-t)^{1-v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \int_0^1 t^{u-1}(1-t)^{-v} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \int_0^1 t^{u-1} \left[ \sum_{k=0}^{\infty} \frac{(v)_k}{k!} t^k \right] \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \sum_{k=0}^{\infty} \frac{(v)_k}{k!} \int_0^1 t^{u+k-1}(1-t)^{1-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \end{aligned}$$

Simplifying, we obtain

$$B_{\alpha;\eta;\xi}(u, 1 - v; p; q) = \sum_{k=0}^{\infty} \frac{(v)_k}{k!} B_{\alpha;\eta;\xi}(u + k, 1; p; q),$$

which proves the stated result (2.29). □

**Theorem 2.14.** *The following summation formula is satisfied by the extended beta function  $B_{\alpha;\eta;\xi}(u, v; p; q)$ :*

$$B_{\alpha;\eta;\xi}(u, v; p; q) = \sum_{k=0}^{\infty} B_{\alpha;\eta;\xi}(u + k, v + 1; p; q), \tag{2.31}$$

where  $p, q \in \mathbb{C}_>$  and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* By using the result, we write

$$(1 - w)^{v-1} = (1 - w)^v \sum_{k=0}^{\infty} w^k, \quad (|w| < 1). \tag{2.32}$$

Substituting (2.32) into equation (2.1), we get

$$\begin{aligned} B_{\alpha;\eta;\xi}(u, v; p; q) &= \int_0^1 t^{u-1} \left[ (1 - t)^v \sum_{k=0}^{\infty} t^k \right] \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_{\alpha}\left(-\frac{q}{1-t}\right) dt. \\ &= \sum_{k=0}^{\infty} \int_0^1 t^{u+k-1} (1 - t)^{v+1-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_{\alpha}\left(-\frac{q}{1-t}\right) dt. \end{aligned}$$

Simplifying, this becomes

$$B_{\alpha;\eta;\xi}(u, v; p; q) = \sum_{k=0}^{\infty} B_{\alpha;\eta;\xi}(u + k, v + 1; p; q),$$

which proves the stated result (2.31). □

**Theorem 2.15.** *For every extended beta function  $B_{\alpha;\eta;\xi}(u, v; p; q)$ , the following relation is valid:*

$$B_{\alpha;\eta;\xi}(u, -u - m; p; q) = \sum_{k=0}^m \binom{m}{k} B_{\alpha;\eta;\xi}(u + k, -u - k; p; q), \quad (m \in \mathbb{N}_0). \tag{2.33}$$

*Proof.* Substituting  $v = -u - m$  into the known result given in equation (2.27) is

$$B_{\alpha;\eta;\xi}(u, v; p; q) = B_{\alpha;\eta;\xi}(u + 1, v; p; q) + B_{\alpha;\eta;\xi}(u, v + 1; p; q),$$

we obtain

$$B_{\alpha;\eta;\xi}(u, -u - m; p; q) = B_{\alpha;\eta;\xi}(u + 1, -u - m; p; q) + B_{\alpha;\eta;\xi}(u, -u - m + 1; p; q).$$

Writing this formula recursively for  $m = 1, 2, 3, \dots$ , we get

$$\begin{aligned} B_{\alpha;\eta;\xi}(u, -u - 1; p; q) &= B_{\alpha;\eta;\xi}(u, -u; p; q) + B_{\alpha;\eta;\xi}(u + 1, -u - 1; p; q), \\ B_{\alpha;\eta;\xi}(u, -u - 2; p; q) &= B_{\alpha;\eta;\xi}(u, -u; p; q) + 2B_{\alpha;\eta;\xi}(u + 1, -u - 1; p; q) \\ &\quad + B_{\alpha;\eta;\xi}(u + 2, -u - 2; p; q), \end{aligned}$$

and so on. Continuing this process and generalizing the pattern leads us to

$$B_{\alpha;\eta;\xi}(u, -u - m; p; q) = \sum_{k=0}^m \binom{m}{k} B_{\alpha;\eta;\xi}(u + k, -u - k; p; q),$$

which proves the stated result (2.33). □

### 2.4 Log-convexity and Turán- type inequality of extended beta function

This subsection focuses on the log-convexity properties and Turán-type inequalities associated with  $B_{\alpha,\eta}(\mu, \nu; p; q)$ . Detailed analysis and results will be provided to explore the underlying mathematical structure and its implications.

**Theorem 2.16** (Log-Convexity). *The following inequalities hold:*

$$B_{\alpha;\eta;\xi}(\lambda u_1 + (1 - \lambda)u_2, v; p; q) \leq B_{\alpha;\eta;\xi}(u_1, v; p; q)^\lambda \cdot B_{\alpha;\eta;\xi}(u_2, v; p; q)^{1-\lambda}, \tag{2.34}$$

where

$$(\lambda \in (0, 1); u_1 < u_2; v \in \mathbb{C}, p, q \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}),$$

and

$$B_{\alpha;\eta;\xi}(u, \lambda v_1 + (1 - \lambda)v_2; p; q) \leq B_{\alpha;\eta;\xi}(u, v_1; p; q)^\lambda \cdot B_{\alpha;\eta;\xi}(u, v_2; p; q)^{1-\lambda}, \tag{2.35}$$

where

$$(\lambda \in (0, 1); v_1 < v_2; v \in \mathbb{C}, p, q \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}).$$

*Proof.* For integrals, Hölder’s inequality asserts that

$$\int_a^b |f(t)h(t)| dt \leq \left( \int_a^b |f(t)|^j dt \right)^{1/j} \cdot \left( \int_a^b |h(t)|^k dt \right)^{1/k}, \tag{2.36}$$

where  $(1/j) + (1/k) = 1$  with  $j, k > 1$ . Applying Hölder’s inequality (2.36) and substituting  $u = \lambda u_1 + (1 - \lambda)u_2$  in equation (2.1), we obtain

$$\begin{aligned} B_{\alpha;\eta;\xi}(\lambda u_1 + (1 - \lambda)u_2, v; p; q) &= \int_0^1 t^{\lambda u_1 + (1-\lambda)u_2 - 1} (1 - t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt. \\ &= \int_0^1 \left( t^{u_1 - 1} (1 - t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) \right)^\lambda \\ &\quad \cdot \left( t^{u_2 - 1} (1 - t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) \right)^{1-\lambda} dt. \end{aligned} \tag{2.37}$$

Let  $j = 1/\lambda$  and  $k = 1/(1 - \lambda)$ . By applying Hölder’s inequality, we get

$$\begin{aligned} B_{\alpha;\eta;\xi}(\lambda u_1 + (1 - \lambda)u_2, v; p; q) &\leq \left( \int_0^1 t^{u_1 - 1} (1 - t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt \right)^\lambda \\ &\quad \cdot \left( \int_0^1 t^{u_2 - 1} (1 - t)^{v-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1-t}\right) dt \right)^{1-\lambda}. \end{aligned}$$

Thus, we arrive at the result

$$B_{\alpha;\eta;\xi}(\lambda u_1 + (1 - \lambda)u_2, v; p; q) \leq B_{\alpha;\eta;\xi}(u_1, v; p; q)^\lambda \cdot B_{\alpha;\eta;\xi}(u_2, v; p; q)^{1-\lambda}. \tag{2.38}$$

This proves the inequality in equation (2.34). A similar argument with  $v = \lambda v_1 + (1 - \lambda)v_2$  in equation (2.1) proves the inequality in equation (2.35).  $\square$

**Corollary 2.17** (Turán- type inequality). *The following inequalities hold true:*

$$\begin{aligned} B_{\alpha;\eta;\xi}\left(\frac{u_1 + u_2}{2}, v; p; q\right)^2 - B_{\alpha;\eta;\xi}(u_1, v; p; q) \cdot B_{\alpha;\eta;\xi}(u_2, v; p; q) &\leq 0, \\ (u_1 < u_2; v \in \mathbb{C}, p, q \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}), &\tag{2.39} \end{aligned}$$

and

$$B_{\alpha;\eta;\xi} \left( u, \frac{v_1 + v_2}{2}; p; q \right)^2 - B_{\alpha;\eta;\xi}(u, v_1; p; q) \cdot B_{\alpha;\eta;\xi}(u, v_2; p; q) \leq 0, \tag{2.40}$$

$$(v_1 < v_2; v \in \mathbb{C}, p, q \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}).$$

*Proof.* For  $\lambda = 1/2$  in inequalities (2.34) and (2.35), the above equations (2.39) and (2.40) can be obtained, respectively.  $\square$

**Remark 2.18.** The particular case of  $\alpha = 1$  and  $\xi = \eta$  in equation (2.34) corresponds to the result presented by Luo et al. [20, Eq. (15)]. Moreover, by taking  $p = 0 = q$ , the well-known inequality

$$B(\lambda\mu_1 + (1 - \lambda)\mu_2, \nu) \leq B(\mu_1, \nu)^\lambda \cdot B(\mu_2, \nu)^{1-\lambda} \tag{2.41}$$

is obtained.

### 3 Generalized extension of Gauss and confluent hypergeometric functions

Using  $B_{\alpha;\eta;\xi}(u, v; p; q)$ , we extend Gauss and confluent hypergeometric functions as outlined in this section

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} (\delta)_k \frac{x^k}{k!}, \tag{3.1}$$

$$(p, q \geq 0; \beta, \gamma \in \mathbb{C}_{>>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}),$$

and

$$\Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} \frac{x^k}{k!}, \tag{3.2}$$

$$(p, q \geq 0, |x| < 1; \beta, \gamma \in \mathbb{C}_{>>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}).$$

The functions  $F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q)$  and  $\Phi_{\alpha;\eta}(\gamma; \beta; x; p; q)$  represent a further generalization of the extended Gauss hypergeometric function and a further generalization of the extended confluent hypergeometric function, respectively.

#### 3.1 Integral representations of Gauss and confluent hypergeometric functions

**Theorem 3.1.** For  $p, q \geq 0$ , the following integral formula holds:

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{1}{B(\gamma, \beta - \gamma)} \left[ \int_0^1 t^{\gamma-1} (1-t)^{\beta-\gamma-1} (1-xt)^{-\delta} \Phi \left( \eta; \xi; -\frac{p}{t} \right) E_\alpha \left( -\frac{q}{1-t} \right) dt \right], \tag{3.3}$$

where  $|\arg(1-x)| < \pi$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* By using the definition of  $B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)$  from (2.1) in place of  $B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)$  in (3.1), it is straightforward to derive the integral formula:

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\beta-\gamma-1} \Phi \left( \eta; \xi; -\frac{p}{t} \right) E_\alpha \left( -\frac{q}{1-t} \right) \left( \sum_{k=0}^{\infty} \frac{(\delta)_k (xt)^k}{k!} \right) dt, \tag{3.4}$$

where  $|x| < 1$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ . Using the extended binomial expansion,  $(1-xt)^{-\delta} = \sum_{k=0}^{\infty} \frac{(\delta)_k (xt)^k}{k!}$ , in (3.4), we obtain the integral in (3.3).  $\square$

**Theorem 3.2.** For  $p, q \geq 0$ , the following integral formulae are valid:

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{1}{B(\gamma, \beta - \gamma)} \int_0^\infty \varpi^{\gamma-1} (1 + \varpi)^{\delta-\beta} [1 + \varpi(1 - x)]^{-\delta} \times \Phi\left(\eta; \xi; -\frac{p(1 + \varpi)}{\varpi}\right) E_\alpha[-q(1 + \varpi)] d\varpi, \tag{3.5}$$

where  $|\arg(1 - x)| < \pi$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{2}{B(\gamma, \beta - \gamma)} \int_0^{\pi/2} \sin^{2\gamma-1} \varpi \cos^{2\beta-2\gamma-1} \varpi (1 - x \sin^2 \varpi)^{-\delta} \times \Phi_{\eta,\xi}(-p \csc^2 \varpi) E_\alpha(-q \sec^2 \varpi) d\varpi, \tag{3.6}$$

where  $|\arg(1 - x)| < \pi$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = 2^{1-\delta-\beta} \frac{1}{B(\gamma, \beta - \gamma)} \int_{-1}^1 (1 + \varpi)^{\gamma-1} (1 - \varpi)^{\beta-\gamma-1} [2 - x(1 + \varpi)]^{-\delta} \times \Phi\left(\eta; \xi; -\frac{2p}{1 + \varpi}\right) E_\alpha\left(-\frac{2q}{1 - \varpi}\right) d\varpi, \tag{3.7}$$

where  $|\arg(1 - x)| < \pi$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Setting  $t = \varpi/(1 + \varpi)$ ,  $t = \sin^2 \varpi$ , and  $t = (1 + \varpi)/2$  in (3.3), we derive the integrals in (3.5), (3.6), and (3.7), respectively.  $\square$

**Remark 3.3.** By employing reasoning similar to that used in equation (3.2), the integral representation of  $\Phi_{\alpha;\eta}$  can be obtained as follows:

$$\Phi_{\alpha;\eta}(\gamma; \xi; x; p; q) = \frac{1}{B(\gamma, \xi - \gamma)} \int_0^1 t^{\gamma-1} (1 - t)^{\xi-\gamma-1} e^{xt} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1 - t}\right) dt, \tag{3.8}$$

where  $p, q \geq 0$ ,  $\xi, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta \in \mathbb{C}_{>-}$ .

**Remark 4.4.** For the special case  $\alpha = 1$  and  $\xi = \eta$ , the integral representations in equations (3.3)–(3.8) correspond to those presented by Choi et al. [10]. Also, by setting  $\alpha = 1$ ,  $\xi = \eta$ , and  $p = q$ , the integral representations given in Chaudhry et al.[9] can be recovered. Furthermore, when  $p = 0 = q$ , the integrals in equations (3.3)–(3.8) simplify to the corresponding special cases discussed in [35].

**Theorem 3.4.** The following relation for  $F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q)$  holds true:

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = F_{\alpha;\eta}(\delta, \gamma + 1; \beta; x; p; q) + F_{\alpha;\eta}(\delta, \gamma; \beta + 1; x; p; q), \tag{3.9}$$

where  $p, q \geq 0$ ,  $|x| < 1$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* From (3.3), we get  $F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) =$

$$\frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma-1} (1 - t)^{\beta-\gamma-1} (1 - xt)^{-\delta} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1 - t}\right) dt,$$

where  $B(\gamma, \beta - \gamma)$  is the beta function.

Now, consider the following decomposition.

$$\begin{aligned}
 F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) &= \\
 &= \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^\gamma (1 - t)^{\beta - \gamma - 1} (1 - xt)^{-\delta} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1 - t}\right) dt. \\
 &+ \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma - 1} (1 - t)^{\beta + 1 - \gamma - 1} (1 - xt)^{-\delta} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_\alpha\left(-\frac{q}{1 - t}\right) dt. \\
 &= F_{\alpha;\eta}(\delta, \gamma + 1; \beta; x; p; q) + F_{\alpha;\eta}(\delta, \gamma; \beta + 1; x; p; q).
 \end{aligned}$$

Thus, the result is established. □

**Remark 3.5.** Similar reasoning can be applied to establish the following relation for  $\Phi_{\alpha;\eta}$

$$\Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = \Phi_{\alpha;\eta}(\gamma + 1; \beta; x; p; q) + \Phi_{\alpha;\eta}(\gamma; \beta + 1; x; p; q), \tag{3.10}$$

where  $p, q \geq 0$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta \in \mathbb{C}_{>-}$ .

### 3.2 Mellin transforms

**Theorem 3.6.** For  $p, q \in \mathbb{C}_{>}$ ,  $\epsilon_1 > 0$ ,  $\epsilon_2 > 0$ ,  $\beta, \gamma \in \mathbb{C}_{>>}$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ , the Mellin-Barnes integral formulae listed below are valid

$$\begin{aligned}
 F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) &= \frac{1}{(2\pi i)^2 B(\gamma, \beta - \gamma)} \int_{\epsilon_1 + i\infty}^{\epsilon_1 - i\infty} \int_{\epsilon_2 + i\infty}^{\epsilon_2 - i\infty} \\
 &\left[ \frac{\Gamma(\varphi)\Gamma(\xi)\Gamma(\eta - \varphi)}{\Gamma(\eta)\Gamma(\xi - \varphi)} \frac{\Gamma(\psi)\Gamma(1 - \psi)}{\Gamma(1 - \alpha\psi)} \right] \\
 &\times \frac{B(\gamma + \varphi, \beta + \psi - \gamma)}{B(\gamma, \beta - \gamma)} F(\delta, \gamma + \varphi; \beta + \varphi + \psi; x) p^{-\varphi} q^{-\psi} d\varphi d\psi,
 \end{aligned} \tag{3.11}$$

$$\begin{aligned}
 \Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) &= \frac{1}{(2\pi i)^2 B(\gamma, \beta - \gamma)} \int_{\epsilon_1 + i\infty}^{\epsilon_1 - i\infty} \int_{\epsilon_2 + i\infty}^{\epsilon_2 - i\infty} \\
 &\left[ \frac{\Gamma(\varphi)\Gamma(\xi)\Gamma(\eta - \varphi)}{\Gamma(\eta)\Gamma(\xi - \varphi)} \frac{\Gamma(\psi)\Gamma(1 - \psi)}{\Gamma(1 - \alpha\psi)} \right] \\
 &\times B(\gamma + \varphi, \beta + \psi - \gamma) \Phi(\gamma + \varphi; \beta + \varphi + \psi; x) p^{-\varphi} q^{-\psi} d\varphi d\psi.
 \end{aligned} \tag{3.12}$$

*Proof.* Multiplying both sides in (3.3) by  $p^{\varphi - 1} q^{\psi - 1}$ , then integrating the resultant identity from

0 to  $\infty$  with respect to  $p$  and  $q$ , we obtain

$$\begin{aligned}
 M\{F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q); \varphi, \psi\} &= \int_0^\infty \int_0^\infty p^{\varphi-1} q^{\psi-1} F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) dpdq \\
 &= \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma-1} (1-t)^{\beta-\gamma-1} (1-xt)^{-\delta} \\
 &\quad \times \left[ \int_0^\infty p^{\varphi-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) dp \int_0^\infty q^{\psi-1} E_\alpha\left(-\frac{q}{1-t}\right) dq \right] dt,
 \end{aligned} \tag{3.13}$$

$$\begin{aligned}
 M\{F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q); \varphi, \psi\} &= \frac{\Gamma(\varphi)\Gamma(\xi)\Gamma(\eta - \varphi)}{\Gamma(\eta)\Gamma(\xi - \varphi)} \frac{\Gamma(\psi)\Gamma(1 - \psi)}{\Gamma(1 - \alpha\psi)} \\
 &\quad \times \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\gamma+\varphi-1} (1-t)^{\beta+\psi-\gamma-1} (1-xt)^{-\delta} dt,
 \end{aligned} \tag{3.14}$$

$$\begin{aligned}
 M\{F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q); \varphi, \psi\} &= \frac{\Gamma(\varphi)\Gamma(\xi)\Gamma(\eta - \varphi)}{\Gamma(\eta)\Gamma(\xi - \varphi)} \frac{\Gamma(\psi)\Gamma(1 - \psi)}{\Gamma(1 - \alpha\psi)} \\
 &\quad \times \frac{B(\gamma + \varphi, \beta + \psi - \gamma)}{B(\gamma, \beta - \gamma)} F(\delta, \gamma + \varphi; \beta + \varphi + \psi; x).
 \end{aligned} \tag{3.15}$$

Now, (3.11) is proved by obtaining the inverse Mellin transforms of each side of (3.15). An analogous argument will prove (3.12).  $\square$

**Remark 3.7.** For the particular case  $\alpha = 1$  and  $\xi = \eta$ , the results in the equations (3.11) and (3.12) are reduced to the results presented by Choi et al. (see [10, Eq. (10.1) and (10.2)]).

### 3.3 Differentiation formulae

Differentiating equations (3.1) and (3.2) for  $x$  and using the following formulae will provide the differentiation formulae for generalized extended hypergeometric functions

$$B(\gamma, \beta - \gamma) = \frac{\beta}{\gamma} B(\gamma + 1, \beta - \gamma) \quad \text{and} \quad (\chi)_{k+1} = \chi(\chi + 1)_k.$$

**Theorem 3.8.** *The differentiation formulae listed below are valid*

$$\frac{d}{dx} F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{\delta\gamma}{\beta} F_{\alpha;\eta}(\delta + 1, \gamma + 1; \beta + 1; x; p; q), \tag{3.16}$$

$$\frac{d^k}{dx^k} F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{(\delta)_k (\gamma)_k}{(\beta)_k} F_{\alpha;\eta}(\delta + k, \gamma + k; \beta + k; x; p; q) \quad (k \in \mathbb{N}_0), \tag{3.17}$$

$$\frac{d^k}{dx^k} \Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = \frac{(\gamma)_k}{(\beta)_k} \Phi_{\alpha;\eta}(\gamma + k; \beta + k; x; p; q) \quad (k \in \mathbb{N}_0). \tag{3.18}$$

*Proof.* When we differentiate (3.1) with regard to  $x$ , we obtain

$$\frac{d}{dx} F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \sum_{k=1}^\infty \frac{B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} \frac{(\delta)_k x^{k-1}}{(k-1)!}.$$

This demonstrates (3.16) after substituting  $k$  by  $k + 1$  along with (3.3). This procedure repeatedly results from the generic form (3.17). An analogous argument demonstrates (3.18) for  $\Phi_{\alpha;\eta}(\gamma; \beta; x; p; q)$ .  $\square$

**Remark 3.9.** The specific cases  $\alpha = 1$  and  $\xi = \eta$  in equations (3.17) and (3.18) yield results that are documented by Choi et al. [10]. In [9], the authors presented the corresponding results for the special case  $p = q$  in equations (3.17) and (3.18), after setting  $\alpha = 1$  and  $\xi = \eta$ . Moreover, it is evident that the equivalent formulas for hypergeometric functions, as described in [35], are recovered in the particular case  $p = 0 = q$  of equations (3.17) and (3.18).

**3.4 Transformation formulae**

**Theorem 3.10.** *The transformation formulae shown below are valid:*

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = (1 - x)^{-\delta} F_{\alpha;\eta} \left( \delta, \beta - \gamma; \beta; \frac{-x}{1 - x}; q; p \right) \\ (p, q \geq 0; |\arg(1 - x)| < \pi; \beta, \gamma \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}), \tag{3.19}$$

and

$$\Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = e^x \Phi_{\alpha;\eta}(\beta - \gamma; \beta; -x; q; p) \\ (p, q \geq 0; \beta, \gamma \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}). \tag{3.20}$$

*Proof.* After substituting  $t$  by  $(1 - t)$  in (3.3) and using

$$[1 - x(1 - t)]^{-\delta} = (1 - x)^{-\delta} \left( 1 + \frac{x}{1 - x} t \right)^{-\delta},$$

we get

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = \frac{(1 - x)^{-\delta}}{B(\gamma, \beta - \gamma)} \int_0^1 t^{\beta - \gamma - 1} (1 - t)^{\gamma - 1} \left( 1 + \frac{x}{1 - x} t \right)^{-\delta} \\ \times \Phi \left( \eta; \xi; -\frac{p}{1 - t} \right) E_{\alpha} \left( -\frac{q}{t} \right) dt. \tag{3.21}$$

Thus,

$$F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q) = (1 - x)^{-\delta} F_{\alpha;\eta} \left( \delta, \beta - \gamma; \beta; \frac{-x}{1 - x}; q; p \right). \tag{3.22}$$

We also establish (3.20) similarly. □

**Theorem 3.11.** *For  $F_{\alpha;\eta}(\delta, \gamma; \beta; x; p; q)$ , the following generating function is valid*

$$\sum_{m=0}^{\infty} \frac{(\delta)_m F_{\alpha;\eta}(\delta + m, \gamma; \beta; x; p; q) w^m}{m!} = (1 - w)^{-\delta} F_{\alpha;\eta} \left( \delta, \gamma; \beta; \frac{x}{1 - w}; p; q \right). \tag{3.23}$$

*Proof.* Assume that the left-hand side of (3.23) is  $\Psi$ . Because of (3.1), we get:

$$\Psi = \sum_{m=0}^{\infty} (\delta)_m \left[ \sum_{k=0}^{\infty} \frac{(\delta + m)_k B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q) x^k}{B(\gamma, \beta - \gamma) k!} \right] \frac{w^m}{m!}. \tag{3.24}$$

In the foregoing formula, we can now obtain using the identity  $(a)_m (a + m)_k = (a)_k (a + k)_m$  as:

$$\Psi = \sum_{k=0}^{\infty} \frac{(\delta)_k B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} \left[ \sum_{m=0}^{\infty} \frac{(\delta + k)_m w^m}{m!} \right] \frac{x^k}{k!}. \tag{3.25}$$

On employing the binomial theorem to the inner summation, we obtain

$$\Psi = \sum_{k=0}^{\infty} \frac{(\delta)_k B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} \frac{(1 - w)^{-\delta - k} x^k}{k!}. \tag{3.26}$$

Simplifying further

$$\Psi = (1 - w)^{-\delta} \sum_{k=0}^{\infty} \frac{(\delta)_k B_{\alpha;\eta;\xi}(\gamma + k, \beta - \gamma; p; q)}{B(\gamma, \beta - \gamma)} \frac{1}{k!} \left( \frac{x}{1 - w} \right)^k. \tag{3.27}$$

Thus

$$\Psi = (1 - w)^{-\delta} F_{\alpha;\eta} \left( \delta, \gamma; \beta; \frac{x}{1 - w}; p; q \right), \tag{3.28}$$

which is the right-hand side of (3.23). Therefore,

$$\Psi = (1 - w)^{-\delta} F_{\alpha;\eta} \left( \delta, \gamma; \beta; \frac{x}{1 - w}; p; q \right), \tag{3.29}$$

where  $(p, q \geq 0; \beta, \gamma \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}; |w| < 1)$ . □

**Theorem 3.12.** *The transformation formulae shown below are valid*

$$F_{\alpha;\eta} \left( \delta, \gamma; \beta; 1 - \frac{1}{x}; p; q \right) = x^\delta F_{\alpha;\eta} (\delta, \beta - \gamma; \beta; 1 - x; q; p), \tag{3.30}$$

$$F_{\alpha;\eta} \left( \delta, \gamma; \beta; \frac{x}{1 + x}; p; q \right) = (1 + x)^\delta F_{\alpha;\eta} (\delta, \beta - \gamma; \beta; -x; q; p), \tag{3.31}$$

where  $p, q \geq 0; |\arg(1 - x)| < \pi; \beta, \gamma \in \mathbb{C}_{>}; \alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

*Proof.* Replacing  $x$  with  $1 - (1/x)$  and  $x/(1 + x)$  in (3.19) gives (3.30) and (3.31), respectively. □

**3.5 Differential and difference relations**

**Theorem 3.13.** *The subsequent relations are valid*

$$\Delta_a F_{\alpha;\eta} (\delta, \gamma; \beta; x; p; q) = \frac{\gamma x}{\beta} F_{\alpha;\eta} (\delta + 1, \gamma + 1; \beta + 1; x; p; q), \tag{3.32}$$

and

$$\frac{d}{dx} F_{\alpha;\eta} (\delta, \gamma; \beta; x; p; q) = \frac{\delta}{x} \Delta_\delta F_{\alpha;\eta} (\delta, \gamma; \beta; x; p; q). \tag{3.33}$$

The difference operator defined by  $\Delta_\theta$  is indicated here as

$$\Delta_\theta f(\theta, \dots) = f(\theta + 1, \dots) - f(\theta, \dots).$$

*Proof.* The difference operator that is obtained from (3.3) gives

$$\begin{aligned} \Delta_a F_{\alpha;\eta} (\delta, \gamma; \beta; x; p; q) &= F_{\alpha;\eta} (\beta + 1, \gamma; \beta; x; p; q) - F_{\alpha;\eta} (\delta, \gamma; \beta; x; p; q) \\ &= \frac{x}{B(\gamma, \beta - \gamma)} \int_0^1 t^\gamma (1 - t)^{\beta - \gamma - 1} (1 - xt)^{-\delta - 1} \\ &\quad \cdot \left[ \Phi \left( \eta; \xi; -\frac{p}{t} \right) E_\alpha \left( -\frac{q}{1 - t} \right) \right] dt. \end{aligned} \tag{3.34}$$

By substituting  $\delta + 1, \gamma + 1,$  and  $\beta + 1$  for the parameters  $\delta, \gamma,$  and  $\beta$  in (3.3), respectively, we obtain

$$\begin{aligned} F_{\alpha;\eta} (\delta + 1, \gamma + 1; \beta + 1; x; p; q) &= \frac{\beta}{\gamma} \frac{1}{B(\gamma, \beta - \gamma)} \int_0^1 t^\gamma (1 - t)^{\beta - \gamma - 1} (1 - xt)^{-\delta - 1} \\ &\quad \cdot \Phi \left( \eta; \xi; -\frac{p}{t} \right) E_\alpha \left( -\frac{q}{1 - t} \right) dt. \end{aligned} \tag{3.35}$$

Now, using (3.35) in (3.34) seems to yield (3.32). Subsequently, (3.33) is proved by applying the differentiation formula (3.16) in (3.32). □

**Remark 3.14.** By applying an argument similar to that used in Theorem 3.13, the following relations are obtained

$$\gamma \Delta_\gamma \Phi_{\alpha;\eta}(\gamma; \beta + 1; x; p; q) + \beta \Delta_\beta \Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = 0, \tag{3.36}$$

$$\frac{d}{dx} \Phi_{\alpha;\eta}(\gamma; \beta; x; p; q) = \frac{\gamma}{\beta} \Phi_{\alpha;\eta}(\gamma; \beta + 1; x; p; q) - \Delta_\beta \Phi_{\alpha;\eta}(\gamma; \beta; x; p; q). \tag{3.37}$$

**Remark 3.15.** The special case  $\alpha = 1$  and  $\xi = \eta$  in the equations from Theorem 3.13 and Note 3.14 corresponds to the results presented by Choi et al. [10]. Solving these equations for  $\alpha = 1$  and  $\xi = \eta$ , with the additional condition  $p = q$ , leads to the results reported by Chaudhry et al. [9].

**3.6 Summation formula**

Gauss formulated the following summation formula (see [10, Eq. 13.1]):

$${}_2F_1(\delta, \gamma; \beta; 1) = \frac{\Gamma(\beta)\Gamma(\beta - \delta - \gamma)}{\Gamma(\beta - \delta)\Gamma(\beta - \gamma)} = \frac{B(\gamma, \beta - \delta - \gamma)}{B(\gamma, \beta - \gamma)}, \tag{3.38}$$

where  $(\beta - \delta - \gamma) \in \mathbb{C}_{>}$ .

In 2014, Choi et al. (see[10, Eq. 13.2] ) obtained the following summation formula

$$F_{p,q}(\delta, \gamma; \beta; 1) = \frac{B(\gamma, \beta - \delta - \gamma; p, q)}{B(\gamma, \beta - \gamma)}, \tag{3.39}$$

where  $(p, q \geq 0; (\beta - \delta - \gamma) \in \mathbb{C}_{>})$ .

**Theorem 3.16.** *The summation formula shown below is valid*

$$F_{\alpha;\eta}(\delta, \gamma; \beta; 1; p; q) = \frac{B_{\alpha;\eta;\xi}(\gamma, \beta - \delta - \gamma; p; q)}{B(\gamma, \beta - \gamma)}, \tag{3.40}$$

where  $(p, q \geq 0; (\beta - \delta - \gamma) \in \mathbb{C}_{>>}; \alpha, \eta, \xi \in \mathbb{C}_{>>})$ .

*Proof.* The summation formula (3.40) is produced by taking  $x = 1$  in (3.3) and applying (2.1). □

**Remark 3.17.** To derive equation (3.39), consider the specific case  $\alpha = 1$  and  $\xi = \eta$  in equation (3.40). After setting  $\alpha = 1$  and  $\xi = \eta$ , the particular case  $p = q$  in equation (3.40) results in a corresponding outcome reported in Chaudhry et al. [9]. Furthermore, it is evident that equation (3.40), under the condition  $p = 0 = q$ , reduces equation (3.38), which represents Gauss’ summation formula.

**4 Application of extended beta distribution in statistics**

The generalized extended beta distribution is introduced within the framework of statistical distribution theory. Its probability density function is expressed as follows

$$f(v) = \begin{cases} \frac{1}{B_{\alpha;\eta;\xi}(u,v;p;q)} t^{u-1} (1-t)^{v-1} \Phi(\eta; \xi; -\frac{p}{t}) E_\alpha\left(-\frac{q}{1-t}\right), & \text{if } 0 < t < 1, \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

where  $p, q > 0; -\infty < u, v < \infty; \alpha, \eta, \xi \in \mathbb{C}_{>-}$ . In the following sections, we explore various fundamental properties of this distribution, as defined by (4.1). The  $k$ -th moment, denoted as  $E(X^k)$ , for any real  $k$ , can be obtained as

$$E(X^k) = \frac{B_{\alpha;\eta;\xi}(u+k, v; p; q)}{B_{\alpha;\eta;\xi}(u, v; p; q)},$$

where the parameters  $u, v \in \mathbb{R}, p, q \in \mathbb{R}^+$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ . For  $k = 1$ , the mean is obtained as

$$E(X) = \frac{B_{\alpha;\eta;\xi}(u + 1, v; p; q)}{B_{\alpha;\eta;\xi}(u, v; p; q)}.$$

The variance is expressed as

$$\begin{aligned} \text{Var}(X) &= \text{Var}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 = \mathbb{E}[(X - \mathbb{E}(X))^2] \\ &= \frac{B_{\alpha;\eta;\xi}(u + 2, v; p; q)B_{\alpha;\eta;\xi}(u, v; p; q) - [B_{\alpha;\eta;\xi}(u + 1, v; p; q)]^2}{[B_{\alpha;\eta;\xi}(u, v; p; q)]^2}. \end{aligned}$$

The coefficient of variation, which quantifies the ratio of the standard deviation to the mean, is given as

$$\text{C.V.} = \sqrt{\frac{B_{\alpha;\eta;\xi}(u + 2, v; p; q)B_{\alpha;\eta;\xi}(u, v; p; q)}{B_{\alpha;\eta;\xi}(u + 1, v; p; q)} - 1}.$$

Additional characteristics include the moment generating function (m.g.f.), computed as

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) = \frac{1}{B_{\alpha;\eta;\xi}(u, v; p; q)} \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta;\xi}(u + k, v; p; q)}{k!} t^k.$$

The characteristic function of the proposed distribution can be computed as follows

$$\begin{aligned} E(e^{it\mu}) &= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E(X^k), \\ E(e^{it\mu}) &= \frac{1}{B_{\alpha;\eta}(\mu, \nu; p, q)} \sum_{k=0}^{\infty} \frac{B_{\alpha;\eta}(\mu + k, \nu; p, q)}{k!} (it)^k. \end{aligned}$$

The cumulative distribution function (CDF) of the extended beta distribution is defined as

$$\begin{aligned} F(a) &= P(X < a) = \int_0^a f(t) dt, \\ F(a) &= \frac{B_{\alpha;\eta;a}(\mu, \nu; p, q)}{B_{\alpha;\eta}(\mu, \nu; p, q)}, \end{aligned}$$

where  $B_{\alpha;\eta;a}(\mu, \nu; p, q)$  is the incomplete extended beta function expressed as

$$B_{\alpha;\eta;a}(\mu, \nu; p, q) = \int_0^a t^{\mu-1} (1-t)^{\nu-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_{\alpha}\left(-\frac{q}{1-t}\right) dt,$$

with the conditions  $p, q \in \mathbb{C}_{>}, -\infty < \mu, \nu < \infty$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

The reliability function, which complements the cumulative distribution function, is given by

$$\begin{aligned} R(a) &= P(X \geq a) = 1 - F(a) = \int_a^{\infty} f(t) dt, \\ R(a) &= \frac{\hat{B}_{\alpha;\eta;a}(\mu, \nu; p, q)}{B_{\alpha;\eta}(\mu, \nu; p, q)}. \end{aligned}$$

Here, the upper incomplete extended beta function  $\hat{B}_{\alpha;\eta;a}(\mu, \nu; p, q)$  is defined as

$$\hat{B}_{\alpha;\eta;a}(\mu, \nu; p, q) = \int_a^{\infty} t^{\mu-1} (1-t)^{\nu-1} \Phi\left(\eta; \xi; -\frac{p}{t}\right) E_{\alpha}\left(-\frac{q}{1-t}\right) dt,$$

with the parameters  $p, q > 0, -\infty < \mu, \nu < \infty$ , and  $\alpha, \eta, \xi \in \mathbb{C}_{>-}$ .

## 5 Conclusion remarks

This work extends generalized beta and hypergeometric functions utilizing Kummer's and Mittag-Leffler's functions. Provides integral representations, summation formulas, Mellin transformations, and recurrence relations for the extended beta function. It also establishes features such as log-convexity and Turán-type inequality. In addition, we provide transformation, differentiation, and difference correlations for the extended hypergeometric functions. A statistical application defines a generalized extended beta distribution, methodically obtaining essential features such as variance, coefficient of variation, moment-generating function, characteristic function, cumulative distribution function (CDF), and complementary CDF.

## References

- [1] R. P. Agarwal, A. Kılıçman, R. K. Parmar, and A. K. Rathie, *Certain generalized fractional calculus formulas and integral transforms involving  $(p, q)$ -Mathieu-type series*, Adv. Difference Equ., **221**, 11, (2019). <https://doi.org/10.1186/s13662-019-2142-0>
- [2] M. Ali, *A further extensions of beta and related functions*, Palest. J. Math. **12**, no. 3, 65–73, (2023).
- [3] S. Bhattar, Nishant, Shyamsunder, S. D. Purohit, and D. L. Suthar, *A study of incomplete I-functions relating to certain fractional integral operators*, Appl. Math. Sci. Eng., **31** (1), 2252996, (2023). <https://doi.org/10.1080/27690911.2023.2252996>.
- [4] S. Bhattar, S. D. Purohit and Shyamsunder, *Several computational-based expansions for incomplete N -function using the Leibniz rule*, Advances in Mathematical Modelling and Applied Analysis and Computation (ICMMAAC 2023), (2), 306–314, (2023). [https://doi.org/10.1007/978-3-031-56304-1\\_20](https://doi.org/10.1007/978-3-031-56304-1_20).
- [5] S. Bhattar, K. Jangid, Shyamsunder, S. D. Purohit, *Determining glucose supply in blood using the incomplete I-function*, Partial Differ. Equ. Appl. Math., **10** (1), 100729, (2024). <https://doi.org/10.1016/j.padiff.2024.100729>.
- [6] R. N. Bracewell, *The Fourier transform and its applications*, third edition, McGraw-Hill Series in Electrical Engineering. Circuits and Systems, McGraw-Hill, New York, (1986).
- [7] B. C. Carlson, *Special functions of applied mathematics*, Academic Press, New York-London, (1977).
- [8] M. A. Chaudhry and S. M. Zubair, *On a class of incomplete gamma functions with applications*, Chapman & Hall/CRC, Boca Raton, FL, (2002). <https://doi.org/10.1201/9781420036046>
- [9] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, *Extension of Euler's beta function*, J. Comput. Appl. Math. **78** (1), 19–32, (1997). [https://doi.org/10.1016/S0377-0427\(96\)00102-1](https://doi.org/10.1016/S0377-0427(96)00102-1)
- [10] J. Choi, A. K. Rathie and R. K. Parmar, *Extension of extended beta, hypergeometric and confluent hypergeometric functions*, Honam Math. J. **36** (2), 357–385, (2014). <http://dx.doi.org/10.5831/HMJ.2014.36.2.357>
- [11] J. Choi and R. K. Parmar, *Fractional integration and differentiation of the  $(p, q)$ -extended Bessel function*, Bull. Korean Math. Soc., **55** (2), 599–610, (2018). <https://doi.org/10.4134/BKMS.b170193>
- [12] J. Choi, R. K. Parmar and P. Chopra, *The incomplete Srivastava's triple hypergeometric functions  $\gamma_B^H$  and  $\Gamma_B^H$* , Filomat, **30**, 1779–1787, (2016). <https://doi.org/10.2298/FIL1607779C>
- [13] J. Choi, R. K. Parmar, P. Chopra, *Extended Mittag-Leffler function and associated fractional calculus operators*, Georgian Math. J., **27** (2), 199–209, (2020). <https://doi.org/10.1515/gmj-2019-2030>.
- [14] J. Choi, R. K. Parmar and T. K. Pogány, *Mathieu-type series built by  $(p, q)$ -extended Gaussian hypergeometric function*, Bull. Korean Math. Soc. **54** (3), 789–797, (2017). <https://doi.org/10.4134/BKMS.b160313>
- [15] J. Choi and R. K. Parmar, *An extension of the generalized Hurwitz-Lerch Zeta function of two variables*, Filomat, **31** (1), 91–96, (2017). <https://doi.org/10.2298/FIL1701091>.
- [16] D. J. Maširević, R. K. Parmar and T. K. Pogány, *On  $(p, q)$ -extended Bessel and modified Bessel functions of the first kind*, Results Math. **72** (2), 617–632, (2017). <https://doi.org/10.1007/s00025-016-0649-1>
- [17] J. Choi and R. K. Parmar, *The incomplete Lauricella and fourth appell functions*, Far East J. Math. Sci., **96**, 315–328, (2015). <https://doi.org/10.5831/HMJ.2014.36.3.531>
- [18] J. Choi and R. K. Parmar, *The incomplete Srivastava's triple hypergeometric functions  $\gamma_A^H$  and  $\Gamma_A^H$* , Miskolc Math. Notes, **19**, 191–200, (2018). <https://doi.org/10.18514/MMN.2018.1503>
- [19] N. Khan and S. W. Khan, *A new extension of the Mittag-Leffler function*, Palest. J. Math. **9**, no. 2, 1–7, (2020).

- [20] M.-J. Luo, R. K. Parmar, and R. K. Raina, *On extended Hurwitz-Lerch zeta function*, J. Math. Anal. Appl. **448** (2), 1281–1304, (2017). <https://doi.org/10.1016/j.jmaa.2016.11.046>
- [21] R. K. Parmar, *Extended  $\tau$ -Hypergeometric Functions and Associated Properties*, Comptes Rendus Mathématique, **353** (5), 421–426, (2015). <https://doi.org/10.1016/j.crma.2015.01.016>.
- [22] R. K. Parmar and T. K. Pogány, *On Mathieu-type series for the unified Gaussian hypergeometric functions*, Appl. Anal. Discrete Math. **14** (1), 138–149, (2020). <https://doi.org/10.2298/AADM190525014P>
- [23] R. K. Parmar and R. K. Raina, *On a certain extension of the Hurwitz-Lerch zeta function*, An. Univ. Vest Timiș. Ser. Mat.-Inform. **52** (2), 157–170, (2014). <https://doi.org/10.2478/awutm-2014-0017>
- [24] R. K. Parmar, J. Choi and S. D. Purohit, *Further generalization of the extended Hurwitz-Lerch zeta functions*, Bol. Soc. Parana. Mat. **37** (1), 177–190, (2019). <https://doi.org/10.5269/bspm.v37i1.31842>
- [25] R. K. Parmar, P. Chopra and R. B. Paris, *On an extension of extended beta and hypergeometric functions*, J. Class. Anal. **11** (2), 91–106, (2017). <https://dx.doi.org/10.7153/jca-2017-11-07>
- [26] R. K. Parmar, G. V. Milovanović and T. K. Pogány, *Multi-parameter Mathieu, and alternating Mathieu series*, Appl. Math. Comput. **400**, 126099, 27, (2021). <https://doi.org/10.1016/j.amc.2021.126099>
- [27] R. K. Parmar, T. K. Pogány and R. K. Saxena, *On properties and applications of  $(p, q)$ -extended  $\tau$ -hypergeometric functions*, Comptes rendus. Mathématique, **356** (3), 278–282, (2018). <https://doi.org/10.1016/j.crma.2017.12.014>.
- [28] R. K. Parmar and T. K. Pogány, *On  $(p, q)$ -an extension of further members of Bessel-Struve functions class*, Miskolc Math. Notes **20** (1), 451–463, (2019). <https://doi.org/10.18514/MMN.2019.2608>
- [29] R. K. Parmar and T. K. Pogány, *Extended Srivastava's triple hypergeometric  $H_{A,p,q}$  function and related bounding inequalities*, J. Contemp. Math. Anal., **52** (6), 261–272, (2017). <https://doi.org/10.3103/S1068362317060036>.
- [30] R. K. Parmar and R. K. Saxena, *Incomplete extended Hurwitz-Lerch zeta functions and associated properties*, Commun. Korean Math. Soc., **32**, 287–304, (2017). <https://doi.org/10.4134/CKMS.c150227>.
- [31] R. K. Parmar and R. K. Saxena, *The Incomplete generalized  $\tau$ -Hypergeometric and second  $\tau$ -appell functions*, J. Korean Math. Soc., **53**, 363–379, (2016). <https://doi.org/10.4134/JKMS.2016.53.2.363>.
- [32] P. I. Pucheta, *An extended  $p - k$  Mittag-Leffler function*, Palest. J. Math. **9**, no. 2, 1–7, (2020).
- [33] E. D. Rainville, *Special functions*, The Macmillan Company, New York, 1960.
- [34] Shoukat Ali, Naresh Kumar Regar, and Subrat Parida, *On generalized extended beta and hypergeometric functions*, Honam Math. J. **46** (2), 313–334, (2024). <https://doi.org/10.5831/HMJ.2024.46.2.313>
- [35] L. J. Slater, *Generalized hypergeometric functions*, Cambridge Univ. Press, Cambridge, 1966.
- [36] Shyamsunder, D. Gangwar, *Impact of fractional order on reaction rates: Solutions to kinetic equations with incomplete  $N$ -function*, Comput. Methods Differ. Equ., **2**, 1–9, (2025). <https://doi.org/10.22034/cmde.2025.61995.2704>.
- [37] Shyamsunder, *Solutions of fractional kinetic equation involving incomplete  $N$ -function*, Computational Modeling and Sustainable Energy (ICCMSE 2023), pp 215–230, (2025). [https://doi.org/10.1007/978-981-97-6794-6\\_13](https://doi.org/10.1007/978-981-97-6794-6_13).
- [38] X. J. Yang, *Theory and applications of special functions for scientists and engineers*, Springer, Singapore, (2021).

### Author information

R. Kanaga, Department of Mathematics, Pondicherry University, 605014, India.  
E-mail: kanagavennila75@gmail.com

Received: 2025-01-22

Accepted: 2025-06-20