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ON UNRESTRICTED GENERALIZED LEONARDO QUATERNIONS AND RELATED SPINORS

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Abstract In the present study, we aim to introduce a new generalization of the Leonardo quaternions and related spinors. We define a new quaternion sequence called the unrestricted generalized Leonardo quaternion sequence. Moreover, we investigate some properties of these quaternions, such as the recurrence relations, Binet formula, several generating functions, and Vajda's identity. Then, we introduce a new spinor sequence called the unrestricted generalized Leonardo spinor sequence and obtain some formulas involving these spinors.

1 Introduction

The Fibonacci sequence is a kind of second-order linear recursive integer sequence, where each term is the sum of the two preceding ones. This sequence begins with 0 and 1. The Lucas sequence is defined by the formula that the sum of the two preceding numbers generates the succeeding Lucas number, and the Lucas sequence starts with 2 and 1. Obviously, the Lucas sequence shares recursive similarity with the Fibonacci sequence, and this leads to them having many similar identities. The reader may refer to references [14, 17, 23] for further information concerning the Fibonacci and Lucas sequences.

The Leonardo sequence, an integer sequence that is closely related to the Fibonacci sequence, has attracted attention from researchers in recent years. While both the Lucas and Leonardo sequences are closely related to the Fibonacci sequence, the Leonardo sequence differs from the Lucas sequence because it is defined by a non-homogeneous extension of the Fibonacci sequence's recurrence relation. Also, many properties of the Leonardo sequence have been given in [1, 6]. To date, the generalizations of the integer sequences have been examined in many ways by several authors. One of the generalizations examined for integer sequences is given by Kuhapatanakul and Chobsorn in [18] for the Leonardo sequence. The authors of [18] introduced a one-parameter generalization of the Leonardo sequence and obtained some significant formulas for this newly generalized Leonardo sequence. Also, with a different approach from the study of Kuhapatanakul and Chobsorn [18], a one-parameter generalization of the Leonardo sequence was introduced by Gökbaş in [9].

Quaternions, as 4-dimensional hyper-complex numbers, are important and useful mathematical tools and play an essential role in many fields, such as mathematics, physics, and engineering. The development of quaternions is attributed to the Irish mathematician W.R. Hamilton [13] in 1843. In the literature, we have seen many interesting studies involving quaternions with integer number components. For example, Horadam [15] defined the Fibonacci and Lucas quaternions by taking Fibonacci and Lucas numbers as coefficients instead of real numbers in quaternions, respectively. Then, Iyer [16] derived relations connecting the Fibonacci and Lucas quaternions. After many years, Halici [10] studied the Fibonacci quaternions and obtained the Binet formula for these numbers. Following that, the generalizations and properties of the Fibonacci and Lucas quaternions have been examined by many authors, some of which can be found in [11, 12, 20, 21, 22, 26]. More recently, Beites and Catarino defined and studied the Leonardo quaternions in [2]. In addition, Kumari et al. [19] defined the generalized Leonardo quaternions and investigated some of their algebraic properties.

A characteristic that all of these above mentioned studies have in common is that the coefficients of the analyzed quaternion sequences are composed of successive terms from the chosen integer sequences. However, there are also some studies in which the coefficients of quaternions are randomly selected from integer sequences. For instance, Daşdemir and Bilgici [7] introduced a generalization of the Fibonacci and Lucas quaternions by choosing arbitrary Fibonacci and Lucas numbers as coefficients, respectively. The authors of [7] called these new quaternions as the unrestricted Fibonacci and unrestricted Lucas quaternions, respectively. In [3], Bhati and Kumar defined the unrestricted Fibonacci and Lucas octonions in a similar way to the study in [7].

Spinors emerged from the study of the French mathematician E. Cartan on the representations of simple Lie algebra [4]. Furthermore, Vivarelli [24] defined a relationship between quaternions and spinors. In 2020, Erişir and Güngör [8] introduced the Fibonacci and Lucas spinors, which are spinor representations of the Fibonacci and Lucas quaternions. Then, Kumari et al. [19] introduced the generalized Leonardo spinors.

Inspired by some of the above studies, in this paper, we propose a new generalization of the Leonardo and generalized Leonardo quaternions, referred to as unrestricted generalized Leonardo quaternions. The unrestricted generalized Leonardo quaternions will be formed by choosing arbitrary generalized Leonardo numbers as the coefficients instead of real numbers in quaternions. In addition, we investigate some of their properties. Then, we present unrestricted generalized Leonardo spinors by establishing a relationship between the unrestricted generalized Leonardo quaternions and spinors.

2 Preliminaries

In this section, some definitions and background information that we use in this paper are presented.

The Fibonacci sequence $\{F_n\}_{n\geq 0}$ [17], Lucas sequence $\{L_n\}_{n\geq 0}$ [17], and Leonardo sequence $\{\mathcal{L}_n\}_{n\geq 0}$ [6] are defined, respectively, by the relations

$$\begin{aligned} &F_0 = 0, F_1 = 1; \quad F_{n+2} = F_{n+1} + F_n, \\ &L_0 = 2, L_1 = 1; \quad L_{n+2} = L_{n+1} + L_n, \\ &\mathcal{L}_0 = 1, \mathcal{L}_1 = 1; \quad \mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n + 1, \end{aligned}$$

where F_n, L_n, and $\mathcal{L}_n$ denote the n-th Fibonacci, Lucas, and Leonardo numbers, respectively.

More recently, for fixed positive integer k, Kuhapatanakul and Chobsorn defined the generalized Leonardo sequence $\{\mathcal{L}_{k,n}\}_{n\geq 0}$ in [18] by the relation

$$\mathcal{L}_{k,0} = 1, \mathcal{L}_{k,1} = 1; \quad \mathcal{L}_{k,n+2} = \mathcal{L}_{k,n+1} + \mathcal{L}_{k,n} + k. \tag{2.1}$$

The Binet formula for the generalized Leonardo numbers is given by Kumari et al. in [19] as

$$\mathcal{L}_{k,n} = (k+1)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right) - k, \tag{2.2}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

A quaternion q is represented by

$$q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3,$$

where q_0, q_1, q_2, and q_3 are real numbers, and $e_0 \cong 1$, e_1, e_2, and e_3 are the quaternionic units that satisfy the rules

$${e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 e_2 e_3 = -1, \quad e_1 e_2 = e_3 = -e_2 e_1, \quad e_2 e_3 = e_1 = -e_3 e_2, \quad e_3 e_1 = e_2 = -e_1 e_3. \tag{2.3}$$

For detailed information about quaternions, we refer to [13, 25].

In 1963, the n-th Fibonacci and Lucas quaternions are defined in [15] by Horadam, respectively, as

$$QF_n = F_n e_0 + F_{n+1} e_1 + F_{n+2} e_2 + F_{n+3} e_3$$

and

$$QL_n = L_n e_0 + L_{n+1} e_1 + L_{n+2} e_2 + L_{n+3} e_3,$$

where e_0, e_1, e_2, and e_3 are the quaternionic units defined as in (2.3).

Furthermore, in a similar manner, Beites and Catarino [2] defined the n-th Leonardo quaternion as

$$Q\mathcal{L}_n = \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3.$$

Then, in [19], Kumari et al. defined the n-th generalized Leonardo quaternion as

$$Q\mathcal{L}_{k,n} = \mathcal{L}_{k,n} e_0 + \mathcal{L}_{k,n+1} e_1 + \mathcal{L}_{k,n+2} e_2 + \mathcal{L}_{k,n+3} e_3.$$

On the other hand, in [7], Daşdemir and Bilgici defined the n-th unrestricted Fibonacci and Lucas quaternions, respectively, by

$$QF_n^{(x,y,z)} = F_n e_0 + F_{n+x} e_1 + F_{n+y} e_2 + F_{n+z} e_3$$

and

$$QL_n^{(x,y,z)} = L_n e_0 + L_{n+x} e_1 + L_{n+y} e_2 + L_{n+z} e_3,$$

where x, y, and z are arbitrary integers.

We now present the definition of the spinors that Cartan [4] mentioned.

For the vector space $\mathbb{C}^3$, assume that $v = (v_1, v_2, v_3) \in \mathbb{C}^3$ is an isotropic vector, that is, ${v_1}^2 + {v_2}^2 + {v_3}^2 = 0$. This vector can be related with two numbers $\mathcal{V}_1$ and $\mathcal{V}_2$ provided by

$$v_1 = {\mathcal{V}_1}^2 - {\mathcal{V}_2}^2, \quad v_2 = i({\mathcal{V}_1}^2 + {\mathcal{V}_2}^2), \quad \text{and} \quad v_3 = -2\mathcal{V}_1\mathcal{V}_2.$$

From the solutions of these above equations, it is obtained that

$$\mathcal{V}_1 = \pm\sqrt{\frac{v_1 - iv_2}{2}} \quad \text{and} \quad \mathcal{V}_2 = \pm\sqrt{\frac{-v_1 - iv_2}{2}}.$$

The 2-dimensional complex vectors described as

$$\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2) \cong \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}$$

are referred to as spinors by Cartan [4].

Let $\mathbb{H}$ and $\mathbb{S}$ denote the set of quaternions and spinors, respectively. Vivarelli [24] presented a relationship between any quaternion $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3$ and spinor $\mathcal{V} = \begin{bmatrix} \mathcal{V}_1 \\ \mathcal{V}_2 \end{bmatrix}$ by the map $f : \mathbb{H} \to \mathbb{S}$, where

$$f(q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3) = \begin{bmatrix} q_3 + iq_0 \\ q_1 + iq_2 \end{bmatrix} \equiv \mathcal{V}. \tag{2.4}$$

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\bar{\mathcal{V}}$ be the complex conjugate of $\mathcal{V}$. Then, the conjugate of spinor $\mathcal{V}$, denoted by $\tilde{\mathcal{V}}$, is introduced by Cartan [4] as

$$\tilde{\mathcal{V}} = iA\bar{\mathcal{V}}.$$

Furthermore, the mate of spinor $\mathcal{V}$, denoted by $\check{\mathcal{V}}$, is defined by Castillo [5] as

$$\check{\mathcal{V}} = -A\bar{\mathcal{V}}.$$

In 2020, Erişir and Güngör introduced the Fibonacci spinor sequence in [8]. They gave the correspondence between the set of Fibonacci quaternions $\mathbb{F}$ and the set of spinors $\mathbb{S}$ by the map $f : \mathbb{F} \to \mathbb{S}$, where

$$f(F_n e_0 + F_{n+1}e_1 + F_{n+2}e_2 + F_{n+3}e_3) = \begin{bmatrix} F_{n+3} + iF_n \\ F_{n+1} + iF_{n+2} \end{bmatrix} \equiv S_n.$$

Then, Kumari et al. [19] gave the correspondence between the set of generalized Leonardo quaternions $\mathbb{L}$ and the set of spinors $\mathbb{S}$ by the map $f : \mathbb{L} \to \mathbb{S}$, where

$$f(\mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+1}e_1 + \mathcal{L}_{k,n+2}e_2 + \mathcal{L}_{k,n+3}e_3) = \begin{bmatrix} \mathcal{L}_{k,n+3} + i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+1} + i\mathcal{L}_{k,n+2} \end{bmatrix} \equiv \mathfrak{L}_n.$$

3 The Unrestricted Generalized Leonardo Quaternions

In this section, we begin with the following necessary definition that describes the general term of the unrestricted generalized Leonardo quaternions.

Definition 3.1. Let n be a non-negative integer. For any integers x, y, and z, the n-th unrestricted generalized Leonardo quaternion is

$$Q\mathcal{L}_{k,n}^{(x,y,z)} = \mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+x}e_1 + \mathcal{L}_{k,n+y}e_2 + \mathcal{L}_{k,n+z}e_3, \tag{3.1}$$

where $\mathcal{L}_{k,n}$ is the n-th generalized Leonardo number, and e_0, e_1, e_2, and e_3 are the quaternionic units defined as in (2.3).

Therefore, $\{Q\mathcal{L}_{k,n}^{(x,y,z)}\}_{n\geq 0}$ is the sequence of quaternions called unrestricted generalized Leonardo quaternion sequence.

Remark 3.2. If we take $x = 1$, $y = 2$, and $z = 3$ in (3.1), we get the n-th generalized Leonardo quaternion in [19]. If we take $k = 1$, $x = 1$, $y = 2$, and $z = 3$ in (3.1), we get the n-th Leonardo quaternion in [2].

From (2.1) and (3.1), for $n \geq 0$, the non-homogeneous recurrence relation of the unrestricted generalized Leonardo quaternions can be obtained as

$$Q\mathcal{L}_{k,n+2}^{(x,y,z)} = Q\mathcal{L}_{k,n+1}^{(x,y,z)} + Q\mathcal{L}_{k,n}^{(x,y,z)} + \delta, \tag{3.2}$$

where $\delta := k(e_0 + e_1 + e_2 + e_3)$.

Moreover, from (3.2), the homogeneous recurrence relation of the unrestricted generalized Leonardo quaternions can be derived as

$$Q\mathcal{L}_{k,n+3}^{(x,y,z)} = 2Q\mathcal{L}_{k,n+2}^{(x,y,z)} - Q\mathcal{L}_{k,n}^{(x,y,z)}. \tag{3.3}$$

From now on, throughout this paper, let $\delta := k(e_0+e_1+e_2+e_3)$, $\alpha^* = e_0+\alpha^x e_1+\alpha^y e_2+\alpha^z e_3$ with $\alpha = \frac{1+\sqrt{5}}{2}$, and $\beta^* = e_0 + \beta^x e_1 + \beta^y e_2 + \beta^z e_3$ with $\beta = \frac{1-\sqrt{5}}{2}$.

Theorem 3.3. *The Binet formula of the unrestricted generalized Leonardo quaternions is*

$$Q\mathcal{L}_{k,n}^{(x,y,z)} = (k+1)\left(\frac{\alpha^{n+1}\alpha^* - \beta^{n+1}\beta^*}{\alpha - \beta}\right) - \delta. \tag{3.4}$$

Proof. By virtue of (2.2) and (3.1), we have

$$\begin{aligned}
Q\mathcal{L}_{k,n}^{(x,y,z)} &= \mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+x}e_1 + \mathcal{L}_{k,n+y}e_2 + \mathcal{L}_{k,n+z}e_3 \\
&= \left((k+1)\left(\frac{\alpha^{n+1}-\beta^{n+1}}{\alpha-\beta}\right)-k\right)e_0 + \left((k+1)\left(\frac{\alpha^{n+x+1}-\beta^{n+x+1}}{\alpha-\beta}\right)-k\right)e_1 \\
&\quad + \left((k+1)\left(\frac{\alpha^{n+y+1}-\beta^{n+y+1}}{\alpha-\beta}\right)-k\right)e_2 + \left((k+1)\left(\frac{\alpha^{n+z+1}-\beta^{n+z+1}}{\alpha-\beta}\right)-k\right)e_3 \\
&= (k+1)\left(\frac{\alpha^{n+1}(e_0+\alpha^x e_1+\alpha^y e_2+\alpha^z e_3)-\beta^{n+1}(e_0+\beta^x e_1+\beta^y e_2+\beta^z e_3)}{\alpha-\beta}\right) \\
&\quad - k(e_0+e_1+e_2+e_3) \\
&= (k+1)\left(\frac{\alpha^{n+1}\alpha^* - \beta^{n+1}\beta^*}{\alpha-\beta}\right) - \delta.
\end{aligned}$$

□

Example 3.4. For $n=3$, $x=2$, $y=1$, and $z=-1$ in (3.1), we get

$$\begin{aligned}
Q\mathcal{L}_{k,3}^{(2,1,-1)} &= \mathcal{L}_{k,3}e_0 + \mathcal{L}_{k,5}e_1 + \mathcal{L}_{k,4}e_2 + \mathcal{L}_{k,2}e_3 \\
&= (3+2k)e_0 + (8+7k)e_1 + (5+4k)e_2 + (2+k)e_3.
\end{aligned}$$

On the other hand, for $n=3$, $x=2$, $y=1$, and $z=-1$ in (3.4), we get

$$\begin{aligned}
Q\mathcal{L}_{k,3}^{(2,1,-1)} &= (k+1)\left(\frac{\alpha^4(e_0+\alpha^2 e_1+\alpha e_2+\alpha^{-1}e_3)-\beta^4(e_0+\beta^2 e_1+\beta e_2+\beta^{-1}e_3)}{\alpha-\beta}\right) \\
&\quad - k(e_0+e_1+e_2+e_3) \\
&= (k+1)\left(\frac{(\alpha^4-\beta^4)e_0+(\alpha^6-\beta^6)e_1+(\alpha^5-\beta^5)e_2+(\alpha^3-\beta^3)e_3}{\alpha-\beta}\right) \\
&\quad - k(e_0+e_1+e_2+e_3) \\
&= (k+1)(\alpha^2+\beta^2)(\alpha+\beta)e_0 + (k+1)(\alpha^3+\beta^3)(\alpha^2+\alpha\beta+\beta^2)e_1 \\
&\quad + (k+1)((\alpha-\beta)^4+5\alpha\beta(\alpha^2-\alpha\beta+\beta^2))e_2 + (k+1)(\alpha^2+\alpha\beta+\beta^2)e_3 \\
&\quad - k(e_0+e_1+e_2+e_3) \\
&= (k+1)(3e_0+8e_1+5e_2+2e_3) - k(e_0+e_1+e_2+e_3) \\
&= (2k+3)e_0 + (7k+8)e_1 + (4k+5)e_2 + (k+2)e_3,
\end{aligned}$$

by virtue of $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.

Theorem 3.5. *The ordinary generating function of the unrestricted generalized Leonardo quaternions is*

$$\begin{aligned}
g(t) &= \sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} t^n \\
&= \frac{Q\mathcal{L}_{k,0}^{(x,y,z)} + (Q\mathcal{L}_{k,1}^{(x,y,z)} - 2Q\mathcal{L}_{k,0}^{(x,y,z)})t + (Q\mathcal{L}_{k,2}^{(x,y,z)} - 2Q\mathcal{L}_{k,1}^{(x,y,z)})t^2}{1-2t+t^3}.
\end{aligned}$$

Proof. From (3.3), we get

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} t^n \\ &= Q\mathcal{L}_{k,0}^{(x,y,z)} + Q\mathcal{L}_{k,1}^{(x,y,z)} t + Q\mathcal{L}_{k,2}^{(x,y,z)} t^2 + \sum_{n=3}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} t^n \\ &= Q\mathcal{L}_{k,0}^{(x,y,z)} + Q\mathcal{L}_{k,1}^{(x,y,z)} t + Q\mathcal{L}_{k,2}^{(x,y,z)} t^2 + \sum_{n=3}^{\infty} \left(2Q\mathcal{L}_{k,n-1}^{(x,y,z)} - Q\mathcal{L}_{k,n-3}^{(x,y,z)}\right) t^n \\ &= Q\mathcal{L}_{k,0}^{(x,y,z)} + Q\mathcal{L}_{k,1}^{(x,y,z)} t + Q\mathcal{L}_{k,2}^{(x,y,z)} t^2 + 2t \sum_{n=3}^{\infty} Q\mathcal{L}_{k,n-1}^{(x,y,z)} t^{n-1} - t^3 \sum_{n=3}^{\infty} Q\mathcal{L}_{k,n-3}^{(x,y,z)} t^{n-3} \\ &= Q\mathcal{L}_{k,0}^{(x,y,z)} + (Q\mathcal{L}_{k,1}^{(x,y,z)} - 2Q\mathcal{L}_{k,0}^{(x,y,z)})t + (Q\mathcal{L}_{k,2}^{(x,y,z)} - 2Q\mathcal{L}_{k,1}^{(x,y,z)})t^2 \\ &\quad + 2t \sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} t^n - t^3 \sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} t^n. \end{aligned}$$

Thus, we obtain

$$g(t)(1 - 2t + t^3) = Q\mathcal{L}_{k,0}^{(x,y,z)} + (Q\mathcal{L}_{k,1}^{(x,y,z)} - 2Q\mathcal{L}_{k,0}^{(x,y,z)})t + (Q\mathcal{L}_{k,2}^{(x,y,z)} - 2Q\mathcal{L}_{k,1}^{(x,y,z)})t^2$$

which is the desired result. □

Theorem 3.6. *The exponential generating function of the unrestricted generalized Leonardo quaternions is*

$$\sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} \frac{t^n}{n!} = (k+1)\left(\frac{\alpha\alpha^* e^{\alpha t} - \beta\beta^* e^{\beta t}}{\alpha - \beta}\right) - \delta e^t.$$

Proof. By virtue of (3.4), we have

$$\begin{aligned} \sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \left((k+1)\left(\frac{\alpha^{n+1}\alpha^* - \beta^{n+1}\beta^*}{\alpha - \beta}\right) - \delta\right) \frac{t^n}{n!} \\ &= \frac{(k+1)\alpha\alpha^*}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{(\alpha t)^n}{n!} - \frac{(k+1)\beta\beta^*}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{(\beta t)^n}{n!} - \delta \sum_{n=0}^{\infty} \frac{t^n}{n!} \\ &= \frac{(k+1)\alpha\alpha^*}{\alpha - \beta} e^{\alpha t} - \frac{(k+1)\beta\beta^*}{\alpha - \beta} e^{\beta t} - \delta e^t \\ &= (k+1)\left(\frac{\alpha\alpha^* e^{\alpha t} - \beta\beta^* e^{\beta t}}{\alpha - \beta}\right) - \delta e^t. \end{aligned}$$

□

Corollary 3.7. *The Poisson generating function of the unrestricted generalized Leonardo quaternions is*

$$\sum_{n=0}^{\infty} Q\mathcal{L}_{k,n}^{(x,y,z)} \frac{t^n}{n!} e^{-t} = (k+1)\left(\frac{\alpha\alpha^* e^{\alpha(t-1)} - \beta\beta^* e^{\beta(t-1)}}{\alpha - \beta}\right) - \delta.$$

Theorem 3.8. *(Vajda's Identity) For non-negative integers m, n, and r, we have*

$$\begin{aligned} &Q\mathcal{L}_{k,n+m}^{(x,y,z)} \times Q\mathcal{L}_{k,n+r}^{(x,y,z)} - Q\mathcal{L}_{k,n}^{(x,y,z)} \times Q\mathcal{L}_{k,n+m+r}^{(x,y,z)} \\ &\quad = (k+1)^2(-1)^{n+1} F_m \left(\frac{\alpha^r \beta^* \alpha^* - \beta^r \alpha^* \beta^*}{\alpha - \beta}\right) \\ &\qquad + \left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+m}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,n+m+r}^{(x,y,z)} - Q\mathcal{L}_{k,n+r}^{(x,y,z)}\right), \end{aligned}$$

where F_m is the m-th Fibonacci number.

Proof. Using (3.4) to the left-hand side (LHS), we get

$$\begin{aligned}
LHS &= \left((k+1)\left(\frac{\alpha^{n+m+1}\alpha^* - \beta^{n+m+1}\beta^*}{\alpha-\beta}\right)-\delta\right)\left((k+1)\left(\frac{\alpha^{n+r+1}\alpha^* - \beta^{n+r+1}\beta^*}{\alpha-\beta}\right)-\delta\right)\\
&\quad-\left((k+1)\left(\frac{\alpha^{n+1}\alpha^* - \beta^{n+1}\beta^*}{\alpha-\beta}\right)-\delta\right)\left((k+1)\left(\frac{\alpha^{n+m+r+1}\alpha^* - \beta^{n+m+r+1}\beta^*}{\alpha-\beta}\right)-\delta\right)\\
&= \frac{(k+1)^2}{(\alpha-\beta)^2}(\alpha\beta)^{n+1}\alpha^*\beta^*\beta^r(\beta^m-\alpha^m) + \frac{(k+1)^2}{(\alpha-\beta)^2}(\alpha\beta)^{n+1}\beta^*\alpha^*\alpha^r(\alpha^m-\beta^m)\\
&\quad- Q\mathcal{L}_{k,n+m}^{(x,y,z)}\delta - \delta Q\mathcal{L}_{k,n+r}^{(x,y,z)} + Q\mathcal{L}_{k,n}^{(x,y,z)}\delta + \delta Q\mathcal{L}_{k,n+m+r}^{(x,y,z)}\\
&= \frac{(k+1)^2}{(\alpha-\beta)^2}(\alpha\beta)^{n+1}(\alpha^m-\beta^m)(\alpha^r\beta^*\alpha^*-\beta^r\alpha^*\beta^*)\\
&\quad+\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+m}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,n+m+r}^{(x,y,z)} - Q\mathcal{L}_{k,n+r}^{(x,y,z)}\right)\\
&= (k+1)^2(-1)^{n+1}F_m\left(\frac{\alpha^r\beta^*\alpha^*-\beta^r\alpha^*\beta^*}{\alpha-\beta}\right)\\
&\quad+\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+m}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,n+m+r}^{(x,y,z)} - Q\mathcal{L}_{k,n+r}^{(x,y,z)}\right).
\end{aligned}$$

Here, we use the Binet formula $F_m = \frac{\alpha^m-\beta^m}{\alpha-\beta}$ (see [17]). □

Corollary 3.9. *(Catalan's Identity) For $m \to -r$ in Theorem 3.8, we have*

$$\begin{aligned}
Q\mathcal{L}_{k,n-r}^{(x,y,z)} \times Q\mathcal{L}_{k,n+r}^{(x,y,z)} - \left(Q\mathcal{L}_{k,n}^{(x,y,z)}\right)^2 &= (k+1)^2(-1)^{n+r}F_r\left(\frac{\alpha^r\beta^*\alpha^*-\beta^r\alpha^*\beta^*}{\alpha-\beta}\right)\\
&\quad+\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n-r}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+r}^{(x,y,z)}\right).
\end{aligned}$$

Here, we use the formula $F_{-r} = (-1)^{r+1}F_r$ in [17].

Corollary 3.10. *(Cassini's Identity) For $m \to -r$, $r = 1$ in Theorem 3.8, we have*

$$\begin{aligned}
Q\mathcal{L}_{k,n-1}^{(x,y,z)} \times Q\mathcal{L}_{k,n+1}^{(x,y,z)} - \left(Q\mathcal{L}_{k,n}^{(x,y,z)}\right)^2 &= (k+1)^2(-1)^{n+1}\left(\frac{\alpha\beta^*\alpha^*-\beta\alpha^*\beta^*}{\alpha-\beta}\right)\\
&\quad+\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n-1}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+1}^{(x,y,z)}\right).
\end{aligned}$$

Corollary 3.11. *(d'Ocagne's Identity) For $r \to s-n$, $m = 1$ in Theorem 3.8, we have*

$$\begin{aligned}
&Q\mathcal{L}_{k,n+1}^{(x,y,z)} \times Q\mathcal{L}_{k,s}^{(x,y,z)} - Q\mathcal{L}_{k,n}^{(x,y,z)} \times Q\mathcal{L}_{k,s+1}^{(x,y,z)}\\
&\quad= (k+1)^2(-1)^{n+1}\left(\frac{\alpha^{s-n}\beta^*\alpha^*-\beta^{s-n}\alpha^*\beta^*}{\alpha-\beta}\right)\\
&\quad+\left(Q\mathcal{L}_{k,n}^{(x,y,z)} - Q\mathcal{L}_{k,n+1}^{(x,y,z)}\right)\delta + \delta\left(Q\mathcal{L}_{k,s+1}^{(x,y,z)} - Q\mathcal{L}_{k,s}^{(x,y,z)}\right).
\end{aligned}$$

4 The Unrestricted Generalized Leonardo Spinors

In this section, we give the definition of the unrestricted generalized Leonardo spinors and examine some fundamental properties of them.

Definition 4.1. Let $Q\mathcal{L}_{k,n}^{(x,y,z)} = \mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+x}e_1 + \mathcal{L}_{k,n+y}e_2 + \mathcal{L}_{k,n+z}e_3$ be the n-th unrestricted generalized Leonardo quaternion, and $\mathbf{L}$ denotes the set of unrestricted generalized Leonardo quaternions. Considering the linear transformation between the spinors and quaternions in (2.4), the following linear transformation occurs:

$f : \mathbf{L} \to \mathbb{S}$, where

$$f(\mathcal{L}_{k,n}e_0 + \mathcal{L}_{k,n+x}e_1 + \mathcal{L}_{k,n+y}e_2 + \mathcal{L}_{k,n+z}e_3) = \begin{bmatrix} \mathcal{L}_{k,n+z} + i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+x} + i\mathcal{L}_{k,n+y} \end{bmatrix} \equiv \mathfrak{L}_{k,n}.$$

Therefore, $\{\mathfrak{L}_{k,n}\}_{n\geq 0}$ is the sequence of spinors associated with the unrestricted generalized Leonardo quaternions, which we call the unrestricted generalized Leonardo spinor sequence.

Moreover, for $n \geq 0$, considering the recurrence relation (2.1), we have

$$\mathfrak{L}_{k,n+2} = \mathfrak{L}_{k,n+1} + \mathfrak{L}_{k,n} + k\mathcal{A}, \tag{4.1}$$

where $\mathcal{A} = \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}$.

From the relation (4.1), we can obtain the following:

$$\mathfrak{L}_{k,n+3} = 2\mathfrak{L}_{k,n+2} - \mathfrak{L}_{k,n}.$$

Let $\mathfrak{L}_{k,n}$ be the n-th unrestricted generalized Leonardo spinor. Then, we have

$$\bar{\mathfrak{L}}_{k,n} = \begin{bmatrix} \mathcal{L}_{k,n+z} - i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+x} - i\mathcal{L}_{k,n+y} \end{bmatrix} \qquad (Complex \quad Conjugate),$$

$$\tilde{\mathfrak{L}}_{k,n} = \begin{bmatrix} \mathcal{L}_{k,n+y} + i\mathcal{L}_{k,n+x} \\ -\mathcal{L}_{k,n} - i\mathcal{L}_{k,n+z} \end{bmatrix} \qquad (Spinor \quad Conjugate),$$

$$\breve{\mathfrak{L}}_{k,n} = \begin{bmatrix} -\mathcal{L}_{k,n+x} + i\mathcal{L}_{k,n+y} \\ \mathcal{L}_{k,n+z} - i\mathcal{L}_{k,n} \end{bmatrix} \qquad (Mate \quad of \quad Spinor).$$

Theorem 4.2. *Let* $\mathcal{A} = \begin{bmatrix} 1+i \\ 1+i \end{bmatrix}$. *For* $n \geq 0$, *the Binet-like formula of the unrestricted generalized Leonardo spinors is*

$$\mathfrak{L}_{k,n} = \frac{k+1}{\alpha-\beta}\left(\alpha^{n+1}\begin{bmatrix} \alpha^z + i \\ \alpha^x + i\alpha^y \end{bmatrix} - \beta^{n+1}\begin{bmatrix} \beta^z + i \\ \beta^x + i\beta^y \end{bmatrix}\right) - k\mathcal{A}. \tag{4.2}$$

Proof. By virtue of (2.2), we get

$$\begin{aligned}
\mathfrak{L}_{k,n} &= \begin{bmatrix} \mathcal{L}_{k,n+z} + i\mathcal{L}_{k,n} \\ \mathcal{L}_{k,n+x} + i\mathcal{L}_{k,n+y} \end{bmatrix} \\
&= \begin{bmatrix} (k+1)\left(\frac{\alpha^{n+z+1} - \beta^{n+z+1}}{\alpha-\beta}\right) - k + i\left((k+1)\left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha-\beta}\right) - k\right) \\ (k+1)\left(\frac{\alpha^{n+x+1} - \beta^{n+x+1}}{\alpha-\beta}\right) - k + i\left((k+1)\left(\frac{\alpha^{n+y+1} - \beta^{n+y+1}}{\alpha-\beta}\right) - k\right) \end{bmatrix} \\
&= \frac{k+1}{\alpha-\beta}\begin{bmatrix} \alpha^{n+z+1} - \beta^{n+z+1} + i\left(\alpha^{n+1} - \beta^{n+1}\right) \\ \alpha^{n+x+1} - \beta^{n+x+1} + i\left(\alpha^{n+y+1} - \beta^{n+y+1}\right) \end{bmatrix} - k\begin{bmatrix} 1+i \\ 1+i \end{bmatrix} \\
&= \frac{k+1}{\alpha-\beta}\left(\alpha^{n+1}\begin{bmatrix} \alpha^z + i \\ \alpha^x + i\alpha^y \end{bmatrix} - \beta^{n+1}\begin{bmatrix} \beta^z + i \\ \beta^x + i\beta^y \end{bmatrix}\right) - k\mathcal{A}.
\end{aligned}$$

□

Theorem 4.3. *The ordinary generating function of the unrestricted generalized Leonardo spinors is*

$$G(t) = \sum_{n=0}^{\infty} \mathfrak{L}_{k,n}t^n = \frac{\mathfrak{L}_{k,0} + (\mathfrak{L}_{k,1} - 2\mathfrak{L}_{k,0})t + (\mathfrak{L}_{k,2} - 2\mathfrak{L}_{k,1})t^2}{1 - 2t + t^3}.$$

Proof. Similar to the proof of Theorem 3.5. □

Theorem 4.4. *The exponential generating function of the unrestricted generalized Leonardo spinors is*

$$\sum_{n=0}^{\infty}\mathfrak{L}_{k,n}\frac{t^n}{n!}=\frac{k+1}{\alpha-\beta}\left(\alpha\begin{bmatrix}\alpha^z+i\\ \alpha^x+i\alpha^y\end{bmatrix}e^{\alpha t}-\beta\begin{bmatrix}\beta^z+i\\ \beta^x+i\beta^y\end{bmatrix}e^{\beta t}\right)-k\mathcal{A}e^t.$$

Proof. Using the Binet-like formula in (4.2), the proof can be done in a similar way to Theorem 3.6. □

Corollary 4.5. *The Poisson generating function of the unrestricted generalized Leonardo spinors is*

$$\sum_{n=0}^{\infty}\mathfrak{L}_{k,n}\frac{t^n}{n!}e^{-t}=\frac{k+1}{\alpha-\beta}\left(\alpha\begin{bmatrix}\alpha^z+i\\ \alpha^x+i\alpha^y\end{bmatrix}e^{(\alpha-1)t}-\beta\begin{bmatrix}\beta^z+i\\ \beta^x+i\beta^y\end{bmatrix}e^{(\beta-1)t}\right)-k\mathcal{A}.$$

5 Conclusions

In this study, the unrestricted generalized Leonardo quaternion sequence is defined and studied. This sequence generalizes previously introduced quaternion sequences given in [2] and [19]. Subsequently, by establishing a correspondence between any unrestricted generalized Leonardo quaternion and spinor, the unrestricted generalized Leonardo spinor sequence is introduced. Furthermore, several identities involving the unrestricted generalized Leonardo quaternions and related spinors are presented.

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