Palestine Journal of Mathematics

Vol 14(4)(2025) , 496–503

Amalgamation of rings defined by clean-like conditions

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Communicated by: Ayman Badawi

MSC 2020 Classifications: Primary 15A03, 13A15; Secondary 13B25, 13D05, 13F05.

Keywords and phrases: Units, nilpotents, idempotents, tripotent, 2-nil-clean ring, n-clean ring, nil-good ring, fine ring.

The authors would like to express their sincere thanks to the anonymous referee for his/her insightful suggestions toward the improvement of the paper.

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Abstract. This paper investigates necessary and sufficient conditions for an amalgamated algebra to inherit the 2-nil-clean as well as the nil-good and the fine properties. The new results recovers different settings of other constructions such as duplications and trivial ring extensions. All results are used to build new and illustrative examples arising as amalgamations.

1 Introduction

Everywhere in the text of the current paper, all our rings R are assumed to be commutative unless otherwise stated and containing the identity element 1, which in general differs from the zero element 0. A rings R is called a $clean\ ring$ if every element is the sum of a unit and an idempotent. Von Neumann regular rings and local rings are trivial examples of clean rings. This notion was introduced by Nicholson [19], and further studied by many authors. In the past ten years, there have been many investigations concerning variants of the clean properties. Additionally, several authors have studied versions of such properties in the case of $nil\text{-}clean\ rings$. In [13] Diesl introduced and he gave some basic examples and basic results of this notion. Following [13], an element $a \in R$ is called nil-clean if there is an idempotent $e \in R$ and a nilpotent element $b \in R$ such that $a = b + e$. A ring R is called nil-clean if every element in R is nil-clean. A ring R is 2-$nil\text{-}clean$ provided that every element in R is the sum of two idempotents and a nilpotent (see [6]). In [5] Călugăreanu and Lam defined the class of so-called $fine\ rings$ that are rings for which each non-zero element can be written as the sum of a unit and a nilpotent, if their presentation is unique the rings are called $uniquely\ fine$. It is apparent that division rings are fine as well as reduced fine rings are division. Some principal results concerning these rings are the following: (1) A local ring is fine if and only if it is a division ring; (2) A ring is uniquely fine if and only if it is a division ring. In [12] Danchev generalize that notion and introduce the notion of $nil\text{-}good$ as follows: R is called nil-good rings if for any $a \in R$, $a = w + u$, where w is a nilpotent element of R and u is either a unit or zero.

The amalgamation algebras along an ideal, introduced and studied by D'Anna, Finocchiaro and Fontana in [11] and defined as follows:

Let A and B be two rings, J an ideal of B and let $f : A \longrightarrow B$ be a ring homomorphism. In this setting, we can consider the following subring of $A \times B$:

$$A \bowtie^f J = \{(a, f(a) + j) \ / a \in A, j \in J\}$$

called the amalgamation of A and B along J with respect to f (see [2, 3, 8, 14, 18]). In particular, they have studied amalgamations in the frame of pullbacks which allowed them to establish numerous (prime) ideal and ring-theoretic basic properties for this new construction. This construction is a generalization of the amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [9, 10]). See for instance [1, 4, 15]. Moreover, other classical constructions (such as the $A + XB[X]$, $A + XB[[X]]$, and the $D + M$ constructions) can be studied as particular cases of the amalgamation [11, Examples 2.5 and 2.6].

In [7], Chhiti, Mahdou and Tamekkante studied the transfer of the clean property to amalgamated algebras along an ideal. In [4], Bakkari and Es-Saidi established necessary and sufficient conditions for the amalgamations to be nil-cleans. The purpose of this paper is to investigate the transfer of the notions of 2-nil-clean rings, nil-good rings and fine rings to amalgamations rings. Our work is motivated by an attempt to generate new families of these rings.

We denote $Nilp(R)$ for the set of all nilpotent elements in R; $Idem(R)$ the set of all idempotent elements of R and $U(R)$ is the group of all invertible elements of R.

2 Main result

In order to prove our results, we will use a characterization of 2-nil-clean rings which was established by Chen and Sheibani [6]

Theorem 2.1. *[6, Theorem 2.3] Let R be a ring. Then the following are equivalent:*

(i) R is 2-nil-clean.

(ii) For all $a \in R$, $a - a^3 \in Nilp(R)$.

(iii) For all $a \in R$, $a^2 \in R$ is nil-clean.

Taking into account Theorem 2.1, we can prove our first result.

Theorem 2.2. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B.*

(i) $A \bowtie^f J$ is 2-nil-clean if and only if A and $f(A) + J$ are 2-nil-clean.

(ii) Assume that $\{0\} \times J$ is a nil ideal of a ring $A \bowtie^f J$. Then $A \bowtie^f J$ is 2-nil-clean if and only if A is 2-nil-clean.

(iii) Assume that $f^{-1}(J) \times \{0\}$ is a nil ideal of a ring $A \bowtie^f J$. Then $A \bowtie^f J$ is 2-nil-clean if and only if $f(A) + J$ is 2-nil-clean.

Proof. (i) "Necessity". By [11, Proposition 5.1], we have $\frac{A\bowtie^f J}{\{0\}\times J} \cong A$ and $\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}} \cong f(A) + J$ then [6, Lemma 3.1] allows us to conclude that A and $f(A) + J$ are 2-nil-clean. "As for the sufficiency", let $(a, j) \in A \times J$. As A and $f(A) + J$ are 2-nil-clean, by Theorem 2.1, $a - a^3 \in Nilp(A)$ and $(f(a) + j) - (f(a) + j)^3 \in Nilp(f(A) + J)$. This shows that $(a, f(a) + j) - (a, f(a) + j)^3$ is nilpotent, which completes the proof.
2) and 3) are clear (by [6, Lemma 3.1]), completing the proof of Theorem 2.2. □

Remark 2.3. Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B.

(i) If $B = J$ then, $A \bowtie^f J$ is 2-nil-clean if and only if A and B are 2-nil-clean since $A \bowtie^f J = A \times B$.

(ii) If $f^{-1}(J) = \{0\}$ then, $A \bowtie^f J$ is 2-nil-clean if and only if $f(A) + J$ is 2-nil-clean (by [6, Proposition 5.1(3)]).

Let I be a proper ideal of A. The (amalgamated) duplication of A along I is a special amalgamation given by:

$$A \bowtie I := A \bowtie^{id_A} J = \{(a, a + i)/a \in A, i \in I\}$$

The next corollary is a consequence of Theorem 2.2 on the transfer of 2-nil-clean property to duplications.

Corollary 2.4. *Let A be a ring and I be an ideal of A. Then, $A \bowtie I$ is 2-nil-clean if and only if so is A.*

Proof. In this case, we have $f(A) + I = A + I = A$ and Theorem 2.2 completes the proof. □

In the next proposition, we present more conditions where the amalgamation ring is 2-nil-clean ring.

Proposition 2.5. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B. Then, $A \bowtie^f J$ is 2-nil-clean if one of the following two properties holds:*

(i) A is 2-nil-clean and $J \subseteq Nilp(B)$.

(ii) A is 2-nil-clean and $J \subseteq Idem(B)$.

Proof. By Theorem 2.2 (i), it suffices to show that $f(A) + J$ is 2-nil-clean.

(i) Let $(a, j) \in A \times J$. We can write $a = e_1 + e_2 + n$ where $e_1, e_2 \in Idem(A)$ and $n \in Nilp(B)$, since A is 2-nil-clean. As $J \subseteq Nilp(B)$, we get $f(a) + j = f(e_1) + f(e_2) + f(n) + j$ is sum of two idempotents $f(e_1)$, $f(e_2)$ and a nilpotent $f(n) + j$, as desired.

(ii) First, notice that $2J = 0$, since $J \subset Idem(B)$. Therefore, $f(e) + j$ is a idempotent of $f(A) + J$, for all $(e, j) \in Idem(A) \times J$. Next, let $(a, j) \in A \times J$. So, there exist two idempotents e_3, e_4 of A and a nilpotent n such that $a = e_3 + e_4 + n'$, since A is 2-nil-clean, hence $f(a) + j = (f(e_3) + j) + f(e_4) + f(n')$ is 2-nil-clean in $f(A) + J$.

□

Now, we give necessary and sufficient conditions for amalgamation to be 2-nil-clean, where $J \cap Nilp(B) = \{0\}$ and $J \cap Idem(B) = \{0\}$.

Corollary 2.6. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B.*

(i) If $J \cap Nilp(B) = \{0\}$ and $2J = 0$. Then, $A \bowtie^f J$ is 2-nil-clean if and only if A is 2-nil-clean and $J \subseteq Idem(B)$.

(ii) If $J \cap Idem(B) = \{0\}$. Then, $A \bowtie^f J$ is 2-nil-clean if and only if A is 2-nil-clean and $J \subseteq Nilp(B)$.

Proof. (i) By Theorem 2.2 (i) and Proposition 2.5 (ii), it remains to show that if $A \bowtie^f J$ is 2-nil-clean, then $J \subset Idem(B)$. First, note that if $(e, f(e) + j)$ is an idempotent of $A \bowtie^f J$, then j is an idempotent de B. Let $j \in J$. Without loss of generality, we may assume that $j \neq 0$. Hence, $(0, j)$ is 2-nil-clean element of $A \bowtie^f J$ and so by [7, Lemma 2.10], $(0, j) = (e_1, f(e_1) + k_1) + (e_2, f(e_2) + k_2) + (n, f(n))$ where $e_1, e_2 \in Idem(A)$ and $n \in Nilp(B)$. It follows that $e_1 + e_2 + n = 0$. Thus, $f(e_1) + f(e_2) + f(n) = 0$, hence $j = f(e_1) + k_1 + f(e_2) + k_2 + f(n) = k_1 + k_2$. As we see in the first lines of the proof $k_1, k_2 \in Idem(B)$ and so $j = k_1 + k_2 \in Idem(B)$ since $2J = 0$.

(ii) By Theorem 2.2 (i) and Proposition 2.5 (i), we need only prove that if $A \bowtie^f J$ is 2-nil-clean, then $J \subseteq Nilp(B)$. Let $j \in J$. By applying the [7, Lemma 2.5] and [7, Lemma 2.10] we get that $(0, j) = (e', f(e')) + (e'', f(e'')) + (n', f(n') + k')$ for some $e', e'' \in Idem(A)$ and $(n', k') \in Nilp(B) \times (J \cap Nilp(B))$. This imply that $0 = e' + e'' + n'$. Hence, $j = f(e') + f(e'') + f(n') + k' = k'$ which is a nilpotent element of B. Therefore, $J \subseteq Nilp(B)$ and this completes the proof of corollary.

□

Corollary 2.6 allows us to construct new original example of 2-nil-clean ring which is not nil-clean ring.

Example 2.7. Let $A := \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$; $B := \mathbb{Z}/6\mathbb{Z}$; $J := 3\mathbb{Z}/6\mathbb{Z}$ and $f : A \longrightarrow B$ be a ring homomorphism. Then, the amalgamation $A \bowtie^f J$ is a 2-nil-clean ring, which is not nil-clean ring.

Proof. Notice first that $J \cap Nilp(B) = \{0\}$ and $2J = 0$. Further, A is 2-nil-clean by [6, Theorem 2.3 (2)], since for all $a \in A$, $a - a^3 \in Nilp(A)$. So, $A \bowtie^f J$ is a 2-nil-clean by Corollary 2.6 (i) as $J \subseteq Idem(B)$. However, $A \bowtie^f J$ is not nil-clean by [4, Theorem 2.1] and [17, Theorem 3], since $(\bar{0}, \bar{2}) - (\bar{0}, \bar{2})^2 = (\bar{0}, \bar{1})$ which is not nilpotent of A. □

An element a of a ring R is called tripotent if $a^3 = a$, and a ring R is tripotent provided that every element in R is tripotent. The next result shows that the characterization for $A \bowtie^f J$ to be 2-nil-clean can be reconducted to the case where $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is tripotent.

Theorem 2.8. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B. Set $\bar{A} = A/Nilp(A)$, $\bar{B} = B/Nilp(B)$, $\pi : B \longrightarrow \bar{B}$ the canonical projection, and $\bar{J} = \pi(J)$. Consider the ring homomorphism $\bar{f} : \bar{A} \longrightarrow \bar{B}$ defined by setting $\bar{f}(\bar{a}) = \overline{f(a)}$. Then, $A \bowtie^f J$ is 2-nil-clean if and only if $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is tripotent.*

Before proving this theorem, we recall the following Lemma.

Lemma 2.9. *[6, Theorem 3.6] A ring R is 2-nil-clean if and only if $R/Nilp(R)$ is tripotent.*

Proof of Theorem 2.8. It is easy to see that $\bar{f}$ is well defined and it is a ring homomorphism. The map:

$$\begin{array}{cccc} \psi & : & A \bowtie^f J/Nilp(A \bowtie^f J) & \rightarrow & \bar{A} \bowtie^{\bar{f}} \bar{J} \\ & & \overline{(a, f(a) + j)} & \mapsto & (\bar{a}, \bar{f}(\bar{a}) + \bar{j}) \end{array}$$

is an isomorphism by [7, Proof of Theorem 2.9].
Assume that $A \bowtie^f J$ is 2-nil-clean. Then by Lemma 2.9, $A \bowtie^f J/Nilp(A \bowtie^f J)$ is tripotent and so $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is tripotent.
Conversely, assume that $\bar{A} \bowtie^{\bar{f}} \bar{J}$ is tripotent. Hence, $A \bowtie^f J/Nilp(A \bowtie^f J)$ is tripotent and so $A \bowtie^f J$ is 2-nil-clean by Lemma 2.9.

Theorem 2.8 enriches the literature with original examples of 2-nil-clean rings. The next corollary shows how to construct such rings.

Corollary 2.10. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B such that $J \subset Nilp(B)$. Then, $A \bowtie^f J$ is 2-nil-clean if and only if A is 2-nil-clean.*

Proof. By Theorem 2.8 $A \bowtie^f J$ is 2-nil-clean if and only if $\bar{A}$ is tripotent since $\bar{A} \bowtie^{\bar{f}} \bar{0} \cong \bar{A}$ (by [7, Proposition 5.1(3)]) and $\bar{J} = \bar{0}$. On the other hand, $\bar{A}$ is tripotent if and only if A is 2-nil-clean by Lemma 2.9. Therefore, $A \bowtie^f J$ is 2-nil-clean if and only if so is A. □

Example 2.11. Let R be an 2-nil-clean. Then by [6, Corollary 2.6 (2)]

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} ; a \in R \right\} \text{ and } B = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} ; a, b \in R \right\}$$

are 2-nil-clean rings. Let $f : A \longrightarrow B$ be defined by $f(\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and

$$J := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} ; b \in R \right\} \text{ be an ideal of } B$$

Then $A \bowtie^f J \simeq f(A) + J$ is an 2-nil-clean ring.

Remark 2.12. Let E be an A-module and set $B := A \ltimes E$. Let $\iota : A \hookrightarrow B$ be the canonical embedding. After identifying E with $0 \ltimes E$, E becomes an ideal of B. $A \ltimes E$ coincides with $A \bowtie^{\iota} E$ (cf.[11, Remark 2.8]).

Corollary 2.13. *With the above notation, the ring $A \ltimes E$ is 2-nil-clean if and only if A is 2-nil-clean.*

Proof. This result follows immediately from Corollary 2.10 since $(0 \ltimes E)^2 = (0)$. □

The following result establishes another necessary and sufficient conditions under which the amalgamation will be 2-nil-clean.

Theorem 2.14. *Let $f : A \longrightarrow B$ be a ring homomorphism and let J be an ideal of B. Then the following are equivalent:*

(i) $A \bowtie^f J$ *is* 2*-nil-clean.*

(ii) Any proper homomorphic image of $A \bowtie^f J$ *is* 2*-nil-clean.*

Proof. Assume that $A \bowtie^f J$ is 2-nil-clean. It is well known that any proper homomorphic image of a 2-nil-clean is again 2-nil-clean by [6, Lemma 3.1].
Conversely, assume that any proper homomorphic image of $A \bowtie^f J$ is 2-nil-clean. Therefore, A and $f(A) + J$ which are proper homomorphic images of $A \bowtie^f J$, are 2-nil-clean by [11, Proposition 5.1] and so $A \bowtie^f J$ is 2-nil-clean by Theorem 2.2, as desired. □

Given a positive integer n, R is n-clean if every element of R can be written as the sum of n units and an idempotent in R [17]. As an example, $\mathbb{Z}_{(p)} = \{\frac{m}{n}/m, n \in \mathbb{Z} \ and \ pgcd(p, n) = 1\}$ is 1-clean for every prime number p. Let $g(x)$ be a fixed polynomial in $R[x]$. An element $\alpha \in R$ is called $(n, g(x))$-clean if $\alpha = u_1 + u_2 + \cdots + u_n + s$, where $g(s) = 0$ and $u_1, u_2, \cdots, u_n$ are units in R. A ring is called an $(n, g(x))$-clean ring if every element in R is $(n, g(x))$-clean. The class of clean rings and n-clean ring rings is a proper subset of the class of $(n, g(x))$-clean rings [16]. With this in mind, we record the following result.

Theorem 2.15. *Let* $f : A \longrightarrow B$ *be a ring homomorphism and let* J *be an ideal of* B*.*

(i) If $A \bowtie^f J$ *is* n*-clean, then* A *and* $f(A) + J$ *are* n*-clean.*

(ii) Let $g(x) = \sum_{k=0}^{m} (a_k, f(a_k) + j_k)x^k \in (A \bowtie^f J)[x]$ *and* $h(x) = \sum_{k=0}^{m} a_k x^k \in A[x]$*. Then* A *is* $(n, h(x))$*-clean if* $A \bowtie^f J$ *is* $(n, g(x))$*-clean.*

(iii) Let $g(x) = \sum_{k=0}^{m} (a_k, f(a_k) + j_k)x^k \in (A \bowtie^f J)[x]$ *and* $h(x) = \sum_{k=0}^{m} (f(a_k) + j_k x^k) \in (f(A) + J)[x]$*. Then* $f(A) + J$ *is* $(n, h(x))$*-clean if* $A \bowtie^f J$ *is* $(n, g(x))$*-clean.*

Proof. (i) Since a homographic image of n-clean ring is n-clean by [17, Proposition 2.4] and $\frac{A \bowtie^f J}{\{0\} \times J} \cong A$ and $\frac{A \bowtie^f J}{f^{-1}(J) \times \{0\}} \cong f(A) + J$, A n-clean.

(ii) Let $\Phi : A \bowtie^f J \longrightarrow A$ be ring epimorphism defined by $\Phi(a, f(a) + j) = a$. Then by [16, Proposition 2.8], $A \bowtie^f J$ is $(n, g(x))$-clean implies that A is $(n, \Psi(g(x)))$-clean, where $\Psi : (A \bowtie^f J)[x] \longrightarrow A[x]$ defined by $\Psi(\sum_{k=0}^{m} (a_k, f(a_k) + j_k)x^k) = \sum_{k=0}^{m} \Phi((a_k, f(a_k) + j_k))x^k$. But $\Psi(g(x)) = h(x)$, so the result follows.

(iii) Let $\Phi : A \bowtie^f J \longrightarrow f(A) + J$ be ring epimorphism defined by $\Phi(a, f(a) + j) = f(a) + j$. Then by [16, Proposition 2.8], $A \bowtie^f J$ is $(n, g(x))$-clean implies that $f(A) + J$ is $(n, \Psi(g(x)))$-clean, where $\Psi : (A \bowtie^f J)[x] \longrightarrow (f(A)+J)[x]$ defined by $\Psi(\sum_{k=0}^{m} (a_k, f(a_k) + j_k)x^k) = \sum_{k=0}^{m} \Phi((a_k, f(a_k) + j_k))x^k$. But $\Psi(g(x)) = h(x)$, so the result follows.

□

The question will raise now: under what conditions on a ring A and an ideal J, the ring $A \bowtie^f J$ is n-clean or $(n, g(x))$-clean? The following results give partial answer to this question.

Proposition 2.16. *Let* $f : A \longrightarrow B$ *be a ring homomorphism such that* $J = B$*.*

(i) $A \bowtie^f J$ *is* n*-clean if and only if* A *and* B *are* n*-clean.*

(ii) Let $g(x) = \sum_{k=0}^{m} (a_k, f(a_k))x^k \in (A \bowtie^f J)[x]$, $h(x) = \sum_{k=0}^{m} a_k x^k \in A[x]$ *and* $k(x) = \sum_{k=0}^{m} f(a_k)x^k \in B[x]$*. Then* $A \bowtie^f J$ *is* $(n, g(x))$*-clean if and only if* A *is* $(n, h(x))$*-clean and* B *is* $(n, k(x))$*-clean.*

Proof. Note that, if $J = B$, then $A \bowtie^f J \cong A \times B$. □

Theorem 2.17. *Let* $f : A \longrightarrow B$ *be a ring homomorphism and let* J *be an ideal of* B *such that* $J \subset Nilp(B)$*.*

(i) $A \bowtie^f J$ *is* n*-clean if and only if* A *is* n*-clean.*

(ii) Let $g(x) = \sum_{k=0}^{m}(a_k, f(a_k))x^k \in (A \bowtie^f J)[x]$*,* $h(x) = \sum_{k=0}^{m} a_k x^k \in A[x]$ *and* $k(x) = \sum_{k=0}^{m} f(a_k)x^k \in (f(A)+J)[x]$*. Then* $A \bowtie^f J$ *is* $(n, g(x))$*-clean if and only if* A *is* $(n, h(x))$*-clean and* $f(A)+J$ *is* $(n, k(x))$*-clean.*

Proof. (i) Let $(a, f(a)+j) \in A \bowtie^f J$. Since A is n-clean, then there exist $u_1, u_2, ..., u_n \in U(A)$ and $e \in Idem(A)$ satisfy $a = u_1 + u_2 + ... + u_n + e$ and $(u_1, f(u_1)+j) \in U(A \bowtie^f J)$ by [1, Lemma 2.3] since $J \subset Nilp(B)$. So, $(a, f(a)+j) = (u_1, f(u_1)+j) + (u_2, f(u_2)) + ... + (u_n, f(u_n)) + (e, f(e))$ is a sum of n-units and an idempotent. Thus $A \bowtie^f J$ is n-clean.

(ii) is obvious.

□

By [5], a nonzero ring is said to be fine if every nonzero element in it is a sum of a unit and a nilpotent element, while if their presentation is unique the rings are called uniquely fine. In [5, Proposition 2.5], the authors shows that if $Nilp(R)$ is a nonzero additive subgroup in a ring R, then R is not a fine ring. For this, we examine the transfer of the property of fine to the amalgamation of noncommutative rings.

Theorem 2.18. *Let* $f : A \longrightarrow B$ *be a ring homomorphism and let* J *be an ideal of* B*.*

(i) If $A \bowtie^f J$ *is a fine ring, then* A *and* $f(A)+J$ *are fine rings.*

(ii) Assume that $\frac{f(A)+J}{J}$ *is uniquely fine. Then,* $A \bowtie^f J$ *is a fine ring if and only if* A *and* $f(A)+J$ *are fine rings.*

Proof. (i) Assume that R is fine. By [5, Theorem 2.3 (3)] every homomorphic image of a fine ring is fine. So A and $f(A)+J$ are fine rings by [11, Proposition 5.1].

(ii) By (1), it suffices to prove " if" assertion. Suppose that A and $f(A)+J$ are fine rings and consider $(a, j) \in A \times J$. Since A is fine, we can write $a = u + n$, where $u \in U(A)$ and $n \in Nilp(A)$. On the other hand, since $f(A)+J$ is fine, $f(a)+j = f(x)+j_1+f(y)+j_2$ with $f(x)+j_1 \in U(f(A)+J)$ and $f(y)+j_2 \in Nilp(f(A)+J)$. It is clear $\overline{f(a)} = \overline{f(a)+j} \in \frac{f(A)+J}{J}$, $\overline{f(x)} = \overline{f(x)+j_1} \in U(\frac{f(A)+J}{J})$ and $\overline{f(y)} = \overline{f(y)+j_2} \in \frac{Nilp(f(A)+J)+J}{J} \subseteq Nilp(\frac{f(A)+J}{J})$. From $a = u + n$ we have $\overline{f(a)} = \overline{f(u)} + \overline{f(n)}$. So $\overline{f(x)} + \overline{f(y)} = \overline{f(a)} = \overline{f(u)} + \overline{f(n)}$. Since $\frac{f(A)+J}{J}$ is uniquely fine, $\overline{f(x)} = \overline{f(u)}$, $\overline{f(y)} = \overline{f(n)}$. So there exist $j', j'' \in J$ such that $f(x) = f(u) + j'$, $f(y) = f(n) + j''$ and then $(a, f(a)+j) = (u, f(u)+j'+j_1) + (n, f(n)+j''+j_2)$.

□

Corollary 2.19. *Let* A *be a fine ring,* $f : A \longrightarrow B$ *be a ring homomorphism and* $f(A)+J$ *be a fine ring such that* J *is an ideal of* B *and maximal of* $f(A)+J$*. Then,* $A \bowtie^f J$ *is fine.*

Proof. Since J is maximal of $f(A)+J$, $\frac{f(A)+J}{J}$ is uniquely fine, by [5, Proposition 2.1] and hence the proof is complete by Theorem 2.18 (2). □

Proposition 2.20. *Let* $f : A \longrightarrow B$ *be a ring homomorphism and* J *an ideal of* B*.*

(i) Assume that $J \subseteq Nilp(B)$*. If* A *is fine, then* $A \bowtie^f J$ *is fine.*

(ii) Assume that $f^{-1}(J) \subseteq Nilp(A)$*. If* $f(A)+J$ *is fine then* $A \bowtie^f J$ *is fine.*

Proof. (i) First, note that if $\overline{(u, f(u)+j)} \in U(\frac{A \bowtie^f J}{\{0\} \times J})$, then $(u, f(u)+j) \in U(A \bowtie^f J)$. Indeed, $\overline{(u, f(u)+j)} \in U(\frac{A \bowtie^f J}{\{0\} \times J})$ implies that there is $\overline{(v, f(v)+e)} \in \frac{A \bowtie^f J}{\{0\} \times J}$ such that $\overline{(u, f(u)+j)(v, f(v)+e))} = \overline{(1,1)}$. It follows that $(uv-1, (f(u)+j)(f(v)+e)-1) \in \{0\} \times J$, that is $uv = 1$ and $(f(u)+j)(f(v)+e) = 1+k)$ with $k \in J$. Since $J \subseteq Nilp(B)$, we get that $1+k \in U(f(A)+J)$. Thus, $(f(u)+j)(f(v)+e)(f(v')+e') = (1+k)(f(v')+e') = 1$, for some $(f(v')+e') \in f(A)+J$. Hence, $(u, f(u)+j)$ is a unit in $A \bowtie^f J$. Next, let $(a, f(a)+e) \in A \bowtie^f J$. Since $\frac{A\bowtie^f J}{\{0\}\times J} \simeq A$ is fine, for $\overline{(a, f(a)+e)} \in \frac{f(A)+J}{\{0\}\times J}$ we can assume that $\overline{(a, f(a)+e)} = \overline{(u, f(u)+j)} + \overline{(n, f(n)+k)}$, for some unit $\overline{(u, f(u)+j)}$ and some nilpotent $\overline{(n, f(n)+k)}$ in $\frac{A\bowtie^f J}{\{0\}\times J}$. So $((a, f(a)+e)+\{0\}\times J) - ((u, f(u)+j)+\{0\}\times J)$=$(n, f(n)+k)+\{0\}\times J$. Therefore $((a, f(a)+e)-(u, f(u)+j))+\{0\}\times J$=$(n, f(n)+k)+\{0\}\times J$. Hence $((a, f(a)+e)-(u, f(u)+j))$ is nilpotent modulo $\{0\}\times J$, since $\{0\}\times J \subseteq Nilp(A\bowtie^f J)$, this means $((a, f(a)+e)-(u, f(u)+j))$ is nilpotent. That is $(a, f(a)+e)$ is fine.

(ii) Assume that $f^{-1}(J) \subseteq Nilp(A)$. By [11, Proposition 5.1] we have $\frac{A\bowtie^f J}{f^{-1}(J)\times\{0\}} \cong f(A)+J$. With a similar argument as in the statement (1), we get that $A \bowtie^f J$ is fine.

□

Corollary 2.21. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B such that $J \subseteq Nilp(B)$.*

(i) $A \bowtie^f J$ is a fine ring if and only if so is A.

(ii) If f is a monomorphism. Then, $A \bowtie^f J$ is a fine ring if and only if so is $f(A)+J$.

Proof. (i) By Theorem 2.18 (i) and Proposition 2.20 (i), we obtain the result.

(ii) As $J \subseteq Nilp(B)$ and f is a monomorphism, $f^{-1}(J) \subseteq Nilp(A)$, we complete the proof by Proposition 2.20 (ii).

□

In the above result, the assumption $J \subseteq Nilp(B)$ is not redundant, as shown by the next example.

Example 2.22. Let A be a fine ring, $B := A[X]$ and $J := XA[X]$. Consider the ring homomorphism $f : A \longrightarrow B$ $(f(a) = a)$. It is easily seen that $J \nsubseteq Nilp(B)$ and $A[X]$ is not fine. So, $A \bowtie^f J \simeq f(A)+J = A[X]$ is not fine.

Now we give the following characterization of nil-good amalgamated rings. Recall from [12], a ring R is nil-good if for any $a \in R$, $a = w+u$, where $w \in Nilp(R)$ and $u \in U(R)\cup\{0\}$.

Proposition 2.23. *Let $f : A \longrightarrow B$ be a ring homomorphism and J an ideal of B.*

(i) If $A \bowtie^f J$ is a nil-good ring, then A and $f(A)+J$ are nil-good rings.

(ii) $A \bowtie^f J$ is nil-good if and only if A is nil-good and $J \subseteq Nilp(B)$.

Proof. (i) By [12, Example 4], every homomorphic image of a nil-good ring is nil-good, so A and $f(A)+J$ are nil-good.

(ii) By (1), we need only prove that if $A \bowtie^f J$ is a nil-good ring, then $J \subseteq Nilp(B)$. Let $j \in J$, we write that $(0, j) = (u, f(u)+e)+(n, f(n)+k)$ $(*)$, where $(u, f(u)+e) \in U(A\bowtie^f J)\cup\{0\}$ and $(n, f(n)+k) \in Nilp(A\bowtie^f J)$. If $(u, f(u)+e) \in U(A\bowtie^f J)$, then the $(*)$ implies that $u = -n \in Nilp(A)$, which contradicts the fact that $A \neq (0)$ and so $(0, j) = (n, f(n)+k)$, forcing $j \in Nilp(B)$. Conversely, let $(a, f(a)+j) \in A \bowtie^f J$ and write $a = u+n$ where $u \in U(A)\cup\{0\}$ and $n \in Nilp(A)$. Then if $u = 0$, $(a, f(a)+j)$ has the nil-good decomposition $(a, f(a)+j) = (n, f(n)+j)$ since $J \subseteq Nilp(B)$ and if $u \in U(A)$, $(a, f(a)+j)$ has the nil-good decomposition $(a, f(a)+j) = (u+n, f(u)+f(n)+j) = (u, f(u))+(n, f(n)+j)$ since $J \subseteq Nilp(B)$.

□

Clearly, fine rings are nil-good rings. But in general, nil-good rings may not be fine. Then to enrich the literature with new examples of nil-good ring which not fine, we propose the following class of examples.

Example 2.24. Let $f : \frac{\mathbb{Z}}{4\mathbb{Z}} \longrightarrow B$ be a ring homomorphism and J an ideal of B such that $J \subseteq Nilp(B)$. Then, the amalgamation $R := A \bowtie^f J$ satisfies the following statements:.

Proof. (i) R is a nil-good ring, by Proposition 2.23 (ii), since $\frac{\mathbb{Z}}{4\mathbb{Z}}$ is a nil-good ring.

(ii) R is not a fine ring by Theorem 2.18 since $\bar{2}$ is not fine element of $\frac{\mathbb{Z}}{4\mathbb{Z}}$.

□

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Received: 2025-01-27
Accepted: 2025-06-24