# Solvability of Non-separated type Integral Boundary value Problems Involving the Combined Caputo Fractional Derivative

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**Abstract**. In this paper, we introduce the existence and uniqueness of solutions for non-separated type integral boundary value problems involving combined Caputo fractional derivatives for orders $\zeta, \sigma \in (1, 2]$. Our results are based on the Banach fixed point theorem which guarantees the existence and uniqueness of fixed points for a certain type of self-maps defined on metric spaces. In addition, the Schaefer and Krasnoselskii fixed point theorems are employed to obtain the existence of solutions, followed by three illustrative numerical examples.

## 1 Introduction

Fractional calculus concerns derivatives and integrals of non-integer order. It has found broad applications in diverse scientific and engineering fields. This field of research has been successfully utilized in areas such as control theory, signal processing, image analysis, chemical kinetics, economics and many other related fields [14, 16, 17, 19, 21, 23]. Recently, the study of fractional calculus has been extended to understand and model complex physical phenomena, especially those involving memory effects.

In the last decade, there has been immense focus on the Riemann-Liouville and Caputo fractional derivatives when dealing with the study of fractional differential equations, more specifically initial and boundary value problems. Despite being used commonly, the operators are naturally one-sided, where Riemann-Liouville derivatives are defined in terms of the lower limit of integration, and Caputo derivatives involve only past data. They consequently fail to achieve a satisfactory depiction of entirely memory-dependent behaviour within systems guided by past as well as future system states. Conversely, the Riesz space fractional derivative represents a two-sided, symmetric operator that efficiently incorporates non-local memory contributions in both time directions. This property allows for more comprehensive and realistic modeling of systems with time-symmetric memory effects. One of the important uses of the Riesz derivative is in anomalous diffusion, where it has been used effectively to describe concentration dynamics that are influenced by both forward and backward memory components [6, 9, 10].

New developments in the theory of fractional differential equations have gone beyond traditional methodologies, bringing new analytical methods and existence results to a wide range of complex systems. For example, in [3], authors used a power series technique named residual power series method for the solution of a certain type fractional order integro-differential equations. The authors considered Caputo sense fractional derivatives for their study. In [12], the authors have provided the inverse nodal problem for the fractional Sturm-Liouville problem and also discussed the stability for this problem as well. They have established asymptotic forms for nodal parameters and the potential function can be reconstructed with a limit of nodal parameters. In [4], the authors designed a scheme for computation of some analytical results for fractional order fuzzy diffusion problem under Atangana-Baleanu and Caputo fractional differential operator. In their article, Shah et al. [20] derived some theoretical and computational

findings for a coupled system of drug therapy process. The discussed problem is studied using the concept of piecewise modeling.

On the other hand, the notion of combined Caputo derivative operator $^C\mathcal{D}^{\zeta,\sigma}$, introduced by Toprakseven [24], offers a unique approach to model various phenomena. By combining left and right Caputo fractional derivatives of orders $\zeta$ and $\sigma$ on interval $(1, 2]$, the author extended the capabilities of fractional calculus and provided a potential alternative to the well-established Riesz space fractional derivative which combines left and right Riemann-Liouville derivatives [13, 18]. While the Riesz derivative [15] has proven effective in modeling non-conservative systems, the combined Caputo derivative provides a potentially valuable alternative for a wider range of applications.

Applications for integral boundary conditions can be found in many applied domains, including population dynamics, chemical engineering, blood flow issues, thermo-elasticity, and underground water flow. To obtain an in-depth explanation of the integral boundary conditions, we refer the readers to the recent articles [1, 2, 11, 22]. For more details on non-local and integral boundary conditions, see [5, 7, 8] and the references therein.

In this study, we investigate the existence and uniqueness results for non-separated type integral boundary value problem involving combined Caputo fractional derivative given by

$$^C_0\mathcal{D}^{\zeta,\sigma}_{\mathcal{L}}\varpi(\tau) = \Im(\tau, \varpi(\tau)), \ \tau \in [0, \mathcal{L}], \ \zeta, \sigma \in (1, 2]$$

$$\alpha\varpi(0) + \beta\varpi'(0) = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi, \ \alpha\varpi(\mathcal{L}) + \beta\varpi'(\mathcal{L}) = \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi, \qquad (1.1)$$

where $^C\mathcal{D}^{\zeta,\sigma}$ denotes the combined Caputo fractional derivative of order $\zeta, \sigma \in (1, 2]$, $\Im : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$, $\zeta_1, \zeta_2 : \mathbb{R} \to \mathbb{R}$ are continuous functions and $\alpha, \beta \geq 0$ are real numbers.

## 2 Preliminaries

Throughout this article, we use $\mathcal{J} = [0, \mathcal{L}]$ and $\mathbb{F} = \mathfrak{C}(\mathcal{J}, \mathbb{R})$, which denotes the set of all continuous functions equipped with the norm $\| \cdot \|$ defined by

$$\|\varpi\| = \sup_{\tau \in \mathcal{J}} |\varpi(\tau)|$$

for all $\tau \in \mathcal{J}$. Further, in this sequel, we require the following definitions and findings which exist in the literature.

**Definition 2.1.** [13] Let $\zeta$ be a positive real number. The left, right and Riemann-Liouville fractional integrals of any function $\Im \in \mathfrak{C}^1(\mathcal{J})$, the space of continuously differentiable functions on the interval $\mathcal{J}$, are given by

$$\mathcal{I}^{\zeta}_0\Im(\tau) = \frac{1}{\Gamma(\zeta)} \int_0^{\tau} (\tau - \varphi)^{\zeta-1}\Im(\varphi)d\varphi,$$
$$_{\mathcal{L}}\mathcal{I}^{\zeta}\Im(\tau) = \frac{1}{\Gamma(\zeta)} \int_{\tau}^{\mathcal{L}} (\varphi - \tau)^{\zeta-1}\Im(\varphi)d\varphi,$$
$$_0\mathcal{I}^{\zeta}_{\mathcal{L}}\Im(\tau) = \frac{1}{\Gamma(\zeta)} \int_0^{\mathcal{L}} |\tau - \varphi|^{\zeta-1}\Im(\varphi)d\varphi.$$

**Definition 2.2.** [24] Let $\zeta, \sigma \in (0, 1], \ \lambda \in [0, 1]$. The combined Caputo fractional integral of a function $\Im \in \mathfrak{C}^1(\mathcal{J})$ of order $\zeta$ is defined as

$$^{\lambda}_{\mathcal{L}}\mathcal{I}^{\zeta,\sigma}_0\Im(\tau) = \mathcal{I}^{\zeta}_0\Im(\tau) + _{\mathcal{L}}\mathcal{I}^{\sigma}\Im(\tau), \quad \tau \in \mathcal{J}.$$

Note that the Riesz fractional integral is an example of the combined Riemann fractional integral operator when $\zeta = \sigma$ and takes the following form

$$^{\lambda}_{\mathcal{L}}\mathcal{I}^{\zeta}_0\Im(\tau) = \frac{1}{\Gamma(\zeta)} \int_0^{\mathcal{L}} |\tau - \varphi|^{\zeta-1}\Im(\varphi)d\varphi, \quad \tau \in \mathcal{J}.$$

**Definition 2.3.** [13] For any function $\Im \in AC(\mathcal{J})$, the space of absolutely continuous functions on the interval $\mathcal{J}$, the left and right Caputo fractional derivatives of order $\zeta$ are defined as follows

$$^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau) = \frac{1}{\Gamma(1-\zeta)} \int_0^{\tau} (\tau - \varphi)^{-\zeta}\Im'(\varphi)d\varphi = \mathcal{I}^{1-\zeta}_0\Im'(\tau),$$
$$^C_{\tau}\mathcal{D}^{\zeta}_{\mathcal{L}}\Im(\tau) = \frac{-1}{\Gamma(1-\zeta)} \int_{\tau}^{\mathcal{L}} (\varphi - \tau)^{-\zeta}\Im'(\varphi)d\varphi = -_{\mathcal{L}}\mathcal{I}^{1-\zeta}\Im'(\tau).$$

**Definition 2.4.** [24] Let $\zeta, \sigma \in (0, 1]$, and $\lambda \in [0, 1]$. The combined Caputo fractional derivative ${}^C_0\mathcal{D}^{\zeta;\sigma}_{\mathcal{L}}$ of order $\zeta, \sigma$ for a function $\Im \in AC(\mathcal{J})$ is denoted and defined by

$$
{}^C_0\mathcal{D}^{\zeta;\sigma}_{\mathcal{L}}\Im(\tau) = \lambda\, {}^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau) + (1-\lambda){}^C_{\tau}\mathcal{D}^{\sigma}_{\mathcal{L}}\Im(\tau) = \lambda\, \mathcal{I}^{1-\zeta}_0\Im'(\tau) - (1-\lambda)_{\mathcal{L}}\mathcal{I}^{1-\sigma}\Im'(\tau).
$$

**Lemma 2.5.** *[13, 24] Let $\Im \in \mathfrak{C}^n(\mathcal{J})$. Then the subsequent relations hold*

$$
\mathcal{I}^{\zeta}_0 {}^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau) = \Im(\tau) - \sum_{k=0}^{n-1} \frac{\Im^k(0)}{k!}(\tau - 0)^k
$$

*and*

$$
{}_{\mathcal{L}}\mathcal{I}^{\sigma}_{\tau} {}^C\mathcal{D}^{\sigma}_{\mathcal{L}}\Im(\tau) = \Im(\tau) - \sum_{k=0}^{n-1} \frac{(-1)^k\Im^{(k)}(\mathcal{L})}{k!}(\mathcal{L} - \tau)^k.
$$

From the above definitions and lemma, if $\zeta, \sigma \in (1, 2]$ and $\Im \in \mathfrak{C}^2(\mathcal{J})$, then

$$
{}_{\mathcal{L}}\lambda\mathcal{I}^{\zeta;\sigma}_0 {}^C_0\mathcal{D}^{\zeta;\sigma}_{\mathcal{L}}\Im(\tau) = \lambda\left(\mathcal{I}^{\zeta}_0 {}^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau) + {}_{\mathcal{L}}\mathcal{I}^{\sigma}_0 {}^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau)\right)
$$

$$
+ (-1)^{2+1}(1-\lambda)\left(\mathcal{I}^{\zeta}_0 {}^C_{\tau}\mathcal{D}^{\sigma}_{\mathcal{L}}\Im(\tau) + {}_{\mathcal{L}}\mathcal{I}^{\sigma}_{\tau} {}^C\mathcal{D}^{\sigma}_{\mathcal{L}}\Im(\tau)\right)
$$

$$
= \lambda\, \mathcal{I}^{\zeta}_0 {}^C_0\mathcal{D}^{\zeta}_{\tau}\Im(\tau) + (-1)^{2+1}(1-\lambda)\, {}_{\mathcal{L}}\mathcal{I}^{\sigma}_{\tau} {}^C\mathcal{D}^{\sigma}_{\mathcal{L}}\Im(\tau).
$$

If $\zeta, \sigma \in (1, 2]$, then we have the following simplified form

$$
{}_{\mathcal{L}}\lambda\mathcal{I}^{\zeta;\sigma}_0 {}^C_0\mathcal{D}^{\zeta;\sigma}_{\mathcal{L}}\Im(\tau) = (2\lambda - 1)\Im(\tau) - \lambda\Im(0) + (1-\lambda)\Im(\mathcal{L}) - \lambda\tau\Im'(0) - (1-\lambda)\Im'(\mathcal{L})(\mathcal{L} - \tau).
$$

## 3 Main Results

**Lemma 3.1.** *Assume that $\hbar \in \mathfrak{C}(\mathcal{J}, \mathbb{R})$ and $\varpi \in \mathfrak{C}^2(\mathcal{J})$. Then the boundary value problem*

$$
{}^C_0\mathcal{D}^{\zeta;\sigma}_{\mathcal{L}}\varpi(\tau) = \hbar(\tau),\ \tau \in [0, \mathcal{L}],\ \zeta, \sigma \in (1, 2]
$$

$$
\alpha\varpi(0) + \beta\varpi'(0) = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi,\ \alpha\varpi(\mathcal{L}) + \beta\varpi'(\mathcal{L}) = \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi,
$$

(3.1)

*can be expressed as the integral equation*

$$
\varpi(\tau) = \left(\frac{x + \mathcal{L}u + \tau y}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi + \left(\frac{m + u\tau}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi
$$

$$
+ \left(\frac{x + y\tau - w\alpha\mathcal{L}}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\right)\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi
$$

$$
+ \left(\frac{k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\right)\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi
$$

$$
+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)}\int_0^{\tau} (\tau - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi
$$

$$
+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)}\int_{\tau}^{\mathcal{L}} (\varphi - \tau)^{\sigma-1}\hbar(\varphi)d\varphi,
$$

*where*

$$
\left(\frac{\lambda}{2\lambda - 1}\right) = v,\ \left(\frac{1 - \lambda}{2\lambda - 1}\right) = w,
$$

$$
(\beta w - \beta v) = m,\ (\beta v - \beta w) = x,
$$

$$
(\beta^2 v - \beta^2 w) = k,\ \alpha v - \alpha w = u,
$$

$$
\alpha w - \alpha v = y,\ \alpha\beta w - \alpha\beta v = h,
$$

$$
u + y = a.
$$

*Proof.* By virtue of Lemma 2.5, the equation (3.1) converts to the following form

$$\varpi(\tau) = \left(\frac{\lambda}{2\lambda - 1}\right)\varpi(0) - \left(\frac{1 - \lambda}{2\lambda - 1}\right)\varpi(\mathcal{L}) + \left(\frac{\lambda\tau}{2\lambda - 1}\right)\varpi'(0) + \left(\frac{1 - \lambda}{2\lambda - 1}\right)\varpi'(\mathcal{L})$$

$$(\mathcal{L} - \tau) + \left(\frac{1}{2\lambda - 1}\right)(_0\mathcal{I}_{\mathcal{L}}^{\zeta;\sigma}\hbar(\tau)). \tag{3.2}$$

Differentiating (3.2), we have

$$\varpi'(\tau) = \left(\frac{\lambda}{2\lambda - 1}\right)\varpi'(0) - \left(\frac{1 - \lambda}{2\lambda - 1}\right)\varpi'(\mathcal{L}) + \frac{1}{(2\lambda - 1)\Gamma(\zeta - 1)}\int_0^\tau (\tau - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi$$

$$- \frac{1}{(2\lambda - 1)\Gamma(\sigma - 1)}\int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma-2}\hbar(\varphi)d\varphi. \tag{3.3}$$

Put $\tau = \mathcal{L}$ in (3.2) and (3.3) to derive

$$\varpi(\mathcal{L}) = \varpi(0) + \mathcal{L}\varpi'(0) + \frac{1}{\lambda\Gamma(\zeta)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi \tag{3.4}$$

and

$$\varpi'(\mathcal{L}) = \varpi'(0) + \frac{1}{\lambda\Gamma(\zeta - 1)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.5}$$

Using (3.4) and (3.5) in the boundary condition

$$\alpha\varpi(\mathcal{L}) + \beta\varpi'(\mathcal{L}) = \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi,$$

we get

$$\int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi = \alpha\varpi(0) + \alpha\mathcal{L}\varpi'(0) + \frac{\alpha}{\lambda\Gamma(\zeta)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \beta\varpi'(0) + \frac{\beta}{\lambda\Gamma(\zeta - 1)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.6}$$

Then, we employ the other boundary condition

$$\alpha\varpi(0) + \beta\varpi'(0) = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi$$

on (3.6) to get

$$\int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi - \beta\varpi'(0) + (\alpha\mathcal{L} + \beta)\varpi'(0) + \frac{\alpha}{\lambda\Gamma(\zeta)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \frac{\beta}{\lambda\Gamma(\zeta - 1)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi$$

$$\Rightarrow \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi + \alpha\mathcal{L}\varpi'(0) + \frac{\alpha}{\lambda\Gamma(\zeta)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \frac{\beta}{\lambda\Gamma(\zeta - 1)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.7}$$

On simplifying (3.7), we get

$$\varpi'(0) = -\frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi + \frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi))d\varphi - \frac{1}{\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$- \frac{\beta}{\alpha\mathcal{L}\Gamma(\zeta - 1)}\int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.8}$$

Now, replacing (3.8) in the other boundary condition

$$\alpha\varpi(0) + \beta\varpi'(0) = \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi))d\varphi,$$

we get

$$\alpha\varpi(0) = -\beta\left[-\frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi + \frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi - \frac{1}{\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi\right.$$

$$\left. -\frac{\beta}{\alpha\lambda\mathcal{L}\Gamma(\zeta-1)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi\right] + \int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi$$

$$\Rightarrow \varpi(0) = \frac{\beta}{\alpha^2\mathcal{L}}\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi - \frac{\beta}{\alpha^2\mathcal{L}}\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi + \frac{\beta}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \frac{\beta^2}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi + \frac{1}{\alpha}\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi$$

$$\Rightarrow \varpi(0) = \left(\frac{\beta+\alpha\mathcal{L}}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi - \frac{\beta}{\alpha^2\mathcal{L}}\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi + \frac{\beta}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \frac{\beta^2}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.9}$$

By substituting (3.8) and (3.9) in (3.4) and (3.5), we get

$$\varpi(\mathcal{L}) = \frac{\beta}{\alpha^2\mathcal{L}}\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi + \left(\frac{-\beta+\alpha\mathcal{L}}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi + \left(\frac{\beta}{\alpha\mathcal{L}\lambda\Gamma(\zeta)}\right)\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \left(\frac{\beta^2-\beta\alpha\mathcal{L}}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\right)\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi \tag{3.10}$$

and

$$\varpi'(\mathcal{L}) = -\frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi + \frac{1}{\alpha\mathcal{L}}\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi - \frac{1}{\lambda\mathcal{L}\Gamma(\alpha)}\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \left(\frac{-\beta+\alpha\mathcal{L}}{\alpha\mathcal{L}\lambda\Gamma(\zeta-1)}\right)\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi. \tag{3.11}$$

Finally by substituting the values of (3.8),(3.9),(3.10) and (3.11) in (3.2) we get

$$\varpi(\tau) = \left(\frac{x+\mathcal{L}u+\tau y}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi + \left(\frac{m+u\tau}{\alpha^2\mathcal{L}}\right)\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi$$

$$+ \left(\frac{x+y\tau-w\alpha\mathcal{L}}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\right)\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi$$

$$+ \left(\frac{k+h\tau+w\alpha^2\mathcal{L}^2-w\alpha^2\mathcal{L}\tau}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\right)\int_0^{\mathcal{L}}(\mathcal{L}-\varphi)^{\zeta-2}\hbar(\varphi)d\varphi$$

$$+ \frac{1}{(2\lambda-1)\Gamma(\zeta)}\int_0^{\tau}(\tau-\varphi)^{\zeta-1}\hbar(\varphi)d\varphi + \frac{1}{(2\lambda-1)\Gamma(\sigma)}\int_{\tau}^{\mathcal{L}}(\varphi-\tau)^{\sigma-1}\hbar(\varphi)d\varphi.$$

This completes the proof. □

Now, for our further discussions, we define an integral operator $\mathcal{T}: \mathbb{F} \to \mathbb{F}$ for all $\varpi \in \mathbb{F}$ as

follows

$$
(\mathcal{T}\varpi)(\tau) = \frac{1}{\alpha^2 \mathcal{L}} \left( (x + \mathcal{L}u + \tau y) \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi)) d\varphi + (m + u\tau) \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi)) d\varphi \right)
$$
$$
+ \left( \frac{x + y\tau - w\alpha\mathcal{L}}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \hbar(\varphi) d\varphi
$$
$$
+ \left( \frac{k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \hbar(\varphi) d\varphi
$$
$$
+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^\tau (\tau - \varphi)^{\zeta-1} \hbar(\varphi) d\varphi + \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma-1} \hbar(\varphi) d\varphi. \tag{3.12}
$$

### 3.1 Uniqueness Result

**Theorem 3.2.** *Suppose* $\Im : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ *is a continuous function such that the image of every bounded subset of* $\mathcal{J} \times \mathbb{R}$ *under* $\Im$ *is relatively compact in* $\mathbb{R}$. *Additionally, let* $\zeta_1, \zeta_2 : \mathbb{R} \to \mathbb{R}$ *be continuous functions. Furthermore,*

$(H_1)$ *there exists a constant* $L_1 > 0$ *such that*

$$
\|\Im(\tau, \varpi) - \Im(\tau, \eta)\| \le L_1 \|\varpi - \eta\|
$$

*for each* $\tau \in \mathcal{J}$ *and* $\varpi, \eta \in \mathbb{R}$;

$(H_2)$ *there exist positive constants* $L_2, L_3, M_2, M_3$ *satisfying* $\|\zeta_1(\varpi) - \zeta_1(\eta)\| \le L_2 \|\varpi - \eta\|$, $\|\zeta_2(\varpi) - \zeta_2(\eta)\| \le L_3 \|\varpi - \eta\|$ *with* $\|\zeta_1(\varpi)\| \le M_2$ *and* $\|\zeta_2(\varpi)\| \le M_3$ *for all* $\varpi, \eta \in \mathbb{R}$.

*Under these conditions, the boundary value problem (1.1) possesses a unique solution provided*

$$
\frac{|x + (u + y)|\mathcal{L}L_2 + |m + u\mathcal{L}|L_3}{\alpha^2} + \left[ \frac{|x + (y - \alpha w)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)} \mathcal{L}^{\zeta-1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} \mathcal{L}^{\zeta-2} \right.
$$
$$
\left. + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)} \right] L_1 < 1 \tag{3.13}
$$

*with*

$$
L_1 \le \frac{1}{2} \left[ \frac{|x + (y - \alpha w)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)} \mathcal{L}^{\zeta-1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} \mathcal{L}^{\zeta-2} + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)} \right]^{-1}. \tag{3.14}
$$

*Proof.* Using operator $\mathcal{T}$ defined by (3.12), we have

$$
(\mathcal{T}\varpi)(\tau) = \frac{1}{\alpha^2 \mathcal{L}} \left[ (x + \mathcal{L}u + \tau y) \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi)) d\varphi + (m + u\tau) \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi)) d\varphi \right]
$$
$$
+ \left( \frac{x + y\tau - w\alpha\mathcal{L}}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \Im(\varphi, \varpi(\varphi)) d\varphi
$$
$$
+ \left( \frac{k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \Im(\varphi, \varpi(\varphi)) d\varphi
$$
$$
+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^\tau (\tau - \varphi)^{\zeta-1} \Im(\varphi, \varpi(\varphi)) d\varphi
$$
$$
+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma-1} \Im(\varphi, \varpi(\varphi)) d\varphi.
$$

Setting $\sup_{\tau \in \mathcal{J}} \|\Im(\tau, 0)\| = M_1$ (by the assumption on $\Im$) and choosing

$$2\left[\frac{|x + (u + y)\mathcal{L}|\, M_2 + |m + u\mathcal{L}|\, M_3}{\alpha^2} + \left(\frac{|x + (y - \alpha w)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)}\mathcal{L}^{\zeta - 1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)}\mathcal{L}^{\zeta - 2}\right.\right.$$
$$\left.\left. + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)}\right) M_1\right] \leq r,$$

we show that $\mathcal{T}\mathcal{B}_r \subseteq \mathcal{B}_r$, where $\mathcal{B}_r = \{\varpi \in \mathbb{F} : \|\varpi\| \leq r\}$. For $\varpi \in \mathcal{B}_r$, we have

$$\|(\mathcal{T}\varpi)(\tau)\| \leq \frac{1}{\alpha^2\mathcal{L}}\left[|x + \mathcal{L}u + \tau y|\int_0^{\mathcal{L}}\|\zeta_1(\varpi(\varphi))\|\,d\varphi + |m + u\tau|\int_0^{\mathcal{L}}\|\zeta_2(\varpi(\varphi))\|\,d\varphi\right]$$

$$+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta - 1}\|\Im(\varphi, \varpi(\varphi))\|\,d\varphi$$

$$+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta - 2}\|\Im(\varphi, \varpi(\varphi))\|\,d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)}\int_0^{\tau}(\tau - \varphi)^{\zeta - 1}\|\Im(\varphi, \varpi(\varphi))\|\,d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)}\int_{\tau}^{\mathcal{L}}(\varphi - \tau)^{\sigma - 1}\|\Im(\varphi, \varpi(\varphi))\|\,d\varphi$$

$$\leq \frac{1}{\alpha^2\mathcal{L}}\left[|x + \mathcal{L}u + \tau y|\int_0^{\mathcal{L}}\|\zeta_1(\varpi(\varphi))\|\,d\varphi + |m + u\tau|\int_0^{\mathcal{L}}\|\zeta_2(\varpi(\varphi))\|\,d\varphi\right]$$

$$+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta - 1}[\|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, 0)\| + \|\Im(\varphi, 0)\|]\,d\varphi$$

$$+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta - 2}[\|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, 0)\| + \|\Im(\varphi, 0)\|]\,d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)}\int_0^{\tau}(\tau - \varphi)^{\zeta - 1}[\|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, 0)\| + \|\Im(\varphi, 0)\|]\,d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)}\int_{\tau}^{\mathcal{L}}(\varphi - \tau)^{\sigma - 1}[\|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, 0)\| + \|\Im(\varphi, 0)\|]\,d\varphi$$

$$\leq \frac{|x + (u + y)\mathcal{L}|\, M_2 + |m + u\mathcal{L}|\, M_3}{\alpha^2\mathcal{L}} + (L_1 r + M_1)\left[\frac{|x + (y - w\alpha)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)}\mathcal{L}^{\zeta - 1}\right.$$

$$\left. + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)}\mathcal{L}^{\zeta - 2} + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)}\right]$$

$$\leq \frac{|x + (u + y)\mathcal{L}|\, M_2 + |m + u\mathcal{L}|\, M_3}{\alpha^2\mathcal{L}} + L_1\left[\frac{|x + (y - w\alpha)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)}\mathcal{L}^{\zeta - 1}\right.$$

$$\left. + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)}\mathcal{L}^{\zeta - 2} + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)}\right] r + M_1\left[\frac{|x + (y - w\alpha)\mathcal{L}|}{\alpha\lambda\Gamma(\zeta + 1)}\mathcal{L}^{\zeta - 1}\right.$$

$$\left. + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)}\mathcal{L}^{\zeta - 2} + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)}\right] \leq r. \qquad (3.15)$$

Now, for $\varpi, \eta \in \mathbb{F}$ and for each $\tau \in \mathcal{J}$, we obtain

$$\|(\mathcal{T}\varpi)(\tau) - (\mathcal{T}\eta)(\tau)\|$$

$$\leq \frac{1}{\alpha^2 \mathcal{L}} \left[ |x + \mathcal{L}u + \tau y| \int_0^{\mathcal{L}} \|\zeta_1(\varpi(\varphi)) - \zeta_1(\eta(\varphi))\| \, d\varphi \right.$$

$$+ |m + u\tau| \int_0^{\mathcal{L}} \|\zeta_2(\varpi(\varphi)) - \zeta_2(\eta(\varphi))\| \, d\varphi \Bigg]$$

$$+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^{\tau} (\tau - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma-1} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$\leq \frac{1}{\alpha^2 \mathcal{L}} \left[ |x + \mathcal{L}(u + y)| L_2 \|\varpi - \eta\| + |m + u\mathcal{L}| L_3 \|\varpi - \eta\| \right] + \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta + 1)} L_1 \mathcal{L}^{\zeta-1} \|\varpi - \eta\|$$

$$+ \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} L_1 \mathcal{L}^{\zeta-2} \|\varpi - \eta\| + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} L_1 \|\varpi - \eta\| + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)} L_1 \|\varpi - \eta\|$$

$$\leq \left[ \frac{|x + \mathcal{L}(u + y)| L_2 + |m + u\mathcal{L}| L_3}{\alpha^2 \mathcal{L}} + \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta + 1)} L_1 \mathcal{L}^{\zeta-1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} L_1 \mathcal{L}^{\zeta-2} \right.$$

$$\left. + \frac{\mathcal{L}^\zeta L_1}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma L_1}{(2\lambda - 1)\Gamma(\sigma + 1)} \right] \|\varpi - \eta\|$$

$$\leq \kappa \|\varpi - \eta\|_{\mathbb{F}},$$

where

$$\kappa = \frac{|x + \mathcal{L}(u + y)| L_2 + |m + u\mathcal{L}| L_3}{\alpha^2 \mathcal{L}} + \left[ \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta + 1)} \mathcal{L}^{\zeta-1} \right.$$

$$\left. + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} \mathcal{L}^{\zeta-2} + \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)} \right] L_1.$$

By (3.13), we know $\kappa < 1$, which implies that $\mathcal{T}$ is a contraction. Therefore, the conclusion of the theorem follows from the Banach contraction principle. □

## 3.2 First Existence Result

**Theorem 3.3.** *Let* $\Im : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ *be a continuous function. Suppose that the hypothesis* $(H_1)$ *along with the following assumption hold*

$(H_3)$ *there exists a constant* $L_4 > 0$ *such that* $\|\Im(\tau, \varpi)\| \leq L_4$ *for* $\tau \in \mathcal{J}$, *and* $\varpi \in \mathbb{F}$.

*Then there exists at least one solution to the boundary value problem (1.1) on* $\mathcal{J}$.

*Proof.* We break down the proof into the following steps.
**Claim 1:** $\mathcal{T}$ is continuous.
Suppose that $(\varpi_n)$ is a convergent sequence in $\mathbb{F}$ such that $\varpi_n \to \varpi$ as $n \to \infty$. Then, for $\tau \in \mathcal{J}$,

and using assumption $(H_2)$, we obtain

$$
\begin{aligned}
&\|(\mathcal{T}\varpi_n)(\tau) - (\mathcal{T}\varpi)(\tau)\| \\
&= \frac{1}{\alpha^2\mathcal{L}}\left[|x + \mathcal{L}u + \tau y|\int_0^{\mathcal{L}}\|\zeta_1(\varpi_n(\varphi)) - \zeta_1(\varpi(\varphi))\|\,d\varphi\right. \\
&\left. + |m + u\tau|\int_0^{\mathcal{L}}\|\zeta_2(\varpi_n(\varphi)) - \zeta_2(\varpi(\varphi))\|\,d\varphi\right] \\
&+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta-1}\|\Im(\varphi, \varpi_n(\varphi)) - \Im(\varphi, \varpi(\varphi))\|\,d\varphi \\
&+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta-2}\|\Im(\varphi, \varpi_n(\varphi)) - \Im(\varphi, \varpi(\varphi))\|\,d\varphi \\
&+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)}\int_0^{\tau}(\tau - \varphi)^{\zeta-1}\|\Im(\varphi, \varpi_n(\varphi)) - \Im(\varphi, \varpi(\varphi))\|\,d\varphi \\
&+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)}\int_{\tau}^{\mathcal{L}}(\varphi - \tau)^{\sigma-1}\|\Im(\varphi, \varpi_n(\varphi)) - \Im(\varphi, \varpi(\varphi))\|\,d\varphi \\
&\leq \frac{1}{\alpha^2\mathcal{L}}\left[|x + \mathcal{L}u + \tau y|\int_0^{\mathcal{L}}\|\zeta_1(\varpi_n(\varphi)) - \zeta_1(\varpi(\varphi))\|\,d\varphi\right. \\
&\left. + |m + u\tau|\int_0^{\mathcal{L}}\|\zeta_2(\varpi_n(\varphi)) - \zeta_2(\varpi(\varphi))\|\,d\varphi\right] \\
&+ \left[\frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta+1)}\mathcal{L}^{\zeta-1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)}\mathcal{L}^{\zeta-2} + \frac{\mathcal{L}^{\zeta}}{(2\lambda - 1)\Gamma(\zeta+1)}\right. \\
&\left. + \frac{\mathcal{L}^{\sigma}}{(2\lambda - 1)\Gamma(\sigma+1)}\right]\|\Im(\varphi, \varpi_n(\varphi)) - \Im(\varphi, \varpi(\varphi))\|.
\end{aligned}
$$

Since $\Im$ and $\zeta_i$ are continuous, we have $\zeta_i\varpi_n \to \zeta_i\varpi$ for $i = 1, 2$ and $\Im(\varphi, \varpi_n(\varphi)) \to \Im(\varphi, \varpi(\varphi))$, we derive

$$
\|(\mathcal{T}\varpi_n)(\tau) - (\mathcal{T}\varpi)(\tau)\| \to 0, \quad \text{as } n \to \infty
$$

which implies that $\mathcal{T}$ is continuous.

**Claim 2:** $\mathcal{T}$ maps bounded sets into bounded sets.
We have already shown this in (3.15). So we omit the proof.

**Claim 3:** $\mathcal{T}$ maps a bounded set into an equicontinuous set.
Let $\tau_1, \tau_2 \in \mathcal{J}$ with $\tau_1 < \tau_2$, and $\varpi \in \mathcal{B}_r = \{\varpi \in \mathbb{F} : \|\varpi\| \leq r\}$. Then we have

$$
\begin{aligned}
&\|(\mathcal{T}\varpi)(\tau_2) - (\mathcal{T}\varpi)(\tau_1)\| \\
&\leq \left\|\frac{1}{\alpha^2\mathcal{L}}\left(y(\tau_2 - \tau_1)\int_0^{\mathcal{L}}\zeta_1(\varpi(\varphi))d\varphi + u(\tau_2 - \tau_1)\int_0^{\mathcal{L}}\zeta_2(\varpi(\varphi))d\varphi\right)\right\| \\
&+ \left\|\frac{y(\tau_2 - \tau_1)}{\alpha\lambda\mathcal{L}\Gamma(\zeta)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta-1}\Im(\varphi, \varpi(\varphi))d\varphi\right\| \\
&+ \left\|\frac{h(\tau_2 - \tau_1) - w\alpha^2\mathcal{L}(\tau_2 - \tau_1)}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)}\int_0^{\mathcal{L}}(\mathcal{L} - \varphi)^{\zeta-2}\Im(\varphi, \varpi(\varphi))d\varphi\right\| \\
&+ \left\|\frac{1}{(2\lambda - 1)\Gamma(\zeta)}\int_0^{\tau_1}\left((\tau_2 - \varphi)^{\zeta-1} - (\tau_1 - \varphi)^{\zeta-1}\right)\Im(\varphi, \varpi(\varphi))d\varphi\right\|
\end{aligned}
$$

$$+ \left\| \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_{\tau_1}^{\tau_2} (\varphi - \tau_1)^{\sigma-1} \, \Im(\varphi, \varpi(\varphi)) d\varphi \right\|$$

$$+ \left\| \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_{\tau_1}^{\tau_2} (\tau_2 - \varphi)^{\zeta-1} \, \Im(\varphi, \varpi(\varphi)) d\varphi \right\|$$

$$+ \left\| \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_{\tau_2}^{\mathcal{L}} \left( (\varphi - \tau_1)^{\sigma-1} - (\varphi - \tau_2)^{\sigma-1} \right) \Im(\varphi, \varpi(\varphi)) d\varphi \right\|$$

$$\leq \frac{1}{\alpha^2 \mathcal{L}} \left( y(\tau_2 - \tau_1) \int_0^{\mathcal{L}} \|\zeta_1(\varpi(\varphi))\| \, d\varphi + u(\tau_2 - \tau_1) \int_0^{\mathcal{L}} \|\zeta_2(\varpi(\varphi))\| \, d\varphi \right)$$

$$+ \frac{y(\tau_2 - \tau_1)}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{h(\tau_2 - \tau_1) - w\alpha^2\mathcal{L}(\tau_2 - \tau_1)}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^{\tau_1} \left( (\tau_2 - \varphi)^{\zeta-1} - (\tau_1 - \varphi)^{\zeta-1} \right) \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_{\tau_1}^{\tau_2} (\varphi - \tau_1)^{\sigma-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_{\tau_1}^{\tau_2} (\tau_2 - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_{\tau_2}^{\mathcal{L}} \left( (\varphi - \tau_1)^{\sigma-1} - (\varphi - \tau_2)^{\sigma-1} \right) \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$\leq \left| \frac{1}{\alpha^2 \mathcal{L}} [y(\tau_2 - \tau_1)M_2 + u(\tau_2 - \tau_1)M_3] \right| + \frac{\left| y(\tau_2 - \tau_1)\mathcal{L}^{\zeta-1} \right| L_4}{\alpha\lambda\Gamma(\zeta + 1)}$$

$$+ \frac{\left| h(\tau_2 - \tau_1) - w\alpha^2\mathcal{L}(\tau_2 - \tau_1)\mathcal{L}^{\zeta-2} \right| L_4}{\alpha\lambda\Gamma(\zeta)} + \frac{\left| (\tau_2 - \tau_1)^{\zeta} + \left( \tau_1^{\zeta} - \tau_2^{\zeta} \right) \right| L_4}{(2\lambda - 1)\Gamma(\zeta + 1)}$$

$$+ \frac{\left| (\tau_2 - \tau_1)^{\sigma} \right| L_4}{(2\lambda - 1)\Gamma(\sigma + 1)} + \frac{\left| (\tau_2 - \tau_1)^{\zeta} \right| L_4}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\left| ((\mathcal{L} - \tau_1)^{\sigma} - (\tau_2 - \tau_1)^{\sigma}) - (\mathcal{L} - \tau_2)^{\sigma} \right| L_4}{(2\lambda - 1)\Gamma(\sigma + 1)}.$$

The right side of the inequality converges to zero as $\tau_2 - \tau_1 \to 0$ for all $\varpi \in \mathcal{B}_r$. That means $\mathcal{T}$ is equicontinuous and by Arzela-Ascoli theorem, the operator $\mathcal{T} : \mathbb{F} \to \mathbb{F}$ is completely continuous.
**Claim 4:** $\mathcal{T}$ is a priori bounded.
We demonstrate that the set $\Delta$ defined by

$$\Delta = \{\varpi \in \mathbb{F} : \varpi = \mu\mathcal{T}(\varpi), \ 0 < \mu < 1\}$$

is bounded. Consider $\varpi \in \Delta$ for some $\mu \in (0, 1)$. Then for every $\tau \in \mathcal{J}$

$$\|\varpi(\tau)\| \leq \frac{1}{\alpha^2 \mathcal{L}} \left[ |x + \mathcal{L}u + \tau y| \int_0^{\mathcal{L}} \|\zeta_1(\varpi(\varphi))\| \, d\varphi + |m + u\tau| \int_0^{\mathcal{L}} \|\zeta_2(\varpi(\varphi))\| \, d\varphi \right]$$

$$+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^{\tau} (\tau - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma - 1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi$$

$$\leq \frac{|x + \mathcal{L}(u + y)| \, M_2 + |m + u\mathcal{L}| \, M_3}{\alpha^2 \mathcal{L}} + \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta + 1)} L_4 \mathcal{L}^{\zeta - 1}$$

$$+ \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} L_4 \mathcal{L}^{\zeta - 2} + \frac{\mathcal{L}^\zeta L_4}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma L_4}{(2\lambda - 1)\Gamma(\sigma + 1)}$$

$$= G.$$

Hence

$$\|\varpi\| \leq G.$$

This leads to the fact that $\Delta$ is bounded and employing Schafer fixed point theorem, we conclude that the equation (1.1) has at least one solution in $\mathcal{J}$. $\qquad\square$

### 3.3  Second Existence Result

**Theorem 3.4.** *Let $\Im$ be a continuous function from $\mathcal{J} \times \mathbb{R}$ to $\mathbb{R}$. Suppose that the hypotheses $(H_1)$-$(H_2)$ along with the following assumptions hold*

*$(H_4)$ there exists a non-negative function $\omega$ defined on $\mathfrak{C}(\mathcal{J})$ such that $\|\Im(\tau, \varpi(\tau))\| \leq \omega(\tau)$ for all $(\tau, \varpi) \in \mathcal{J} \times \mathbb{R}$;*

*$(H_5)$ $\delta = \frac{|x+\mathcal{L}(u+y)|L_2+|m+u\mathcal{L}|L_3}{\alpha^2\mathcal{L}} + \frac{|x+\mathcal{L}(y-w\alpha)|}{\alpha\lambda\Gamma(\zeta+1)} L_1\mathcal{L}^{\zeta-1} + \frac{|k+h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} L_1\mathcal{L}^{\zeta-2} < 1.$*

*Then the boundary value problem (1.1) has at least one solution in $\mathcal{J}$.*

*Proof.* We define the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ on $\mathcal{B}_r = \{\varpi \in \mathbb{F} : \|\varpi\| \leq r\}$ as

$$(\mathcal{T}_1\varpi)(\tau) = \frac{1}{(2\lambda - 1)\Gamma(\zeta)} \int_0^\tau (\tau - \varphi)^{\zeta - 1} \Im(\varphi, \varpi(\varphi)) d\varphi$$

$$+ \frac{1}{(2\lambda - 1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi - \tau)^{\sigma - 1} \Im(\varphi, \varpi(\varphi)) d\varphi$$

and

$$(\mathcal{T}_2\varpi)(\tau) = \frac{1}{\alpha^2\mathcal{L}} \left[ (x + \mathcal{L}u + \tau y) \int_0^{\mathcal{L}} \zeta_1(\varpi(\varphi)) d\varphi + (m + u\tau) \int_0^{\mathcal{L}} \zeta_2(\varpi(\varphi)) d\varphi \right]$$

$$+ \left( \frac{x + y\tau - w\alpha\mathcal{L}}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta - 1} \Im(\varphi, \varpi(\varphi)) d\varphi$$

$$+ \left( \frac{k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta - 1)} \right) \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta - 2} \Im(\varphi, \varpi(\varphi)) d\varphi.$$

Let us fix

$$r \geq \|\omega\|_{L^1} \left[ \frac{\mathcal{L}^\zeta}{(2\lambda - 1)\Gamma(\zeta + 1)} + \frac{\mathcal{L}^\sigma}{(2\lambda - 1)\Gamma(\sigma + 1)} + \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta + 1)} \mathcal{L}^{\zeta - 1} + \frac{|k + h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} \mathcal{L}^{\zeta - 2} \right]$$

$$+ \frac{|x + \mathcal{L}(u + y)| \, M_2 + |m + u\mathcal{L}| \, M_3}{\alpha^2\mathcal{L}}.$$

**Claim 1:**  $\mathcal{T}_1\varpi(\tau) + \mathcal{T}_2\eta(\tau) \in \mathcal{B}_r.$

For any $\varpi, \eta \in \mathcal{B}_r$ and for each $\tau \in \mathcal{J}$, we derive

$$
\begin{aligned}
\|(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\eta)(\tau)\| \leq &\frac{1}{(2\lambda-1)\Gamma(\zeta)} \int_0^\tau (\tau-\varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{1}{(2\lambda-1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi-\tau)^{\sigma-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{|x+y\tau-w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \int_0^{\mathcal{L}} (\mathcal{L}-\varphi)^{\zeta-1} \|\Im(\varphi, \eta(\varphi))\| \, d\varphi \\
&+ \frac{|k+h\tau+w\alpha^2\mathcal{L}^2-w\alpha^2\mathcal{L}\tau|}{\alpha^2\lambda\mathcal{L}\Gamma(\zeta-1)} \int_0^{\mathcal{L}} (\mathcal{L}-\varphi)^{\zeta-2} \|\Im(\varphi, \eta(\varphi))\| \, d\varphi \\
&+ \frac{1}{\alpha^2\mathcal{L}} \left[ |x+\mathcal{L}u+\tau y| \int_0^{\mathcal{L}} \|\zeta_1(\eta(\varphi))\| \, d\varphi + |m+u\tau| \int_0^{\mathcal{L}} \|\zeta_2(\eta(\varphi))\| \, d\varphi \right] \\
\leq &\|\omega\|_{L^1} \left[ \frac{\mathcal{L}^\zeta}{(2\lambda-1)\Gamma(\zeta+1)} + \frac{\mathcal{L}^\sigma}{(2\lambda-1)\Gamma(\sigma+1)} + \frac{|x+\mathcal{L}(y-w\alpha)|}{\alpha\lambda\Gamma(\zeta+1)} \mathcal{L}^{\zeta-1} \right. \\
&\left. + \frac{|k+h\mathcal{L}|}{\alpha^2\lambda\Gamma(\zeta)} \mathcal{L}^{\zeta-2} \right] + \frac{|x+\mathcal{L}(u+y)|\, M_2 + |m+u\mathcal{L}|\, M_3}{\alpha^2\mathcal{L}}. \quad (3.16)
\end{aligned}
$$

This shows

$$\|(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\eta)(\tau)\| \leq r$$

which leads to

$$(\mathcal{T}_1\varpi)(\tau) + (\mathcal{T}_2\eta)(\tau) \in \mathcal{B}_r.$$

**Claim 2:** $\mathcal{T}_1$ is continuous and compact.

The continuity of $\Im$ implies the continuity of $\mathcal{T}_1$. Also $\mathcal{T}_1$ is uniformly bounded on $\mathcal{B}_r$ as

$$
\begin{aligned}
\|(\mathcal{T}_1\varpi)(\tau)\| \leq &\frac{1}{(2\lambda-1)\Gamma(\zeta)} \int_0^\tau (\tau-\varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{1}{(2\lambda-1)\Gamma(\sigma)} \int_\tau^{\mathcal{L}} (\varphi-\tau)^{\sigma-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
\leq &\|\omega\|_{L^1} \left[ \frac{\mathcal{L}^\zeta}{(2\lambda-1)\Gamma(\zeta+1)} + \frac{\mathcal{L}^\sigma}{(2\lambda-1)\Gamma(\sigma+1)} \right].
\end{aligned}
$$

Now, we prove the compactness of the operator $\mathcal{T}_1$. For $\tau_1, \tau_2 \in \mathcal{J}, \tau_1 < \tau_2$, we have

$$
\begin{aligned}
&\|(\mathcal{T}_1\varpi)(\tau_1) - (\mathcal{T}_1\varpi)(\tau_2)\| \\
&\leq \frac{1}{(2\lambda-1)\Gamma(\zeta)} \int_0^{\tau_1} \left( (\tau_2-\varphi)^{\zeta-1} - (\tau_1-\varphi)^{\zeta-1} \right) \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{1}{(2\lambda-1)\Gamma(\sigma)} \int_{\tau_1}^{\tau_2} (\varphi-\tau_1)^{\sigma-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{1}{(2\lambda-1)\Gamma(\zeta)} \int_{\tau_1}^{\tau_2} (\tau_2-\varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&+ \frac{1}{(2\lambda-1)\Gamma(\sigma)} \int_{\tau_2}^{\mathcal{L}} \left( (\varphi-\tau_1)^{\sigma-1} - (\varphi-\tau_2)^{\sigma-1} \right) \|\Im(\varphi, \varpi(\varphi))\| \, d\varphi \\
&\leq \|\omega\|_{L^1} \left| \frac{(\tau_2-\tau_1)^\zeta + \left(\tau_1^\zeta - \tau_2^\zeta\right)}{(2\lambda-1)\Gamma(\zeta+1)} + \frac{(\tau_2-\tau_1)^\sigma}{(2\lambda-1)\Gamma(\sigma+1)} + \frac{(\tau_2-\tau_1)^\zeta}{(2\lambda-1)\Gamma(\zeta+1)} \right. \\
&\left. + \frac{((\mathcal{L}-\tau_1)^\sigma - (\tau_2-\tau_1)^\sigma) - (\mathcal{L}-\tau_2)^\sigma}{(2\lambda-1)\Gamma(\sigma+1)} \right| L_4.
\end{aligned}
$$

It is evident that the right-hand side of the above inequality approaches zero as $\tau_2$ approaches $\tau_1$, uniformly with respect to $\varpi \in \mathcal{B}_r$. Thus $\mathcal{T}_1$ is equicontinuous, and so $\mathcal{T}_1$ is relatively compact

on $\mathcal{B}_r$. Therefore, by the conclusion of the Arzela-Ascoli theorem, the operator $\mathcal{T}_1$ is continuous and compact on $\mathcal{B}_r$.

**Claim 3:**    $\mathcal{T}_2$ is a contraction.

Let $\varpi, \eta \in \mathbb{F}$, and for each $\tau \in \mathcal{J}$, we have

$$\|(\mathcal{T}_2\varpi)(\tau) - (\mathcal{T}_2\eta)(\tau)\|$$

$$\leq \frac{1}{\alpha^2 \mathcal{L}} \left[ |x + \mathcal{L}u + \tau y| \int_0^{\mathcal{L}} \|\zeta_1(\varpi(\varphi)) - \zeta_1(\eta(\varphi))\| \, d\varphi + |m + u\tau| \int_0^{\mathcal{L}} \|\zeta_2(\varpi(\varphi)) - \zeta_2(\eta(\varphi))\| \, d\varphi \right]$$

$$+ \frac{|x + y\tau - w\alpha\mathcal{L}|}{\alpha\lambda\mathcal{L}\Gamma(\zeta)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-1} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$+ \frac{|k + h\tau + w\alpha^2\mathcal{L}^2 - w\alpha^2\mathcal{L}\tau|}{\alpha^2 \lambda\mathcal{L}\Gamma(\zeta-1)} \int_0^{\mathcal{L}} (\mathcal{L} - \varphi)^{\zeta-2} \|\Im(\varphi, \varpi(\varphi)) - \Im(\varphi, \eta(\varphi))\| \, d\varphi$$

$$\leq \left[ \frac{|x + \mathcal{L}(u + y)| \, L_2 + |m + u\mathcal{L}| \, L_3}{\alpha^2 \mathcal{L}} + \frac{|x + \mathcal{L}(y - w\alpha)|}{\alpha\lambda\Gamma(\zeta+1)} L_1 \mathcal{L}^{\zeta-1} + \frac{|k + h\mathcal{L}|}{\alpha^2 \lambda\Gamma(\zeta)} L_1 \mathcal{L}^{\zeta-2} \right] \|\varpi - \eta\|$$

which implies that

$$\|(\mathcal{T}_2\varpi)(\tau) - (\mathcal{T}_2\eta)(\tau)\| \leq \delta\|\varpi - \eta\|.$$

Thus, all the assumptions of Krasnoselskii fixed point theorem are satisfied. Therefore, the boundary value problem (1.1) has at least one solution on $\mathcal{J}$.      □

## 4   Illustrations

**Example 4.1.** Consider the following boundary value problem

$$^C\mathcal{D}^{\zeta,\sigma}\varpi(\tau) = \frac{e^{-\tau}}{(4\sqrt{\pi} + 16e^{-\tau})} \frac{|\varpi(\tau)|}{1 + |\varpi(\tau)|}, \quad \tau \in [0, 1], \ \zeta, \sigma \in (1, 2]$$

$$\varpi(0) + \frac{1}{3}\varpi'(0) = \int_0^1 \frac{|\varpi(\tau)|}{5 + |\varpi(\tau)|} d\tau, \quad \varpi(1) + \frac{1}{3}\varpi'(1) = \int_0^1 \frac{|\varpi(\tau)|}{3 + |\varpi(\tau)|} d\tau. \quad (4.1)$$

Here, we consider $\Im(\tau, \varpi(\tau)) = \frac{e^{-\tau}}{(4\sqrt{\pi} + 16e^{-\tau})} \frac{|\varpi(\tau)|}{1 + |\varpi(\tau)|}$, $\zeta_1(\varpi(\tau)) = \frac{|\varpi(\tau)|}{5 + |\varpi(\tau)|}$, $\zeta_2(\varpi(\tau)) = \frac{|\varpi(\tau)|}{3 + |\varpi(\tau)|}$, and $\alpha = 1$, $\beta = \frac{1}{3}$. Therefore we have,

$$\|\Im(\tau, \varpi) - \Im(\tau, \eta)\| \leq \frac{1}{4\sqrt{\pi} + 16} \|\varpi - \eta\|$$

and

$$\|\zeta_1(\varpi) - \zeta_1(\eta)\| \leq \frac{1}{5} \|\varpi - \eta\| \ \text{ and } \ \|\zeta_2(\varpi) - \zeta_2(\eta)\| \leq \frac{1}{3} \|\varpi - \eta\|.$$

Hence, the assumptions $(H_1)$ and $(H_2)$ are satisfied with $L_1 = \frac{1}{4\sqrt{\pi}+16}$, $L_2 = \frac{1}{5}$, $L_3 = \frac{1}{3}$, $M_2 = 1$, $M_3 = 1$, $\lambda = 0.1$. For the above mentioned constants, we further confirm that the conditions (3.13) and (3.14) are also satisfied. Thus, by Theorem 3.2, the boundary value problem (4.1) has a unique solution on $[0, 1]$.

**Example 4.2.** Consider the following boundary value problem

$$^C\mathcal{D}^{\zeta,\sigma}\varpi(\tau) = \frac{\cos^2\tau + 1}{3} \sin^{-1}\left(\frac{|\varpi(\tau)|}{|\varpi(\tau)| + 1}\right), \quad \tau \in [0, 1], \ \zeta, \sigma \in (1, 2]$$

$$\varpi(0) + \frac{1}{2}\varpi'(0) = \int_0^1 \frac{|\varpi(\tau)|}{9 + |\varpi(\tau)|} d\tau, \quad \varpi(1) + \frac{1}{2}\varpi'(1) = \int_0^1 \frac{|\varpi(\tau)|}{11 + |\varpi(\tau)|} d\tau. \quad (4.2)$$

Here, we consider $\Im(\tau, \varpi(\tau)) = \frac{\cos^2 \tau + 1}{3} \sin^{-1}\left(\frac{|\varpi(\tau)|}{|\varpi(\tau)|+1}\right)$, $\zeta_1(\varpi(\tau)) = \frac{|\varpi(\tau)|}{9+|\varpi(\tau)|}$, $\zeta_2(\varpi(\tau)) = \frac{|\varpi(\tau)|}{11+|\varpi(\tau)|}$, and $\alpha = 1$, $\beta = \frac{1}{2}$. Therefore, we have

$$\|\Im(\tau, \varpi) - \Im(\tau, \eta)\| \leq \frac{2}{3}\|\varpi - \eta\|$$

and

$$\|\zeta_1(\varpi) - \zeta_1(\varpi)\| \leq \frac{1}{9}\|\varpi - \eta\| \text{ and } \|\zeta_2(\varpi) - \zeta_2(\eta)\| \leq \frac{1}{11}\|\varpi - \eta\|.$$

Further, we observe that $\|\Im(\tau, \varpi(\tau))\| \leq \frac{\pi}{3}$. Hence, the assumption $(H_3)$ is satisfied for $L_4 = \frac{\pi}{3}$. Also, it can be verified that $(H_2)$ is satisfied for $L_2 = \frac{1}{9}$, $L_3 = \frac{1}{11}$, $M_2 = 1$, $M_3 = 1$. Therefore, the boundary value problem (4.2) can be solved uniquely on $[0, 1]$ by using the Theorem 3.3.

**Example 4.3.** Consider the following boundary value problem

$$^C\mathcal{D}^{\zeta,\sigma}\varpi(\tau) = \frac{1}{(5+\tau)}\frac{|\varpi(\tau)|}{(2+|\varpi(\tau)|)}, \quad \tau \in [0,1], \ \zeta, \sigma \in (1,2]$$

$$\varpi(0) + \frac{1}{2}\varpi'(0) = \int_0^1 \frac{|\varpi(\tau)|}{8+|\varpi(\tau)|}d\tau, \quad \varpi(1) + \frac{1}{2}\varpi'(1) = \int_0^1 \frac{|\varpi(\tau)|}{6+|\varpi(\tau)|}d\tau. \quad (4.3)$$

Here, we consider $\Im(\tau, \varpi(\tau)) = \frac{1}{5+\tau}\frac{|\varpi(\tau)|}{(2+|\varpi(\tau)|)}$, $\zeta_1(\varpi(\tau)) = \frac{|\varpi(\tau)|}{8+|\varpi(\tau)|}$, $\zeta_2(\varpi(\tau)) = \frac{|\varpi(\tau)|}{6+|\varpi(\tau)|}$ and $\alpha = 1$, $\beta = \frac{1}{2}$. Therefore, we have

$$\|\Im(\tau, \varpi) - \Im(\tau, \eta)\| \leq \frac{1}{5}\|\varpi - \eta\|,$$

and

$$\|\zeta_1(\varpi) - \zeta_1(\eta)\| \leq \frac{1}{8}\|\varpi - \eta\| \text{ and } \|\zeta_2(\varpi) - \zeta_2(\eta)\| \leq \frac{1}{6}\|\varpi - \eta\|.$$

Hence $(H_1)$ and $(H_2)$ are satisfied with $L_1 = \frac{1}{5}$, $L_2 = \frac{1}{8}$, $L_3 = \frac{1}{6}$, $M_2 = 1$, $M_3 = 1$. It is also clear that $\Im$ is bounded, and for $\lambda = 0.1$ with the aforementioned constants the hypothesis $(H_5)$ is also satisfied. Thus, by Theorem 3.4, the boundary value problem (4.3) has a unique solution on $[0, 1]$.

## 5 Conclusion

This research effectively deals with the existence and uniqueness of solutions for non-separated type integral boundary value problems governed by the combined Caputo fractional derivative. The definition of combined Caputo derivatives broadens the range of fractional calculus. Strong analytical techniques, such as the Banach fixed point theorem, Schaefer, and Krasnoselskii fixed point theorems, are employed to establish the results. The validity of the theoretical findings is demonstrated through numerical examples.

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