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Markov Triples with Padovan and Perrin components

Fatih Erduvan and Merve Güney Duman

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Corresponding Author: M. GÜNEY DUMAN

Abstract In this paper, we find that all Padovan and Perin numbers are Markov triples.

1 Introduction

Let $(P_n)_{n\geq 0}$ and $(R_n)_{n\geq 0}$ be the sequences of Padovan and Perrin numbers, respectively, defined by

$$P_0 = 0, P_1 = P_2 = 1;\ P_n = P_{n-2} + P_{n-3} \text{ for all } n \geq 3$$

and

$$R_0 = 3, R_1 = 0, R_2 = 2;\ R_n = R_{n-2} + R_{n-3} \text{ for all } n \geq 3.$$

The roots of the characteristic equation

$$x^3 - x - 1 = 0$$

has roots α,β and γ. Here

$$\alpha = \sqrt[3]{(9+\sqrt{69})/18} + \sqrt[3]{(9-\sqrt{69})/18},$$

and

$$\overline{\gamma} = \beta = -\left((1+i\sqrt{3})/2\right)\sqrt[3]{(9-\sqrt{69})/18} - \left((1-i\sqrt{3})/2\right)\sqrt[3]{(9+\sqrt{69})/18}.$$

Binet formulas for these numbers are

$$P_n = t\cdot\alpha^n + s\cdot\beta^n + r\cdot\gamma^n$$

and

$$R_n = \alpha^n + \beta^n + \gamma^n,$$

where

$$t = \frac{\alpha(\alpha+1)}{2\alpha+3}, s = \frac{\beta(\beta+1)}{2\beta+3}, r = \frac{\gamma(\gamma+1)}{2\gamma+3}.$$

Moreover, with a simple calculation, it can be shown that the following estimates hold:

$$1.32 < \alpha < 1.33,\ \ 0.86 < |\beta| = |\gamma| < \alpha^{-1/2} < 0.87,$$

and

$$0.54 < t < 0.55,\ \ 0.28 < |s| = |r| < 0.29.$$

Put

$$P_n = t\cdot\alpha^n + e_n \text{ or } P_n = t\cdot\alpha^n(1+x_n). \tag{1.1}$$

Here,

$$|e_n| < \frac{0.58}{\alpha^{n/2}} < 0.51 \text{ for all } n \geq 1, \tag{1.2}$$

and so

$$|x_n| = \frac{|e_n|}{t \cdot \alpha^n} < \frac{0.51}{0.54 \cdot \alpha^n} < \frac{0.95}{\alpha^n}. \tag{1.3}$$

Let

$$R_n = \alpha^n + e_n^{'} \text{ or } R_n = \alpha^n(1 + x_n^{'}). \tag{1.4}$$

Then,

$$|e_n^{'}| < \frac{2}{\alpha^{n/2}} < 1.74 \text{ for all } n \geq 1. \tag{1.5}$$

Moreover, we can say

$$|x_n^{'}| = \frac{|e_n^{'}|}{\alpha^n} < \frac{1.74}{\alpha^n}. \tag{1.6}$$

The relationship between P_n, R_n and α is given by

$$\alpha^{n-3} \leq P_n \leq \alpha^{n-1} \text{ for all } n \geq 1, \tag{1.7}$$

and

$$\alpha^{n-2} < R_n < \alpha^{n+1} \text{ for all } n \geq 2. \tag{1.8}$$

If the positive integers x, y and z satisfy the equation

$$x^2 + y^2 + z^2 = 3xyz,$$

then they are called markov triples. In recent years, finding Markov triples in terms of linear recurrence sequences has become a popular problem. In [1], authors determined Markov involving with Fibonacci components. Later, the same problem was studied for Pell numbers [2], k-generalized Fibonacci numbers [3] and k-generalized Pell numbers [4]. In this note, we solved the following equations

$$P_s^2 + P_m^2 + P_n^2 = 3P_sP_mP_n \tag{1.9}$$

and

$$R_s^2 + R_m^2 + R_n^2 = 3R_sR_mR_n \tag{1.10}$$

with Padovan and Perrin numbers, respectively.

2 Main Theorems

Before stating the following theorem, we note that $P_1 = P_2 = P_3 = 1$. Thus, we will take $n \geq m \geq s \geq 3$ to avoid trivial solutions.

Theorem 2.1. *All the solutions (s, m, n) to the Diophantine equation (1.9) are*

$$(s, m, n) \in \{(3, 3, 3), (3, 3, 4), (3, 3, 5), (3, 4, 8), (3, 5, 8)\}$$

in positive integers (s, m, n) with $3 \leq s \leq m \leq n$.

Proof. Assume that the equation (1.9) holds. First, let's examine the relationships among s, m and n. Combining (1.7) and (1.9), we can write

$$\alpha^{2(n-3)} \leq P_n^2 < 3P_sP_mP_n \leq 3\alpha^{s+m+n-3} < \alpha^{s+m+n+1}$$

and

$$3\alpha^{s+m+n-9} < 3P_sP_mP_n < 3P_n^2 < 3\alpha^{2(n-1)}.$$

The above inequalities give us

$$|n - (s + m)| \leq 6. \tag{2.1}$$

From (1.1) and (1.9), we can write

$$P_s^2 + P_m^2 + (t \cdot \alpha^n + e_n)^2 = 3t^3\alpha^{s+m+n}(1 + x), \tag{2.2}$$

where

$$1 + x = (1 + x_s)(1 + x_m)(1 + x_n).$$

Using the above equality and (1.3), we get $|x| < 6.42 \cdot \alpha^{-s}$. By the equation (2.2), we conclude that

$$t^2\alpha^{2n} - 3t^3\alpha^{s+m+n} = -P_s^2 - P_m^2 - e_n^2 - 2te_n\alpha^n + 3t^3\alpha^{s+m+n}x,$$

and hence,

$$\left|t^2\alpha^{2n} - 3t^3\alpha^{s+m+n}\right| \le P_s^2 + P_m^2 + |e_n|^2 + 2t\,|e_n|\,\alpha^n + 3t^3\alpha^{s+m+n}\,|x|\,.$$

The above inequality tell us

$$\begin{aligned}\left|(3t)^{-1}\alpha^{n-(s+m)} - 1\right| &\le \frac{2\alpha^{2(m-1)}}{3t^3\alpha^{s+m+n}} + \frac{|e_n|^2}{3t^3\alpha^{s+m+n}} + \frac{2t\,|e_n|\,\alpha^n}{3t^3\alpha^{s+m+n}} + \frac{6.42}{\alpha^s}\\ &\le \frac{9.44}{\alpha^s}.\end{aligned} \tag{2.3}$$

Here, we have used that $t > 0.54$, $|e_n| < 0.51$ and $n \ge m \ge 3$. Put $u := n - (s+m)$. From (2.1), we have $u \in \{0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6\}$. If $u > 0$, it can be seen that

$$0.79 < 0.6\alpha^1 \le 0.6\alpha^u < (3t)^{-1}\alpha^u < 0.62\alpha^u \le 0.62\alpha^6 < 3.36.$$

Thus, we have

$$\left|(3t)^{-1}\alpha^{n-(s+m)} - 1\right| > 0.21. \tag{2.4}$$

If $u < 0$, we can say

$$0.11 < 0.6\alpha^{-6} \le 0.6\alpha^u < (3t)^{-1}\alpha^u < 0.62\alpha^u \le 0.62\alpha^{-1} < 0.47.$$

From this, it can be easily shown that

$$\left|(3t)^{-1}\alpha^{n-(s+m)} - 1\right| > 0.53. \tag{2.5}$$

Finally, if we treat $u = 0$, then we obtain

$$\left|(3t)^{-1}\alpha^{n-(s+m)} - 1\right| > 0.38. \tag{2.6}$$

Taking into account to inequalities (2.3),(2.4),(2.5) and (2.6), we get $s \le 13$. The inequality (2.1) leads to $0 \le n - m \le 19$. We rewrite the equation (2.2) as

$$P_s^2 + (t \cdot \alpha^m + e_m)^2 + (t \cdot \alpha^n + e_n)^2 = 3P_s t^2\alpha^{m+n}(1 + y)$$

and so

$$\begin{aligned}3P_s t^2\alpha^{m+n} - t^2(\alpha^{2m} + \alpha^{2n}) &= P_s^2 + e_m^2 + e_n^2\\ &\quad + 2t(e_m\alpha^m + e_n\alpha^n) - 3P_s t^2\alpha^{m+n}y.\end{aligned} \tag{2.7}$$

Here,

$$1 + y = (1 + x_n)(1 + x_m) = 1 + x_n + x_m + x_n x_m.$$

The above equality and inequality (1.3) yield to $|y| < 2.81 \cdot \alpha^{-m}$. The equality (2.7) shows that

$$\begin{aligned}\left|1 - (3P_s)^{-1}(\alpha^{n-m} + \alpha^{-(n-m)})\right| &\le \frac{P_s}{3t^2\alpha^{m+n}} + \frac{|e_m|^2 + |e_n|^2}{3P_s t^2\alpha^{m+n}}\\ &\quad + \frac{2\,|e_m|\,\alpha^m}{3P_s t\alpha^{m+n}} + \frac{2\,|e_n|\,\alpha^n}{3P_s t\alpha^{m+n}} + |y|\,.\end{aligned}$$

Using the arguments

$$t > 0.54,\ |e_m| \le |e_n| < 0.51,\ n \ge m \ge 3 \text{ and } 3 \le s \le 13,$$

we find that

$$\left|1-(3P_s)^{-1}(\alpha^{n-m}+\alpha^{-(n-m)})\right| < 14.66\cdot\alpha^{-m}. \tag{2.8}$$

With a simple calculation, we see that

$$\left|1-(3P_s)^{-1}(\alpha^{n-m}+\alpha^{-(n-m)})\right| > 0.05 \tag{2.9}$$

for $0 \le n-m \le 19$ and $3 \le s \le 13$. From the inequalities (2.8) and (2.9), we conclude that $3 \le m \le 20$. On the other hand, we have $3 \le n \le 39$ by (2.1). Consequently, all solutions of the equation (1.9) are represented by

$$\begin{aligned} 1^2+1^2+1^2 &= P_3^2+P_3^2+P_3^2 = 3P_3P_3P_3 = 3\cdot 1\cdot 1\cdot 1,\\ 1^2+1^2+2^2 &= P_3^2+P_3^2+P_4^2 = 3P_3P_3P_4 = 3\cdot 1\cdot 1\cdot 2,\\ 1^2+1^2+2^2 &= P_3^2+P_3^2+P_5^2 = 3P_3P_3P_5 = 3\cdot 1\cdot 1\cdot 2,\\ 1^2+2^2+5^2 &= P_3^2+P_4^2+P_8^2 = 3P_3P_4P_8 = 3\cdot 1\cdot 2\cdot 5, \end{aligned}$$

and

$$1^2+2^2+5^2 = P_3^2+P_5^2+P_8^2 = 3P_3P_5P_8 = 3\cdot 1\cdot 2\cdot 5$$

for $3 \le s \le 13$, $3 \le m \le 20$ and $3 \le n \le 39$. Thus, the proof of our theorem is finished. □

Now, we can give the following result. To avoid trivial solutions we will take $n \ge m \ge s \ge 3$.

Theorem 2.2. *All the solutions (s,m,n) to the Diophantine equation (1.10) are*

$$(s,m,n) \in \{(4,5,12),(4,6,12)\}$$

in positive integers (s,m,n) with $3 \le s \le m \le n$.

Proof. We will largely follow the line of argument given in the proof of Theorem 2.1, omitting some details. Assume that the equation (1.10) holds. From (1.8) and (1.10), we can write

$$\alpha^{2(n-2)} < R_n^2 < 3R_sR_mR_n < 3\alpha^{s+m+n+3} < \alpha^{s+m+n+7}$$

and so

$$3\alpha^{s+m+n-6} < 3R_sR_mR_n < 3R_n^2 < 3\alpha^{2(n+1)}.$$

The above inequalities yield to

$$-7 \le n-(s+m) \le 10. \tag{2.10}$$

From (1.4) and (1.10), we can write

$$R_s^2+R_m^2+(\alpha^n+e_n^{'})^2 = 3\alpha^{s+m+n}(1+x^{'}), \tag{2.11}$$

where

$$1+x^{'} = (1+x_s^{'})(1+x_m^{'})(1+x_n^{'}).$$

The above equality and (1.6) imply that $\left|x^{'}\right| < 19.58\cdot\alpha^{-s}$. We arrange the equation (2.11) as

$$\alpha^{2n}-3\alpha^{s+m+n} = -R_s^2-R_m^2-e_n^{'2}-2e_n\alpha^n+3\alpha^{s+m+n}x^{'}.$$

This follows that

$$\begin{aligned} \left|3^{-1}\alpha^{n-(s+m)}-1\right| &\le \frac{2\alpha^{2(m+1)}}{3\alpha^{s+m+n}}+\frac{\left|e_n^{'}\right|^2}{3\alpha^{s+m+n}}+\frac{2\left|e_n^{'}\right|\alpha^n}{3\alpha^{s+m+n}}+\frac{19.58}{\alpha^s}\\ &\le \frac{21.44}{\alpha^s}, \end{aligned} \tag{2.12}$$

where we have used that $\left|e_n^{'}\right| < 1.74$ and $n \geq m \geq 3$. Let $u := n - (s + m)$. We have $u \in \{0, \pm 1, \pm 2, ..., \pm 6, \pm 7, 8, 9, 10\}$ by (2.10). If $u > 0$, we arrive that

$$\left|3^{-1}\alpha^{n-(s+m)}-1\right| > 0.57. \tag{2.13}$$

If $u < 0$, it is seen that

$$\left|3^{-1}\alpha^{n-(s+m)}-1\right| > 0.75. \tag{2.14}$$

When $u = 0$, we get

$$\left|3^{-1}\alpha^{n-(s+m)}-1\right| > 0.66. \tag{2.15}$$

Comparing (2.12),(2.13),(2.14) and (2.15), we obtain $s \leq 12$. Considering (2.10), we have $0 \leq n - m \leq 22$. We rearrange the equation (1.10) as

$$R_s^2 + (\alpha^m + e_m^{'})^2 + (\alpha^n + e_n^{'})^2 = 3R_s\alpha^{m+n}(1 + y^{'})$$

or

$$3R_s\alpha^{m+n} - (\alpha^{2m} + \alpha^{2n}) = R_s^2 + e_m^{'2} + e_n^{'2} + 2(e_m^{'}\alpha^m + e_n^{'}\alpha^n) - 3R_s\alpha^{m+n}y^{'}, \tag{2.16}$$

where

$$1 + y^{'} = (1 + x_n^{'})(1 + x_m^{'}) = 1 + x_n^{'} + x_m^{'} + x_n^{'}x_m^{'}.$$

The above inequality and (1.6) show that $\left|y^{'}\right| < 6.51 \cdot \alpha^{-m}$. The equality (2.16) implies that

$$\left|1 - (3R_s)^{-1}(\alpha^{-(n-m)} + \alpha^{n-m})\right| < 12.27 \cdot \alpha^{-m}. \tag{2.17}$$

After some calculations, we see that

$$\left|1 - (3R_s)^{-1}(\alpha^{-(n-m)} + \alpha^{n-m})\right| > 0.005 \tag{2.18}$$

for $0 \leq n - m \leq 22$ and $3 \leq s \leq 12$. From the inequalities (2.17) and (2.18), we conclude that $3 \leq m \leq 27$. On the other hand, we have $3 \leq n \leq 49$ by (2.10). Hence, all solutions of the equation (1.10) are represented by

$$\begin{aligned}
1^2 + 1^2 + 1^2 &= R_3^2 + R_3^2 + R_3^2 = 3R_3R_3R_3 = 3 \cdot 1 \cdot 1 \cdot 1, \\
1^2 + 1^2 + 2^2 &= R_3^2 + R_3^2 + R_4^2 = 3R_3R_3R_4 = 3 \cdot 1 \cdot 1 \cdot 2, \\
1^2 + 1^2 + 2^2 &= R_3^2 + R_3^2 + R_5^2 = 3R_3R_3R_5 = 3 \cdot 1 \cdot 1 \cdot 2, \\
1^2 + 2^2 + 5^2 &= R_3^2 + R_4^2 + R_8^2 = 3R_3R_4R_8 = 3 \cdot 1 \cdot 2 \cdot 5,
\end{aligned}$$

and

$$1^2 + 2^2 + 5^2 = R_3^2 + R_5^2 + R_8^2 = 3R_3R_5R_8 = 3 \cdot 1 \cdot 2 \cdot 5,$$

for $3 \leq s \leq 12$, $3 \leq m \leq 27$ and $3 \leq n \leq 49$. Thus, the proof of our theorem is complete. □

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Author information

Fatih Erduvan, MEB, İzmit Namık Kemal Anatolia High School, Türkiye.
E-mail: erduvanmat@hotmail.com

Merve Güney Duman, Sakarya University of Applied Sciences, Department of Engineering Fundamental Science, Türkiye.
E-mail: merveduman@subu.edu.tr

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