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ON MATRIX SOLUTIONS OF NATHANSON'S DIOPHANTINE EQUATIONS

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Abstract. Let m and p be two non-zero positive integers. We show that for all integer k such that $k > -m$, the Diophantine equation $X^m - Y^m = Z^{m+k}$ admits an infinite number of matrix solutions from the set of inversible matrices of order ℓp with positive integer coefficients, where ℓ is the lowest common multiple of m and $m + k$.

1 Introduction

In 2016, Karama studied the Diophantine equation $x^2 - y^2 = z^3$ and predicted that there are no positive integers solution of the equation $x^3 - y^3 = z^4$ (see [7]). In 2017, Nathanson solved this last equation and gave the positive integers solutions for $x \leq 5000$ (see [11]). He proposed to study the Diophantine equation

$$x^m - y^m = z^{m+k}, m \geq 2, \ k \geq 2. \tag{1.1}$$

In 2018, Leung initiated the study of the Diophantine equation (1.1) (see [4]). He showed that for every two positive integers k_1 and k_2, there is no positive integral solution for the Diophantine equation $x^{4k_1} - y^{4k_1} = z^{2k_2}$. However, several authors have been interested in the matrix solutions of Diophantine equations. Indeed, in 1966, Domiaty published a work which explores the solutions to the Diophantine equation $X^4 + Y^4 = Z^4$ using matrices with integer coefficients (see [3]). One can also see [1], [2], for more details. Moreover, in 2021, Mouanda introduced the "Galaxies Number Theory", which allows us to understand the laws and structures of different universes (multiverses). He used Rare matrices to construct some matrix solutions of some Diophantine equations, including Fermat's equation $X^n + Y^n = Z^n, \ n \in \mathbb{N}$ (see [5], [8], [9], [10]).

In this paper, we study the matrix solutions of the Diophantine equation

$$X^m - Y^m = Z^{m+k}, \tag{1.2}$$

where $m \in \mathbb{N}$ and $k \in \mathbb{Z}$. We show that for $k > -m$ and for all non-zero positive integer p, the Diophantine equation (1.2) admits an infinite number of matrix solutions from the set of inversible matrices of order ℓp with positive integer coefficients, where ℓ designates the lowest common multiple of m and $m + k$. Our work is organized as follow. In section 2, we recall some important notions concerning the Rare matrices. In section 3, we show that the Diophantine equation (1.2) admits a finite number of construction structures (matrix geometry) of matrix solutions and every construction structure allows us to construct an infinite number of matrix solutions.

2 Preliminaries

Let n be an integer such that $n \geq 2$. We have the following definitions.

Definition 2.1. (see [6]) A matrix $B \in M_n(\mathbb{N})$ is a construction structure of matrix solutions of Diophantine equations if there exists two positive integers m, β such that $B^m - \beta \times I_n = 0$.

Let us denote

$$D_n(\mathbb{N}) = \{B \in M_n(\mathbb{N}) : B^m - \beta \times I_n = 0, m, \beta \in \mathbb{N}^*\}$$

the set of all construction structures of matrix solutions of Diophantine equations from $M_n(\mathbb{N})$. It is obvious that $D_n(\mathbb{N}) \subset GL_n(\mathbb{N})$, where $GL_n(\mathbb{N}) = \{B \in M_n(\mathbb{N}) : \det(B) \neq 0\}$.

In [9], Mouanda used Rare matrices as construction structures of matrix solutions of Diophantine equations.

Definition 2.2. (see [9]) Let $a, b, c \in \mathbb{C}^*$. The $n \times n$-matrices of the form

$$c\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ a & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \text{ or } c\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & b \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

are called Rare matrices of order n and index 1. The index designates the number of complex coefficients different from 0 and 1.

Note that every Rare matrix of order n is an element of the set $D_n(\mathbb{N})$. There are some interesting properties of Rare matrices.

Remark 2.3. (see [9]) *Let $n \in \mathbb{N}$ such that $n \geq 2$ and*

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_n(\mathbb{C}),\ \alpha \neq 0$$

be a Rare matrix of order n and index 1. Then

$$A_\alpha^n = \begin{pmatrix} \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \alpha \end{pmatrix},\ A_\alpha^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{1}{\alpha} \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_\alpha^{-1} = A_{\frac{1}{\alpha}}^T,\ A_\alpha^n = \alpha I_n,\ (\beta A_\alpha)^{-1} = \frac{1}{\beta} A_\alpha^{-1}, \beta \neq 0.$$

In section 3, we use the Kronecker product of matrices.

Definition 2.4. (see [5]) *Let* $n, m \in \mathbb{N}^*$, $A = (a_{ij})_{1 \le i,j \le n} \in M_n(\mathbb{C})$ *and* $B \in M_m(\mathbb{C})$. *The Kronecker product of* A *and* B *is the matrix denoted* $A \otimes B$ *and defined by*

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{pmatrix} \in M_{mn}(\mathbb{C}).$$

It is well known that for all $A \in M_n(\mathbb{C})$ and all $B \in M_m(\mathbb{C})$, the Kronecker product $A \otimes B$ is not always equal to $B \otimes A$. However, we have

$$I_n \otimes I_m = I_{nm} = I_m \otimes I_n.$$

3 Construction structures of matrix solutions of the Diophantine equation $X^m - Y^m = Z^{m+k}$.

Let $m \in \mathbb{N}^*$, $k \in \mathbb{Z}$ such that $k > -m$. Let ℓ be the lowest common multiple of m and $m + k$, and

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix} \in M_\ell(\mathbb{N}), \ \alpha \neq 0$$

be a Rare matrix of order ℓ and index 1. We have the following result.

Proposition 3.1. *For all non-zero positive integers* x *and* y, *the matrix triple*

$$\left(A^r_{x+y}, A^r_x, A^s_y\right) \in M_\ell(\mathbb{N})^3$$

is a solution of the Diophantine equation $X^m - Y^m = Z^{m+k}$, *where* $r = \dfrac{m+k}{\gcd(m,k)}$ *and* $s = \dfrac{m}{\gcd(m,k)}$.

Proof. By noticing that $\gcd(m,k) = \gcd(m, m+k)$, it follows that

$$r = \frac{m+k}{\gcd(m,k)} \in \mathbb{N}^* \text{ and } s = \frac{m}{\gcd(m,k)} \in \mathbb{N}^*.$$

So, we have

$$rm = s(m+k) = \frac{m(m+k)}{\gcd(m,k)} = \ell.$$

Since for all $x, y \in \mathbb{N}$, one has $A^\ell_x + A^\ell_y = A^\ell_{x+y}$, then $A^{rm}_{x+y} = A^{rm}_x + A^{s(m+k)}_y$, which is equivalent to say that $\left(A^r_{x+y}\right)^m = \left(A^r_x\right)^m + \left(A^s_y\right)^{m+k}$. So, one has

$$\left(A^r_{x+y}\right)^m - \left(A^r_x\right)^m = \left(A^s_y\right)^{m+k}.$$

Consequently, the matrix triple $\left(A^r_{x+y}, A^r_x, A^s_y\right)$ is a solution of the Diophantine equation

$$X^m - Y^m = Z^{m+k}.$$

□

It follows from Proposition 3.1 that the set $\left\{\left(A^r_{x+y}, A^r_x, A^s_y\right), x, y \in \mathbb{N}^*\right\}$ is an infinite set of matrix triple solutions of the Diophantine equation $X^m - Y^m = Z^{m+k}$, from the set $M_\ell(\mathbb{N})$. Moreover, it is well known that from A_α, we deduce ℓ matrices $A_{\alpha,j}, j \in \{1, ..., \ell\}$, by the permutations of α and 1 between the columns, which are $A_{\alpha,1} = A_\alpha$ and

$$A_{\alpha,2} = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \; A_{\alpha,3} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$..., A_{\alpha,\ell-1} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \; A_{\alpha,\ell} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}.$$

Let denote by

$$CS\left(A_\alpha\right) = \left\{A_{\alpha,j}, A^T_{\alpha,j}, j = 1, ..., \ell\right\}.$$

We can see that every element in $CS\left(A_\alpha\right)$ is a construction structure of matrix solutions of the Diophantine equation $X^m - Y^m = Z^{m+k}$, that means $CS\left(A_\alpha\right) \subset D_\ell(\mathbb{N})$. So, there are at least $(2\ell)^3 = 8\ell^3$ construction structures of matrix solutions of the Diophantine equation $X^m - Y^m = Z^{m+k}$, from the set $M_\ell(\mathbb{N})$.

Proposition 3.2. *Let $\alpha, n, p \in \mathbb{N}^*$ such that $n \geq 2$ and*

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \alpha & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in M_n\left(\mathbb{N}\right)$$

be a Rare matrix of order n and index 1. Then

$$\left(A_\alpha \otimes I_p\right)^n = \left(I_p \otimes A_\alpha\right)^n = \alpha I_{np}.$$

Proof.

$a)$ Let us show that $(I_p \otimes A_\alpha)^n = \alpha I_{np}$. We have

$$(I_p \otimes A_\alpha)^n = \begin{pmatrix} A_\alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & A_\alpha & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_\alpha & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_\alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_\alpha \end{pmatrix}^n$$

$$= \begin{pmatrix} \alpha I_n & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \alpha I_n & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha I_n & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha I_n & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \alpha I_n & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \alpha I_n \end{pmatrix}$$

$$= \alpha I_p \otimes I_n = \alpha I_{np}.$$

$b)$ Let us show that $(A_\alpha \otimes I_p)^n = \alpha I_{np}$.

$$A_\alpha \otimes I_p = \begin{pmatrix} 0 & I_p & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & I_p & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_p & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & I_p \\ \alpha I_p & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 1 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 1 \\ \alpha & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & \alpha & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \alpha & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \alpha & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in M_{np}(\mathbb{N}).$$

$A_\alpha \otimes I_p$ is a Rare matrix of order np and index p. So, $A_\alpha \otimes I_p$ is the p^{th} power of a Rare matrix R_α of order np and index 1. Therefore,

$$(A_\alpha \otimes I_p)^n = (R_\alpha^p)^n = R_\alpha^{np} = \alpha I_{np}.$$

□

Remark 3.3. Let $m \in \mathbb{N}^*$, $k \in \mathbb{Z}$ such that $k > -m$. Let ℓ be the lowest common multiple of m and $m+k$, and A_α be a Rare matrix of order ℓ and index 1. For all $j \in \{1, ..., \ell\}$ and all $p \in \mathbb{N}^*$, we have

$$(A_{\alpha,j} \otimes I_p)^\ell = A_{\alpha,j}^\ell \otimes I_p = \alpha I_{\ell p} \text{ and } (I_p \otimes A_{\alpha,j})^\ell = I_p \otimes A_{\alpha,j}^\ell = \alpha I_{\ell p}.$$

Let us denote by

$$B_{\alpha,j}(p) = I_p \otimes A_{\alpha,j} \text{ and } C_{\alpha,j}(p) = A_{\alpha,j} \otimes I_p, \forall j = 1, ..., \ell.$$

From the permutations of α and 1 between the columns of

$$B_{\alpha,1}(p) = \begin{pmatrix} \boxed{A_\alpha} & 0 & \cdots & 0 & 0 \\ 0 & \boxed{A_\alpha} & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & \boxed{A_\alpha} & 0 \\ 0 & 0 & \cdots & 0 & \boxed{A_\alpha} \end{pmatrix} \in M_{\ell p}(\mathbb{N}),$$

the associated matrices for which the operations are made in the blocks A_α, are construction structures of matrix solutions of Diophantine equations. The number of such matrices is ℓ^p.

Example 3.4.

1) Let $\ell = 2$ and $p = 3$. We have

$$B_{\alpha,1}(3) = \begin{pmatrix} A_\alpha & 0 & 0 \\ 0 & A_\alpha & 0 \\ 0 & 0 & A_\alpha \end{pmatrix}.$$

From the permutations of α and 1 in the same block A_α, we obtain

$$B_{\alpha,1,1}(3) = B_{\alpha,1}(3), \quad B_{\alpha,1,2}(3) = \begin{pmatrix} A_\alpha^T & 0 & 0 \\ 0 & A_\alpha & 0 \\ 0 & 0 & A_\alpha \end{pmatrix},$$

$$B_{\alpha,1,3}(3) = \begin{pmatrix} A_\alpha & 0 & 0 \\ 0 & A_\alpha^T & 0 \\ 0 & 0 & A_\alpha \end{pmatrix}, \quad B_{\alpha,1,4}(3) = \begin{pmatrix} A_\alpha & 0 & 0 \\ 0 & A_\alpha & 0 \\ 0 & 0 & A_\alpha^T \end{pmatrix},$$

$$B_{\alpha,1,5}(3) = \begin{pmatrix} A_\alpha^T & 0 & 0 \\ 0 & A_\alpha^T & 0 \\ 0 & 0 & A_\alpha \end{pmatrix}, \quad B_{\alpha,1,6}(3) = \begin{pmatrix} A_\alpha^T & 0 & 0 \\ 0 & A_\alpha & 0 \\ 0 & 0 & A_\alpha^T \end{pmatrix},$$

$$B_{\alpha,1,7}(3) = \begin{pmatrix} A_\alpha & 0 & 0 \\ 0 & A_\alpha^T & 0 \\ 0 & 0 & A_\alpha^T \end{pmatrix}, \quad B_{\alpha,1,8}(3) = \begin{pmatrix} A_\alpha^T & 0 & 0 \\ 0 & A_\alpha^T & 0 \\ 0 & 0 & A_\alpha^T \end{pmatrix}.$$

For every $j = 1, ..., 8$, a simple calculation show that $B_{\alpha,1,j}(3) \in D_6(\mathbb{N})$.

2) Let $\ell = 3$ and $p = 2$. We have

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 0 & 0 \end{pmatrix} \text{ and } B_{\alpha,1}(2) = \begin{pmatrix} A_\alpha & 0 \\ 0 & A_\alpha \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \alpha & 0 & 0 \end{pmatrix}.$$

From the permutations of α and 1 in the same block A_α, we obtain $B_{\alpha,1,1}(2) = B_{\alpha,1}(2)$ and

$$B_{\alpha,1,2}(2) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \alpha & 0 & 0 \end{pmatrix}, \quad B_{\alpha,1,3}(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \alpha & 0 & 0 \end{pmatrix},$$

$$B_{\alpha,1,4}(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_{\alpha,1,5}(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{\alpha,1,6}(2) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_{\alpha,1,7}(2) = \begin{pmatrix} 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$B_{\alpha,1,8}(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad B_{\alpha,1,9}(2) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

For every $j = 1, ..., 9$, a simple calculation show that $B_{\alpha,1,j}(2) \in D_6(\mathbb{N})$.

Remark 3.5. Let $\alpha, \ell, p \in \mathbb{N}^*$ such that $p \geq 2$ and $\ell \geq 2$. All matrices obtained from

$$B_{\alpha,1}(p) = \begin{pmatrix} A_\alpha & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & A_\alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & A_\alpha & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & A_\alpha & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & A_\alpha & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_\alpha \end{pmatrix} \in M_{\ell p}(\mathbb{N}),$$

by the permutations of α and 1 between the columns but not in the same block, are not construction structures of matrix solutions. Indeed, one block will have at least two α and for this, we have $A_\alpha^\ell = \alpha^2 I_\ell$.

From the permutations of α and 1 between the columns of $C_{\alpha,1}(p)$, when the α's are in consecutive columns, the associated matrices are construction structures of matrix solutions of Diophantine equations. This is also true when the α's are in the first η columns ($1 \leq \eta < p\ell$) and the other in the last $p\ell - \eta$ columns. The number of such matrices is $p\ell$.

Example 3.6. Let $\ell = 3$, $p = 2$ and $\alpha \in \mathbb{N}^*$. Then,

$$A_\alpha = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \alpha & 0 & 0 \end{pmatrix} \text{ and } C_{\alpha,1}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \end{pmatrix}.$$

When the α's are in consecutive columns, the associated matrices to $C_{\alpha,1}(2)$ are

$$C_{\alpha,1,1}(2) = C_{\alpha,1}(2), \quad C_{\alpha,1,2}(2) = \begin{pmatrix} 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{\alpha,1,3}(2) = \begin{pmatrix} 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{\alpha,1,4}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C_{\alpha,1,5}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad C_{\alpha,1,6}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

By a simple calculation, we have $C_{\alpha,1,j}(2) \in D_6(\mathbb{N})$ for every $j = 1, ..., 6$.

Let us denote all these construction structures of matrix solutions obtained from $B_{\alpha,1}(p)$ and $C_{\alpha,1}(p)$ by

$$K_{A_\alpha,i}(p), i \in \{1, ..., p\ell + \ell^p\}$$

and let us set

$$CS^*(A_\alpha(p)) = \{K_{A_\alpha,i}(p), i \in \{1, ..., p\ell + \ell^p\}\} \subset D_{\ell p}(\mathbb{N}).$$

It should be noted that the matrices $B_{\alpha,j}(p)$ and $C_{\alpha,j}(p)$, $j = 2, ..., \ell$, are obtained by certain permutations of α and 1 respectively in $B_{\alpha,1}(p)$ and $C_{\alpha,1}(p)$.

There are many other construction structures of matrix solutions obtained by the permutations of α and 1 in $C_{\alpha,1}(p)$. We can also notice that these construction structures do not necessarily have the same geometry for all p.

Example 3.7. Let $\ell = 3$, $p = 2$ and $\alpha \in \mathbb{N}^*$. The following matrices obtained from

$$C_{\alpha,1}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \end{pmatrix}$$

by some permutations of α and 1 between the columns such that the α's are not in consecutive columns, are construction structures of matrix solutions. These are

$$C_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 & 0 & 0 \end{pmatrix}, C_3 = \begin{pmatrix} 0 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The number of these matrices is large when p is large enough.

Proposition 3.8. *Let m, p be two positive integers and let k be an integer such that $k > -m$ and $p \geq 2$. Let ℓ be the lowest common multiple of m and $m+k$. Let A_α be a Rare matrix of order ℓ and index 1. For all non-zero positive integers x, y and for all $P_\alpha, Q_\alpha, R_\alpha \in CS^*(A_\alpha(p))$, the triple $\left(P_{x+y}^r, Q_x^r, R_y^s\right) \in M_{\ell p}(\mathbb{N})^3$ is a matrix solution of the Diophantine equation*

$$X^m - Y^m = Z^{m+k},$$

where $r = \dfrac{m+k}{\gcd(m,k)}$ *and* $s = \dfrac{m}{\gcd(m,k)}$.

Proof. Let $P_\alpha, Q_\alpha, R_\alpha \in CS^*(A_\alpha(p))$ and let $x, y \in \mathbb{N}^*$. We have

$$\begin{aligned}\left(P_{x+y}^r\right)^m - \left(Q_x^r\right)^m &= P_{x+y}^{rm} - Q_x^{rm} = (x+y)I_{\ell p} - xI_{\ell p} = yI_{\ell p} \\ &= R_y^\ell = R_y^{s(m+k)} = \left(R_y^s\right)^{m+k}.\end{aligned}$$

Therefore,

$$\left(P_{x+y}^r\right)^m - \left(Q_x^r\right)^m = \left(R_y^s\right)^{m+k}.$$

Finally, we can see that the triple $\left(P_{x+y}^r, Q_x^r, R_y^s\right)$ is a matrix solution of the Diophantine equation $X^m - Y^m = Z^{m+k}$. □

Consequently, since $CS^*(A_\alpha(p))$ has $\ell(p+\ell^{p-1})$ elements for every $p \geq 2$, there are at least $\ell^3\left(p+\ell^{p-1}\right)^3$ construction structures of matrix solutions, every construction structure allowing us to construct an infinite number of matrix solutions of the Diophantine equation

$$X^m - Y^m = Z^{m+k},$$

from the set $M_{\ell p}(\mathbb{N})$.

Now, we can state our main result.

Theorem 3.9. *Let m and p be two non-zero positive integers and let k be an integer such that $k > -m$. Let ℓ be the lowest common multiple of m and $m+k$. Then, the Diophantine equation $X^m - Y^m = Z^{m+k}$ has an infinite number of matrix triple solutions from the set $GL_{\ell p}(\mathbb{N})$.*

Proof. From Proposition 3.1 and Proposition 3.8, the set

$$\left\{\left(P_{x+y}^r, Q_x^r, R_y^s\right), x, y \in \mathbb{N}^*, P_\alpha, Q_\alpha, R_\alpha \in CS^*(A_\alpha(p))\right\} \subset M_{\ell p}(\mathbb{N})^3$$

is an infinite set of matrix triple solutions of the Diophantine equation $X^m - Y^m = Z^{m+k}$. □

4 Applications to some Diophantine equations

In [4], Leung showed that there is no positive integers solution for the Diophantine equation $x^{4k_1} - y^{4k_1} = z^{2k_2}$, $k_1, k_2 \in \mathbb{N}^*$. He conjectured that there is no positive integral solution for the Diophantine equations $x^6 - y^6 = z^2$ and $x^4 - y^4 = z^7$. In this section we investigate for matrix solutions of these equations.

4.1 Matrix solutions of the equation $X^{4k_1} - Y^{4k_1} = Z^{2k_2}$, $k_1, k_2 \in \mathbb{N}^*$

Let $d = \gcd(2k_1, k_2)$. Here we have

$$m = 4k_1,\ m + k = 2k_2,\ \ell = \frac{4k_1k_2}{d},\ r = \frac{k_2}{d} \text{ and } s = \frac{2k_1}{d}.$$

Let $p \in \mathbb{N}^*$ and let A_α be a Rare matrix of order ℓ and index 1. By taking

$$P_\alpha, Q_\alpha, R_\alpha \in CS^*\left(A_\alpha(p)\right),$$

we have

$$\forall x, y \in \mathbb{N}^*,\ \left(P^r_{x+y}\right)^{4k_1} - \left(Q^r_x\right)^{4k_1} = P^\ell_{x+y} - Q^\ell_x \ \ = (x + y - x)I_{p\ell} = yI_{p\ell} = R^\ell_y = \left(R^s_y\right)^{2k_2}.$$

Therefore, the equality follows and the triple $\left(P^r_{x+y}, Q^r_x, R^s_y\right) \in M_{p\ell}(\mathbb{N})^3$ is a matrix solution of the Diophantine equation $X^{4k_1} - Y^{4k_1} = Z^{2k_2}$, for all non-zero positive integers x and y. Consequently, this equation admits an infinite number of matrix solutions.

4.2 Matrix solutions of the equation $X^6 - Y^6 = Z^2$

In this case, one has $m = 6$ and $k = -4$. It is very easy to see that the lowest common multiple of m and $m+k$ is $\ell = 6$. For $p \geq 2$, there are at least $N_{min} = (6p + 6^p)^3$ construction structures of matrix triple solutions of the Diophantine equation $X^6 - Y^6 = Z^2$ from the set M_{6p}. Let us take

$$P_\alpha = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ \alpha&0&0&0&0&0&0&0&0&0&0&0\\ 0&\alpha&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}, Q_\alpha = \begin{pmatrix} 0&0&\alpha&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&\alpha&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

and

$$R_\alpha = \begin{pmatrix} 0&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&\alpha&0&0&0&0&0\\ 0&0&0&0&0&0&0&\alpha&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1\\ 1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

three elements in $CS^*\left(A_\alpha(2)\right)$. For every two non-zero positive integers x and y, a simple calculation shows that

$$P^6_{x+y} - Q^6_x = \left(R^3_y\right)^2.$$

4.3 Matrix solutions of the equation $X^4 - Y^4 = Z^7$

In this case, we have $m = 4$ and $k = 3$. The lowest common multiple of m and $m + k$ is $\ell = 28$. For all $p \geq 2$, there are at least $N_{min} = (28p + 28^p)^3$ construction structures of matrix triple solutions of the equation $X^4 - Y^4 = Z^7$. As shown above, for $p = 1$, we have $N_{min} = 8 \times 28^3 = 175\,616$. For example, let

$$P_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ \alpha & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, Q_\alpha = \begin{pmatrix} 0 & \alpha & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix},$$

$$R_\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \alpha & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix} \in CS^* \left(A_\alpha(1)\right) = CS\left(A_\alpha\right) \subset D_{28}\left(\mathbb{N}\right).$$

For every non-zero positive integers x and y, the matrix triple $\left(P^7_{x+y}, Q^7_x, R^4_y\right)$ is a solution of the Diophantine equation $X^4 - Y^4 = Z^7$.

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