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A Study of DNA Computation Using the Ring $\Im$

Narendra Kumar

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Corresponding Author: N. Kumar

Abstract. This work's main objective is to go over several significant characteristics of DNA codes that help with information storage and retrieval in synthetic DNA. Here, we take into consideration the algebraic framework for cyclic codes over a ring $\Im = \mathbb{F}_2[\mu, \nu]/\langle \mu^2, \nu^2, \mu\nu = \nu\mu \rangle$ for such discussion. The construction of reversible and reversible complement constraints is discussed. The deletion distance and GC-content constraints are described. In order to validate the primary findings, multiple DNA code examples are given as an application. Additionally, for these samples, we calculate the deletion similarity and Hamming distances.

1 Introduction

Deoxyribonucleic acid (DNA) has attracted significant interest in a variety of research communities in recent years, especially in the areas of information theory and bioinformatics, regarding its potential as a medium for storing user data. Because of its great capacity, endurance, and storage density, DNA-based data storage is significant. These benefits encouraged scholars to investigate the field's evolution. There are several uses for DNA codes in synthetic DNA strands for information storage and retrieval. The physical processes of DNA synthesis and sequencing are used to create and read DNA sequences, respectively [1]. Multiple sorts of errors can arise during synthesis and sequencing, highlighting the significance of studying the minimum distance of DNA code, a property that makes it possible to suppress those errors. In a DNA-based system, every aspect of the data storage procedure is proposed in Figure 1.

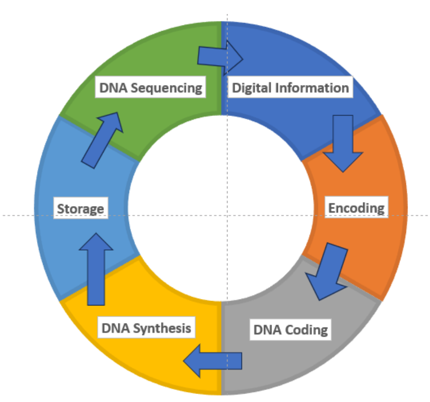

Figure 1. DNA Storage Procedure

For the first time, Adleman [2] used DNA molecules for the solution of the Hamiltonian path problem. Marathe et al. [3] introduced four different constraints for DNA codes, which are explained in Figure 2. A DNA strand is composed of **four nucleotides:** A (adenine), G (guanine), T (thymine), and C (cytosine). The rule of **Watson-Crick complementary** associates these bases as follows: $\overline{T} = A$, $\overline{A} = T$, $\overline{C} = G$, and $\overline{G} = C$.

Numerous scholars have become interested in this intriguing topic since DNA computing has larger storage capacities than silicon-based computing systems. DNA strands that accurately hybridize with their complementary counterparts while minimizing false pairings are required for applications of DNA computing. For example, accurate binding between DNA sequences and their complements is necessary for computation to be successful in Adleman's experiment [2]. False hybridization can happen between a DNA strand and either its reverse complement or the complement of another strand. A further important requirement for designing DNA sequences is to avoid secondary structures, which make strands inactive. When complementary base pairing causes a sequence to fold back into itself, a secondary structure is created [4]. Studies [5] have revealed that the creation of a particular kind of secondary structure could cause thirty percent of read-out efforts in a DNA storage device to fail. Every codeword in a DNA code should ideally have the same CG content in order to guarantee similar melting temperatures.[6]. The WCC pairs

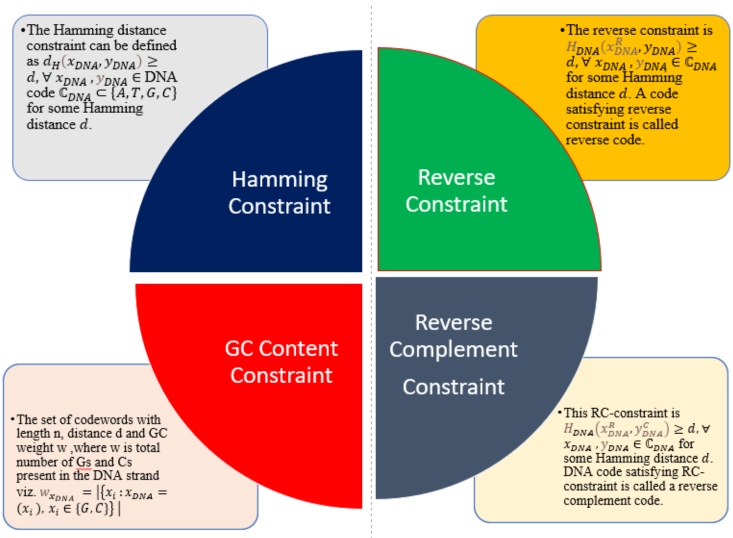

Figure 2. Constraints for DNA Codes

of chemical bonds differ from one another. A pair of hydrogen bonds holds together adenine (A) and thymine (T), whereas three hydrogen bonds hold together cytosine (C) and guanine (G). The quantity of A and T pairings as well as C and G pairs, establishes the DNA molecule's overall energy. A double helix's energy determines how stable it is. In this way, a DNA strand's durability is determined by the quantity of hydrogen bonds it contains. DNA hybridization is studied mathematically using the concept of similarity functions [7], which could be employed to simulate a thermodynamic resemblance between single-stranded DNA sequences. The largest sequence length that appears as a (possibly non-contiguous) subsequence of two sequences, X and Y, is referred to as a deletion similarity $S(X, Y)$ between X and Y. According to the authors [8, 9], deletion similarity measures the quantity of base pair bonds that are created between X and the reverse complement of Y during the hybridization experiment.

Numerous studies have examined cyclic DNA coding over finite rings [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 30]. DNA codes are constructed linearly and are provided in [12]. In their DNA computing study, Siap et al. [21] used codes based on the ring of four elements, whereas Abualrub et al. [10] studied DNA codes using the finite field of 4 elements. In this field of study, the nonchain ring of 16 components was taken into consideration by Bayram et al. [11] for the creation of DNA codes. Using a different type of ring for the same cardinality, Dinh et al. [15] have recently studied DNA codes. Moreover, Narendra and Abhay [23, 24, 29] investigated DNA codes by utilizing the cyclic codes structure with a non-chain ring. They also presented a number of reversible cyclic codes, with $\mathbb{Z}_4$-images that are new distance-optimal

$\mathbb{Z}_4$-linear codes.

The rest of the article is laid out in the following way. Section 2 delves into fundamental definitions of cyclic DNA codes, specifically considering the ring $\Im$. Moving forward, Section 3 focuses on exploring the properties of reversible constraint codes within the context of $\Im$. Our attention then shifts to reversible complement constraint codes over $\Im$ in Section 4. Section 5 investigates the GC Content within DNA cyclic codes. The calculation of several examples of DNA cyclic codes over $\Im$ is covered within Section 6. Section 7 summarizes key findings and contemplates the future avenues for research in this domain.

2 Preliminaries

Let $\Im$ be a ring of size 16, defined as $\mathbb{F}_2 + \mu\mathbb{F}_2 + \nu\mathbb{F}_2 + \mu\nu\mathbb{F}_2$ with $\mu^2 = \nu^2 = 0$ and $\mu\nu = \nu\mu$. A cyclic code is characterized as a type of linear code, denoted by $\mathfrak{C}$, where ring $\Im$ of length n if $(\varsigma_{n-1}, \varsigma_1, \dots, \varsigma_{n-2}) \in \mathfrak{C}$ for each $(\varsigma_0, \varsigma_1, \dots, \varsigma_{n-1}) \in \mathfrak{C}$. A codeword ς of length n has a Hamming weight, $w_H(\varsigma)$, which indicates how many non-zero entries there are in ς. However, the calculation of the Hamming weight of the difference between two codewords $\varsigma_1, \varsigma_2 \in \mathfrak{C}$ yields the Hamming distance $d(\varsigma_1, \varsigma_2)$ between them. The shortest Hamming distance between two different codewords is the minimal distance of a code $\mathfrak{C}$. Assume that Υ is a Grey map that is specified as follows: $\Im$ items are associated with $\mathbb{F}_2^4$ elements.

$$\Upsilon(\mathscr{A} + \mu\mathscr{B} + \nu\mathscr{C} + \mu\nu\mathscr{D}) = (\mathscr{A} + \mathscr{B} + \mathscr{C} + \mathscr{D}, \mathscr{C} + \mathscr{D}, \mathscr{B} + \mathscr{D}, \mathscr{D}),$$

where $\mathscr{A}, \mathscr{B}, \mathscr{C}$, and $\mathscr{D} \in \mathbb{F}_2$.

A collection of codewords $(\varsigma_0, \varsigma_1, \dots, \varsigma_{n-1})$, where $\varsigma_i \in \{A, T, G, C\}$, represents a n-length DNA code. To facilitate representation, we adhere to the mapping

$$0 \to \text{A}, \quad 1 \to \text{G}, \quad \text{u} \to \text{T}, \quad 1{+}\text{u} \to \text{C}.$$

The table uses a Gray map to illustrate a one-to-one link between DNA double pairs and $\Im$ elements. By creating a direct connection between the DNA double pairs and the components of the specified ring $\Im$, this correspondence creates a link between algebraic structures and genetic information.

Let $\hbar(x) = \flat_0 + \flat_1 x + \cdots + \flat_r x^r$ be a polynomial, where with $\flat_r \neq 0$. Then $\hbar^*(x) = x^r \hbar(1/x) = \flat_r + \flat_{r-1} x + \cdots + \flat_0 x^r$ is the reciprocal of $\hbar(x)$. A polynomial $\hbar(x)$ is considered self-reciprocal if, for a constant m, $\hbar^*(x) = m\hbar(x)$. In the case of a codeword $\flat \in \Im^n$, $\flat^r = (\flat_{n-1}, \flat_{n-2}, \dots, \flat_1, \flat_0)$ provides the reverse of $\flat$, $\flat^c = (\overline{\flat_0}, \overline{\flat_1}, \dots, \overline{\flat_{n-1}})$ represents the complement of $\flat$, and $\flat^{rc} = (\overline{\flat_{n-1}}, \overline{\flat_{n-2}}, \dots, \overline{\flat_1}, \overline{\flat_0})$ defines the reverse-complement of $\flat$. The condition that reversing any codeword $\flat \in \mathfrak{C}$ results in another valid codeword $\flat^r$ within the same code is what defines reversible codes, which belong to the family of linear codes $\mathfrak{C}$ of length n. If, for every $\flat \in \mathfrak{C}$, complementing each codeword, $\flat^c$, results in a codeword in $\mathfrak{C}$, then the codeword is termed a complement. It is also called reversible-complement if, for every $\flat \in \mathfrak{C}$, the reverse-complement, $\flat^{rc}$, of each codeword is also in $\mathfrak{C}$. Identifying the ring $\mathbb{F}_2[x]/\langle x^n - 1\rangle$ as the principal ideal ring, it contains the ideals that make up the resulting collection. Accordingly, we obtain

$$\mathfrak{C}_1 = \langle \eth(x) \rangle, \mathfrak{C}_2 = \langle \partial_1(x) \rangle, \mathfrak{C}_3 = \langle \partial_2(x) \rangle, \mathfrak{C}_4 = \langle \partial_3(x) \rangle. \quad (2.1)$$

Lemma 2.1. *A cyclic code $\mathfrak{C} = \langle \flat(x) \rangle$ over $\mathbb{F}_2$ is reversible if and only if the generator polynomial $\flat(x)$ is self-reciprocal.*

3 Reversible Property

In the following section, we look closely at the intricate structure and properties of reversible codes within the mathematical framework defined by the ring $\Im$. Our exploration encompasses a detailed examination of fundamental results, laying the groundwork for subsequent findings and conclusions. The insights gained from these basic results play a significant part in shaping our understanding of reversible codes in this context.

0	(0,0)	AA
v	(1,1)	GG
uv	(u,u)	TT
$v+uv$	(1+u,1+u)	CC
1	(0,1)	AG
$1+v$	(1,0)	GA
$1+uv$	(u,1+u)	TC
$1+v+uv$	(1+u,u)	CT
u	(0,u)	AT
$u+v$	(1,1+u)	GC
$u+uv$	(u,0)	TA
$u+v+uv$	(1+u,1)	CG
$1+u$	(0,1+u)	AC
$1+u+v$	(1,u)	GT
$1+v+uv$	(u,1)	TG
$1+u+v+uv$	(1+u,0)	CA

Table: DNA Correspondence

Lemma 3.1. [28] *Assume that $\hbar(x), \ell(x)$ are two polynomials in $\Im[x]$ with $\deg \hbar(x) \geq \deg \ell(x)$. Then*

- $[\hbar(x)\ell(x)]^* = \hbar^*(x)\ell^*(x)$;
- $[\hbar(x) + \ell(x)]^* = \hbar^*(x) + x^{\deg \hbar - \deg \ell}\ell^*(x)$.

To meet the specific requirements and objectives, we can employ a customized version of Theorem 3.2 and Theorem 3.3 in the context of generating cyclic codes over the ring $\Im$ with a length of n as per [Proposition 3.3, and Theorem 3.4 [27]]. These theorems offer the essential framework and guidelines necessary for constructing cyclic codes tailored to our specific goals and needs.

Theorem 3.2. *Let $\mathfrak{C} = \langle \kappa(x) \rangle$ be a cyclic code over $\Im$, where $\kappa(x) = \eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x), \mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x), \nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x), \mu\nu\partial_3(x)$. For code $\mathfrak{C}$ to be reversible, the necessary and sufficient condition is*

(a) $\eth(x)$, $\partial_1(x)$, $\partial_2(x)$ *and* $\partial_3(x)$ *are self-reciprocal;*

(b) (i) $\partial_2(x) | (x^j\gamma_1^*(x) - \gamma_1(x))$ *and* $\partial_1(x) | (x^i\rho_1^*(x) - \rho_1(x))$ *or*

(ii) $\mu x^i\rho_1^*(x) - \mu\rho_1(x) + \nu x^j\gamma_1^*(x) - \nu\gamma_1(x) + \mu\nu x^k\mathfrak{r}_1^*(x) - \mathfrak{r}_1(x) = \Big(\eth(x) + \mu\rho_1(x) +$

$\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x)\Big)p(x)+\Big(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x)\Big)n_1(x)+\Big(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x)\Big)l_1(x)+$ $\mu\nu\Big(\partial_3(x)k_1(x)+\partial_1(x)n_2(x)\Big)$, *where* $i=\deg\eth(x)-\deg\rho_1(x), j=\deg\eth(x)-\deg\gamma_1(x), k=\deg\eth(x)-\deg\mathfrak{r}_1(x)$, $p(x)\in\Im[x]$ *and* $n_1(x),n_2(x),l_1(x),k_1(x)\in\Im_1[x]$.

Proof. The code $\mathfrak{C}=\langle\kappa(x)\rangle$ is assumed to be reversible over $\Im$. Consider the four ideals that are $\mathfrak{C}$ as $\mathfrak{C}_1=\langle\eth(x)\rangle,\mathfrak{C}_2=\langle\partial_1(x)\rangle,\mathfrak{C}_3=\langle\partial_2(x)\rangle,\mathfrak{C}_4=\langle\partial_3(x)\rangle$, where $\eth(x),\partial_1(x),\partial_2(x),\partial_3(x)\in\mathbb{F}_2[x]$ are monic polynomials. According to Lemma 2.1, the provided polynomials $\eth(x),\partial_1(x),\partial_2(x),\partial_3(x)$ are self-reciprocal. With $\eth(x)$ being self-reciprocal, we might therefore say that

$$\begin{aligned}(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))^*(x)&=\eth^*(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x)\\&=\eth(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x)\end{aligned}\tag{3.1}$$

where $i=\deg\eth(x)-\deg\rho_1(x), j=\deg\eth(x)-\deg\gamma_1(x), k=\deg\eth(x)-\deg\mathfrak{r}_1(x)$. The reversibility of $\mathfrak{C}$ implies that

$$\begin{aligned}(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))^*(x)&=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1)\mathfrak{m}(x)\\&+(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))n(x)\\&+(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))l(x)+\mu\nu\partial_3(x)k(x),\end{aligned}\tag{3.2}$$

where $\mathfrak{m}(x),n(x),l(x),k(x)$ are polynomials over $\Im$. From equations (3.1) and (3.2), we have

$$\begin{aligned}\eth(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x)&=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu r_1)\mathfrak{m}(x)\\&+(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))n(x)\\&+(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))l(x)+\mu\nu\partial_3(x)k(x)\end{aligned}\tag{3.3}$$

Multiply the equation (3.3) by $\mu\nu$ on both sides,

$$\mu\nu\eth(x)(1-\mathfrak{m}(x))=0$$

Since $\mathfrak{m}(x)$ is a polynomial over $\Im$, then $\mathfrak{m}(x)=\mathfrak{m}_1(x)+\nu\mathfrak{m}_2(x)$, where $\mathfrak{m}_1(x),\mathfrak{m}_2(x)\in R_1[x]$. Thus

$$\begin{aligned}\mu\nu\eth(x)(1-\mathfrak{m}_1(x)-\nu\mathfrak{m}_2(x))&=0\\\Rightarrow\mu\nu\eth(x)(1-\mathfrak{m}_1(x))&=0\\\Rightarrow\mu\eth(x)(1-\mathfrak{m}_1(x))&=0\\\Rightarrow 1-\mathfrak{m}_1(x)&\in Ann(\mu\eth(x))\\\Rightarrow\mathfrak{m}_1(x)=1-\left(\mu+\frac{x^n-1}{\eth(x)}\right)&h(x),h(x)\in\Im_1[x]\end{aligned}$$

$$\begin{aligned}\eth(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x)&=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1)\Bigg(1\\&-\left(\mu+\frac{x^n-1}{\eth(x)}\right)h(x)+\nu\mathfrak{m}_2(x)\Bigg)\\&+(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))n(x)+(\nu\partial_2(x)\\&+\mu\nu\mathfrak{r}_3(x))l(x)+\mu\nu\partial_3(x)k(x)\end{aligned}\tag{3.4}$$

This is obtained by multiplying (3.4) by $\mu\eth(x)$.

$$\mu\nu x^j\gamma_1^*(x)\eth(x)=\mu\nu\gamma_1(x)\eth(x)+\mu\nu\eth^2(x)\mathfrak{m}_2(x)+\mu\nu\gamma_2(x)n(x)\eth(x)+\mu\nu\partial_2(x)l(x)\eth(x)$$

$$\Rightarrow \mu\nu\eth(x)\big(x^j\gamma_1^*(x)-\gamma_1(x)\big)=\mu\nu\eth(x)\big(\mathfrak{m}_2(x)\eth(x)+\gamma_2(x)n(x)+\partial_2(x)l(x)\big)$$

As $\partial_2(x)|\eth(x)$ and $\partial_2(x)|\frac{\eth(x)}{\partial_1(x)}\gamma_2(x) \Rightarrow \eth(x)=\partial_2(x).h_1(x), \frac{\eth(x)}{\partial_1(x)}\gamma_2(x)=\partial_2(x).h_2(x)$

$$\Rightarrow \mu\nu\eth(x)\Bigg(x^j\gamma_1^*(x)-\gamma_1(x)\Bigg)=\mu\nu\eth(x)\Bigg(\mathfrak{m}_2(x)\partial_2(x).h_1(x)+\frac{\partial_2(x)h_2(x)}{\frac{\eth(x)}{\partial_1(x)}}+\partial_2(x)l(x)\Bigg)$$

$$\Rightarrow \mu\nu\eth(x)\Bigg(x^j\gamma_1^*(x)-\gamma_1(x)\Bigg)=\mu\nu\eth(x)\partial_2(x)\Bigg(\mathfrak{m}_2(x).h_1(x)+\frac{h_2(x)}{\frac{\eth(x)}{\partial_1(x)}}+l(x)\Bigg)$$

$$\Rightarrow \partial_2(x)|\Bigg(x^j\gamma_1^*(x)-\gamma_1(x)\Bigg)$$

Again, we obtain the following result by multiplying equation (3.4) by ν:

$$\nu\eth(x)+\mu\nu x^i\rho_1^*(x)=\nu\eth(x)+\mu\nu\rho_1(x)+(\nu\eth(x)+\mu\nu\rho_1(x))(\Bigg(\mu+\frac{x^n-1}{\eth(x)}\Bigg)h(x))$$

$$\Rightarrow \mu\nu x^i\rho_1^*(x)-\mu\nu\rho_1(x)=\mu\nu\eth(x)+\mu\nu\rho_1(x)\Bigg(\frac{x^n-1}{\eth(x)}\Bigg)h(x)+\mu\nu\partial_1(x)n(x)$$

As $\partial_1(x)|\eth(x)$ and $\partial_1(x)|\rho_1(x)\frac{x^n-1}{\eth(x)}$

$$\Rightarrow \eth(x)=\partial_1(x).h_1(x),\ \rho_1(x)\frac{x^n-1}{\eth(x)}=\partial_1(x).h_2(x)$$

$$\Rightarrow \mu\nu\eth(x)\Bigg(x^i\rho_1^*(x)-\rho_1(x)\Bigg)=\mu\nu\eth(x)\Bigg(h(x)\partial_1(x).h_1(x)+\partial_1(x).h_2(x)h(x)+\mu\nu\partial_1(x)n(x)\Bigg)$$

$$\Rightarrow \mu\nu\eth(x)\Bigg(x^i\rho_1^*(x)-\rho_1(x)\Bigg)=\mu\nu\eth(x)\partial_1(x)\Bigg(h(x).h_1(x)+h_2(x)h(x)+n(x)\Bigg)$$

$$\Rightarrow \partial_1(x)|\Bigg(x^i\rho_1^*(x)-\rho_1(x)\Bigg).$$

From equation (3.4), we have

$$\begin{aligned}\eth(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x)=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1)\Bigg(1\\ -\Bigg(\mu+\frac{x^n-1}{\eth(x)}\Bigg)h(x)+\nu\mathfrak{m}_2(x)\Bigg)+(\mu\partial_1(x)\\ +\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))\Bigg(n_1(x)+\nu n_2(x)\Bigg)\\ +(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))\Bigg(l_1(x)+\nu l_2(x)\Bigg)\\ +\mu\nu\partial_3(x)\Bigg(k_1(x)+\nu k_2(x)\Bigg).\end{aligned}$$

As $n(x),l(x),k(x)$ belong to the polynomial ring $\Im$, we can express them as $n(x)=n_1(x)+\nu n_2(x), l(x)=l_1(x)+\nu l_2(x)$, and $k(x)=k_1(x)+\nu k_2(x)$, where $n_1(x),n_2(x),l_1(x),l_2(x),k_1(x),k_2(x)$

are polynomials in $\Im_1[x]$.

$$\eth(x) + \mu x^i \rho_1^*(x) + \nu x^j q_1^*(x) + \mu\nu x^k \mathfrak{r}_1^*(x) = (\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1)\Big(1 - \Big(\mu + \frac{x^n - 1}{\eth(x)}\Big)h(x) + \nu\mathfrak{m}_2(x)\Big) + (\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x))n_1(x) + \mu\nu n_2(x)\partial_1(x) + (\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x))l_1(x) + \mu\nu\partial_3(x)k_1(x)$$

$\Rightarrow \mu x^i\rho_1^*(x) - \mu\rho_1(x) + \nu x^j\gamma_1^*(x) - \nu\gamma_1(x) + \mu\nu x^k\mathfrak{r}_1^*(x) - \mathfrak{r}_1(x) = \Big(\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x)\Big)p(x) + \Big(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x)\Big)n_1(x) + \Big(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x)\Big)l_1(x) + \mu\nu\Big(\partial_3(x)k_1(x) + \partial_1(x)n_2(x)\Big)$, where $p(x) \in \Im[x]$ and $n_1(x), l_1(x), k_1(x) \in \Im_1[x]$.

Conversely, however, suppose both assumptions (a) and (b) are satisfied. It suffices to show that in order for $\mathfrak{C}$ to be considered as reversible, $\Big(\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x)\Big)^*$, $\Big(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x)\Big)^*$, $\Big(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x)\Big)^*$ and $\Big(\mu\nu\partial_3(x)\Big)^*$ are in $\mathfrak{C}$. Then

$$\begin{aligned}\Big(\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x)\Big)^* &= \eth^*(x) + \mu x^i\rho_1^*(x) + \nu x^j\gamma_1^*(x) + \mu\nu x^k\mathfrak{r}_1^*(x)\\ &= (\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x))\\ &\quad + \mu x^i\rho_1^*(x) - \mu\rho_1(x) + \nu x^j\gamma_1^*(x) - \nu\gamma_1(x)\\ &\quad + \mu\nu x^k\mathfrak{r}_1^*(x) - \mathfrak{r}_1(x)\\ &= (\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x))(1\\ &\quad + p(x)) + n_1(x)(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x))\\ &\quad + l_1(x)(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x)) + \mu\nu(\partial_3(x)k_1(x)\\ &\quad + \partial_1(x)n_2(x)) \in \mathfrak{C}.\end{aligned}$$

$$\begin{aligned}\Big(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x)\Big)^* &= \mu\partial_1^*(x) + \nu x^{i'}\gamma_2^*(x) + \mu\nu x^{j'}\mathfrak{r}_2^*(x)\\ &= (\mu\partial_1(x) + \nu x^{i'}\gamma_2^*(x) + \mu\nu x^{j'}\gamma_2^*(x)) \in \mathfrak{C},\end{aligned}$$

where $\partial_1(x)$ is self-reciprocal, and $i' = \deg\partial_1(x) - \deg\gamma_2(x), j' = \deg\partial_1(x) - \deg\mathfrak{r}_2(x)$.

$$\Big(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x)\Big)^* = \nu\partial_2^*(x) + \mu\nu x^{j''}\mathfrak{r}_3^*(x) = (\nu\partial_2(x) + \mu\nu x^{j''}\mathfrak{r}_2^*(x)) \in \mathfrak{C},$$

where $\partial_2(x)$ is self-reciprocal, and $j'' = \deg\partial_2(x) - \deg\mathfrak{r}_3(x)$.

$$\Big(\mu\nu\partial_3(x)\Big)^* = \mu\nu\partial_3^*(x) = \mu\nu\partial_3(x) \in \mathfrak{C},,$$

since $\partial_3(x)$ is self-reciprocal, it follows that $\mathfrak{C}$ is reversible. □

Theorem 3.3. *Consider a cyclic code $\mathfrak{C}$ over $\Im$ of odd length n, given by $\mathfrak{C} = \langle\eta(x)\rangle$, an ideal in $\Im_n$, where*

$$\eta(x) = \eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x) + \nu\partial_2(x) + \mu\nu\partial_3(x).$$

A code $\mathfrak{C}$ is reversible if the following condition is both necessary and sufficient:

(a) $\eth(x)$ *and* $\partial_2(x)$ *are self-reciprocal;*

(b) (i) $\partial_1(x)|(x^i\partial_1^*(x) - \partial_1(x))$ *or*

(ii) $\mu x^i\partial_1^*(x) - \mu\partial_1(x) + \mu\nu x^j \mathfrak{r}_1{}^*(x) - \mu\nu\mathfrak{r}_1(x) = (\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))k(x) + (\nu\partial_2(x) + \mu\nu\partial_3(x))n_1(x)$, *where* $i = \deg\eth(x) - \deg\partial_1(x), j = \deg\eth(x) - \deg\mathfrak{r}_1(x)$ *and* $k(x) \in R[x], n_1(x) \in R_1[x]$.

Proof. The code $\mathfrak{C} = \langle\eta(x)\rangle$ is assumed to be reversible over $\Im$. We acquire $Im(\mathfrak{C}) = \langle\eth(x) + \mu\partial_1(x)\rangle$ and $ker(\mathfrak{C}) = \nu\langle\partial_2(x)+\mu\partial_3(x)\rangle$ with $\partial_1(x)|\eth(x)|(x^n-1)$ and $\partial_3(x)|\partial_2(x)|(x^n-1)$. Let code $\mathfrak{C} \mod \langle\mu,\nu\rangle = \langle\eth(x)\rangle$, then $\eth(x)h(x) \in \mathfrak{C} \mod \langle\mu,\nu\rangle$, where $h(x) \in \Im_1[x]$ and $(\eth(x)+\mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))h(x) \in \mathfrak{C}$. Since $\mathfrak{C}$ is reversible, then $((\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))h(x))^* \in C \Rightarrow (\eth^*(x) + \mu x^i\partial_1^*(x) + \mu\nu x^j\mathfrak{r}_1^*(x))h^*(x) \in C \Rightarrow \eth^*(x)h^*(x) \Rightarrow (\eth(x)h(x))^* \in \mathfrak{C}$ which means that $\mathfrak{C} \mod \langle\mu,\nu\rangle = \langle\eth(x)\rangle$ is reversible. Therefore from Theorem 3.2, $\eth(x)$ is self-reciprocal. Let Tor($\mathfrak{C}$) $= \langle\partial_2(x) + \mu\partial_3(x)\rangle$ is a code over $\Im_1$. This implies that Tor($\mathfrak{C}$) mod $\mu = \langle\partial_2(x)\rangle$, then $\partial_2(x)h(x) \in$ Tor($\mathfrak{C}$) mod μ. Since Tor($\mathfrak{C}$) $= \langle\partial_2(x) + \mu\partial_3(x)\rangle$ this implies that $\nu\langle\partial_2(x) + \mu\partial_3(x)\rangle \in \mathfrak{C} \Rightarrow (\nu(\partial_2(x) + \mu\partial_3(x)))h(x) \in \mathfrak{C}$. We are given that $\mathfrak{C}$ is reversible it means $(\nu(\partial_2(x) + \mu\partial_3(x)))^*h^*(x) \in \mathfrak{C} \Rightarrow (\nu(\partial_2^*(x) + \mu x^{j'}\partial_3^*(x)))h^*(x) \in \mathfrak{C}$, where $j' = \deg\partial_2(x) - \deg\partial_3(x) \Rightarrow \partial_2^*(x)h^*(x) = (\partial_2(x)h(x))^* \in$ Tor($\mathfrak{C}$)mod μ. Thus, Tor($\mathfrak{C}$) mod $\mu = \langle\partial_2(x)\rangle$ is reversible. It follows directly from Theorem 3.2 that $\partial_2(x)$ is self-reciprocal. Moreover, the following result may be readily shown using a similar logic to that in Theorem 3.2 because $\eth(x)$ is similarly self-reciprocal. $\mu x^i\partial_1^*(x) - \mu\partial_1(x) + \mu\nu x^j r_1^*(x) - \mu\nu\mathfrak{r}_1(x) = (\eth(x)+\mu\partial_1(x)+\mu\nu\mathfrak{r}_1(x))k(x)+(\nu\partial_2(x)+\mu\nu\partial_3(x))n_1(x)$, where $i = \deg\eth(x) - \deg\partial_1(x), j = \deg\eth(x) - \deg\mathfrak{r}_1(x)$ and $k(x) \in R[x], n_1(x) \in R_1[x]$. where $\mathfrak{m}(x)$ and $n(x)$ are polynomials over $\Im$, and where $k(x) = \left(-\left(\mu + \frac{x^n-1}{\eth(x)}\right)h(x) + \nu\mathfrak{m}_2(x)\right) \in \Im[x]$, hence proved.

Conversely, if (a) and (b) are both met, then $(\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))^*$ and $(\nu\partial_2(x) + \mu\nu\partial_3(x))^*$ are both elements of $\mathfrak{C}$. It now possess

$$\begin{aligned}
(\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))^* &= \eth^*(x) + \mu x^i\partial_1^*(x) + \mu\nu x^j\mathfrak{r}_1^*(x) \\
&= \eth(x) + \mu x^i\partial_1^*(x) + \mu\nu x^j\mathfrak{r}_1^*(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x) \\
&\quad - \mu\partial_1(x) - \mu\nu\mathfrak{r}_1(x) \\
&= (\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x)) + \mu x^i\partial_1^*(x) - \mu\partial_1(x) \\
&\quad + \mu\nu x^j r_1^*(x) - \mu\nu\mathfrak{r}_1(x) \\
&= (\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x)) + (\eth(x) + \mu\partial_1(x) \\
&\quad + \mu\nu\mathfrak{r}_1(x))k(x) + n_1(x)(\nu\partial_2(x) + \mu\nu\partial_3(x)) \\
&= (\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))(1 + k(x)) + n_1(x)(\nu\partial_2(x) \\
&\quad + \mu\nu\partial_3(x)) \in \mathfrak{C}
\end{aligned}$$

Since $\partial_2(x)$ is reciprocal, therefore

$$\nu\partial_2(x) + \mu\nu\partial_3(x))^* = \nu\partial_2^*(x) + \mu\nu x^k\partial_3^*(x) = \nu\partial_2(x) + \mu\nu x^k\partial_3^*(x) \in \mathfrak{C},$$

where $k = \deg\partial_2(x) - \deg\partial_3(x)$. The reversibility of $\mathfrak{C}$ follows. □

4 Property of Reverse Complement

Within the current section, a detailed exploration of the reverse-complement constraint awaits, leveraging insights garnered from the antecedent section. To commence, we revisit fundamental attributes inherent in the elements of $\Im$.

Lemma 4.1. *For any* $\alpha, \beta, \zeta, \delta, \in \Im$*, then*

- $\alpha + \overline{\alpha} = \mu\nu;$
- $\overline{\mu\nu\alpha} = \mu\nu + \mu\nu\alpha;$

- $\overline{\alpha+\beta} = \overline{\alpha} + \overline{\beta} + \mu\nu$;
- $\overline{\alpha+\mu\beta+\mu\nu\delta} = \overline{\alpha} + \mu\beta + \mu\nu\delta$
- $\overline{\nu\alpha+\mu\nu\beta} = \overline{\nu\alpha} + \mu\nu\beta$
- $\overline{\mu\alpha+\nu\beta+\mu\nu\zeta} = \overline{\mu\alpha} + \nu\beta + \mu\nu\zeta$
- $\overline{\alpha+\mu\beta+\nu\zeta+\mu\nu\delta} = \overline{\alpha} + \mu\beta + \nu\zeta + \mu\nu\delta$
- $\overline{\alpha} + \mu\nu = \alpha$.

Theorem 4.2. *Let $\mathfrak{C} = \langle \kappa(x) \rangle$ be a cyclic code over $\Im$. Therefore, $\mathfrak{C}$ possesses the reverse-complement property if and only if*

(a) $\eth(x)$, $\partial_1(x)$, $\partial_2(x)$, $\partial_3(x)$ *are self-reciprocal and* $\mu\nu((1-x^n)/(1-x)) \in \mathfrak{C}$,

(b) (i) $\partial_2(x)|(x^j\gamma_1^*(x) - \gamma_1(x))$ *and* $\partial_1(x)|(x^i\rho_1^*(x) - \rho_1(x))$ *or*

(ii) $\mu x^i\rho_1^*(x) - \mu\rho_1(x) + \nu x^j\gamma_1^*(x) - \nu\gamma_1(x) + \mu\nu x^k\mathfrak{r}_1^*(x) - \mathfrak{r}_1(x) = \Big(\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x)\Big)p(x) + \Big(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x)\Big)n_1(x) + \Big(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x)\Big)l_1(x) + \mu\nu\Big(\partial_3(x)k_1(x) + \partial_1(x)n_2(x)\Big)$, *where* $p(x) \in \Im[x]$ *and* $n_1(x), n_2(x), l_1(x), k_1(x) \in \Im_1[x]$.

Proof. Assuming that the zero codeword must be in $\mathfrak{C}$, its WCC should also be in $\mathfrak{C}$. Nevertheless, Lemma 4.1 gives us

$$\overline{(0,0,\ldots,0)} = (\mu\nu, \mu\nu, \ldots, \mu\nu) = (\mu\nu)\frac{1-x^n}{1-x} \in \mathfrak{C}.$$

Now, let $\eth(x) = \eth_0(x) + \eth_1(x) + \cdots + \eth_{s-1}x^{s-1} + \eth_s x^s$, $\rho_1(x) = a_0 + a_1x + \cdots + a_{r-1}x^{r-1} + a_r x^r$, $\gamma_1(x) = h_0 + h_1x + \cdots + h_{t-1}x^{t-1} + h_t x^t$. $\mathfrak{r}_1(x) = z_0 + z_1x + \cdots + z_{l-1}x^{l-1} + z_l x^l$. Consequently, we can express

$$\begin{aligned}
&\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x)\\
&= \eth_0 + \eth_1 x + \cdots + \eth_{s-1}x^{s-1} + \eth_s x^s + \mu(a_0 + a_1x + \cdots + a_{r-1}x^{r-1} + a_r x^r)\\
&+ \nu(h_0 + h_1x + \cdots + h_{t-1}x^{t-1} + h_t x^t) + \mu\nu(z_0 + z_1x + \cdots + z_{l-1}x^{l-1} + z_l x^l)\\
&= (\eth_0 + \mu a_0 + \nu h_0 + \mu\nu z_0) + (\eth_1 + \mu a_1 + \nu h_1 + \mu\nu z_1)x + \cdots + (\eth_l + \mu a_l\\
&+ \nu h_l + \mu\nu z_l)x^l + (\eth_{l+1} + \mu a_{l+1} + \nu h_{l+1})x^{l+1} + \cdots + (\eth_{t-1} + \mu a_{t-1}\\
&+ \nu h_{t-1})x^{t-1} + (\eth_t + \mu a_t + \nu h_t)x^t + (\eth_{t+1} + \mu a_{t+1})x^{t+1} + \cdots + (\eth_r + \mu a_r)x^r\\
&+ \eth_{r+1}x^{r+1} + \cdots + \eth_s x^s.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))^{rc}\\
&=\mu\nu+\mu\nu x+\cdots+\mu\nu x^{n-s-2}+\overline{\eth_s}x^{n-s-1}+\overline{\eth_{s-1}}x^{n-s}+\cdots+(\overline{\eth_r+\mu a_r})x^{n-r-1}\\
&\ +(\overline{\eth_{r-1}+\mu a_{r-1}})x^{n-r}+\cdots+(\overline{\eth_{t+1}+\mu a_{t+1}})x^{n-t-2}+(\overline{\eth_t+\mu a_t+\nu h_t})x^{n-t-1}\\
&\ +(\overline{\eth_{t-1}+\mu a_{t-1}+\nu h_{t-1}})x^{n-t}+\cdots+(\overline{\eth_{l+1}+\mu a_{l+1}+\nu h_{l+1}})x^{n-l-2}\\
&\ +(\overline{\eth_l+\mu a_l+\nu h_l+\mu\nu z_l})x^{n-l-1}+(\overline{\eth_{l-1}+\mu a_{l-1}+\nu h_{l-1}+\mu\nu z_{l-1}})x^{n-l}\\
&\quad+\cdots+(\overline{\eth_1+\mu a_1+\nu h_1+\mu\nu z_1})x^{n-2}+(\overline{\eth_0+\mu a_0+\nu h_0+\mu\nu z_0})x^{n-1}\\
&=\mu\nu(1+x+\ldots x^{n-s-2})+\overline{\eth_s}x^{n-s-1}+\overline{\eth_{s-1}}x^{n-s}+\cdots+\overline{\eth_r(x)}x^{n-r-1}\\
&\ +\mu a_r x^{n-r-1}+\overline{\eth_{r-1}(x)}x^{n-r}+\mu a_{r-1}x^{n-r}+\cdots+\overline{\eth_{t+1}}x^{n-t-2}+\mu a_{t+1}x^{n-t-2}\\
&\ +\overline{\eth_t}x^{n-t-1}+\mu a_t x^{n-t-1}+\nu h_t x^{n-t-1}+\overline{\eth_{t-1}}x^{n-t}+\mu a_{t-1}x^{n-t}+\nu h_{t-1}x^{n-t}+\ldots\\
&\ +\overline{\eth_{l+1}}x^{n-l-2}+\mu a_{l+1}x^{n-l-2}+\nu h_{l+1}x^{n-l-2}+\overline{\eth_l}x^{n-l-1}+\mu a_l x^{n-l-1}\\
&\ +\nu h_l x^{n-l-1}\mu\nu z_l x^{n-l-1}+\overline{\eth_{l+1}}x^{n-l}+\mu a_{l+1}x^{n-l}+\nu h_{l+1}x^{n-l}\mu\nu z_{l+1}x^{n-l}\\
&\ +\overline{\eth_1}x^{n-2}+\mu a_1 x^{n-2}+\nu h_1 x^{n-2}+\mu\nu z_1 x^{n-2}+\overline{\eth_0}x^{n-1}+\mu a_0 x^{n-1}+\nu h_0 x^{n-1}\\
&\ +\mu\nu z_0 x^{n-1}
\end{aligned}
$$

Being a cyclic code $\mathfrak{C}$, then the following property holds:

$$(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\in\mathfrak{C} \tag{4.1}$$

After that, we obtain

$$
\begin{aligned}
&(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\\
&=\mu\nu(1+x+\dots x^{n-s-2})+\overline{\eth_s}x^{n-s-1}+\overline{\eth_{s-1}}x^{n-s}+\dots+\overline{\eth_r(x)}x^{n-r-1}+\mu a_r x^{n-r-1}\\
&\quad+\overline{\eth_{r-1}(x)}x^{n-r}+\mu a_{r-1}x^{n-r}+\dots+\overline{\eth_{t+1}}x^{n-t-2}+\mu a_{t+1}x^{n-t-2}+\overline{\eth_t}x^{n-t-1}\\
&\quad+\mu a_t x^{n-t-1}+\nu h_t x^{n-t-1}+\overline{\eth_{t-1}}x^{n-t}+\mu a_{t-1}x^{n-t}+\nu h_{t-1}x^{n-t}+\dots+\overline{\eth_{l+1}}x^{n-l-2}\\
&\quad+\mu a_{l+1}x^{n-l-2}+\nu h_{l+1}x^{n-l-2}+\overline{\eth_l}x^{n-l-1}+\mu a_l x^{n-l-1}+\nu h_l x^{n-l-1}+\mu\nu z_l x^{n-l-1}\\
&\quad+\overline{\eth_{l+1}}x^{n-l}+\mu a_{l+1}x^{n-l}+\nu h_{l+1}x^{n-l}+\mu\nu z_{l+1}x^{n-l}+\overline{\eth_1}x^{n-2}+\mu a_1 x^{n-2}+\nu h_1 x^{n-2}\\
&\quad+\mu\nu z_1 x^{n-2}+\overline{\eth_0}x^{n-1}+\mu a_0 x^{n-1}+\nu h_0 x^{n-1}+\mu\nu z_0 x^{n-1}+\mu\nu(1+x+\dots+x^{n-1})\\
&=(\overline{\eth_s}+\mu\nu)x^{n-s-1}+(\overline{\eth_{s-1}}+\mu\nu)x^{n-s}+\dots+(\overline{\eth_r}+\mu\nu)x^{n-r-1}+(\overline{\eth_{r-1}}+\mu\nu)x^{n-r}\\
&\quad+\dots+(\overline{\eth_{t+1}}+\mu\nu)x^{n-t-2}+(\overline{\eth_t}+\mu\nu)x^{n-t-1}+(\overline{\eth_{t-1}}+\mu\nu)x^{n-t}+\dots+(\overline{\eth_{l+1}}\\
&\quad+\mu\nu)x^{n-l-2}+(\overline{\eth_l}+\mu\nu)x^{n-l-1}+(\overline{\eth_{l-1}}+\mu\nu)x^{n-l}+\dots+(\overline{\eth_1}+\mu\nu)x^{n-2}+(\overline{\eth_0}\\
&\quad+\mu\nu)x^{n-1}+\mu a_r x^{n-r-1}+\mu a_{r-1}x^{n-r}+\dots+\mu a_{t+1}x^{n-t-2}+\mu a_t x^{n-t-1}+\mu a_{t-1}x^{n-t}\\
&\quad+\dots+\mu a_{l+1}x^{n-l-2}+\mu a_t x^{n-l-1}+\mu a_{l-1}x^{n-l}+\dots+\mu a_1 x^{n-2}+\mu a_0 x^{n-1}+\nu h_t x^{n-t-1}\\
&\quad+\nu h_{t-1}x^{n-t}+\dots+\nu h_{l+1}x^{n-l-2}+\nu h_l x^{n-l-1}+\nu h_{l+1}x^{n-l}+\dots+\nu h_1 x^{n-2}+\nu h_0 x^{n-1}\\
&\quad+\mu\nu z_l x^{n-l-1}+\mu\nu z_{l+1}x^{n-l}+\dots+\mu\nu z_1 x^{n-2}+\mu\nu z_0 x^{n-1}\\
&=\eth_s x^{n-s-1}+\eth_{s-1}x^{n-s}+\dots+\eth_r x^{n-r-1}+\eth_{r-1}x^{n-r}+\dots+\eth_{t+1}x^{n-t-2}+\eth_t x^{n-t-1}\\
&\quad+\eth_{t-1}x^{n-t}+\dots+\eth_{l+1}x^{n-l-2}+\eth_l x^{n-l-1}+\eth_{l-1}x^{n-l}+\dots+\eth_1 x^{n-2}+\eth_0 x^{n-1}\\
&\quad+\mu a_r x^{n-r-1}+\mu a_{r-1}x^{n-r}+\dots+\mu a_{t+1}x^{n-t-2}+\mu a_t x^{n-t-1}+\mu a_{t-1}x^{n-t}+\dots\\
&\quad+\mu a_{l+1}x^{n-l-2}+\mu a_t x^{n-l-1}+\mu a_{l-1}x^{n-l}+\dots+\mu a_1 x^{n-2}+\mu a_0 x^{n-1}\\
&\quad+\nu h_t x^{n-t-1}+\nu h_{t-1}x^{n-t}+\dots+\nu h_{l+1}x^{n-l-2}+\nu h_l x^{n-l-1}+\nu h_{l+1}x^{n-l}+\dots\\
&\quad+\nu h_1 x^{n-2}+\nu h_0 x^{n-1}+\mu\nu z_l x^{n-l-1}+\mu\nu z_{l+1}x^{n-l}+\dots+\mu\nu z_1 x^{n-2}+\mu\nu z_0 x^{n-1}\\
&=x^{n-s-1}\eth^*(x)+\mu x^{n-r-1}\rho_1^*(x)+\nu x^{n-t-1}\gamma_1^*(x)+\mu\nu x^{n-l-1}\mathfrak{r}_1^*(x)\\
&=x^{n-s-1}(\eth^*(x)+\mu\rho_1^*(x)x^{s-r}+\nu\gamma_1^*(x)x^{s-t}+\mu\nu\mathfrak{r}_1^*(x)x^{s-l})
\end{aligned}
$$

Hence, $\eth^*(x)+\mu\rho_1^*(x)x^{s-r}+\nu\gamma_1^*(x)x^{s-t}+\mu\nu\mathfrak{r}_1^*(x)x^{s-l}\in\mathfrak{C}$. Consequently, we can express $\eth^*(x)+\mu\rho_1^*(x)x^{s-r}+\nu\gamma_1^*(x)x^{s-t}+\mu\nu\mathfrak{r}_1^*(x)x^{s-l}=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu r_1)\mathfrak{m}(x)+(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))n(x)+(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))l(x)+\mu\nu\partial_3(x)k(x)$
It is easily verify $\mathfrak{m}(x)=\left(1-\left(\mu+\frac{x^n-1}{\eth(x)}\right)h(x)+\nu\mathfrak{m}_2(x)\right)$ and $n(x)=n_1(x)+\nu n_2(x), l(x)=l_1(x)+\nu l_2(x), k(x)=k_1(x)+\nu k_2(x)$.

According to Theorem 3.2, it follows that if $\eth(x)=\eth^*(x)$, then $\eth(x)$ is self-reciprocal. Moreover, we obtain that $\partial_2(x)|(x^j\gamma_1^*(x)-\gamma_1(x))$ and $\partial_1(x)|(x^i\rho_1^*(x)-\rho_1(x))$, where $j=s-t$ and $i=s-r$. Now, let $\partial_1(x)=\eth_0(x)+\eth_1(x)+\dots+\eth_{l-1}x^{l-1}+\eth_l x^l$, $\gamma_2(x)=p_0+p_1x+\dots+p_{r-1}x^{r-1}+p_r x^r$, $\mathfrak{r}_2(x)=q_0+q_1x+\dots+q_{t-1}x^{t-1}+q_t x^t$. Consequently, we can express

$$\begin{aligned}\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x) &= \mu(\eth_0+\eth_1x+\dots+\eth_{l-1}x^{l-1}+\eth_lx^l)+\nu(p_0+p_1x+\cdots\\ &\quad +p_{r-1}x^{r-1}+p_rx^r)+\mu\nu(q_0+q_1x+\dots+q_tx^t+q_tx^t)\\ &=(\mu\eth_0+\nu p_0+\mu\nu q_0)+(\mu\eth_1+\nu p_1+\mu\nu q_1)x+\dots\\ &\quad +(\mu\eth_{t-1}+\nu p_{t-1}+\mu\nu q_{t-1})x^{t-1}+(\mu\eth_t+\nu p_t+\mu\nu)x^t\\ &\quad +(\mu\eth_{t+1}+\nu p_{t+1})x^{t+1}+\dots+(\mu\eth_{t-1}+\nu p_{t-1})x^{t-1}\\ &\quad +(\mu\eth_t+\nu p_t)x^t+\mu\eth_{t+1}x^{t+1}+\dots+\mu\eth_{l-1}x^{l-1}+\mu\eth_lx^l.\end{aligned}$$

Consequently,

$$\begin{aligned}&(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))^{rc}\\ &=\mu\nu+\mu\nu x+\dots+\mu\nu x^{n-s-2}+\overline{\mu\eth_s}x^{n-l-1}+\overline{\mu\eth_{l-1}}x^{n-l}+\dots+(\overline{\mu\eth_r+\nu p_r})x^{n-r-1}\\ &\quad+(\overline{\mu\eth_{r-1}+\nu p_{r-1}})x^{n-r}+\dots+(\overline{\mu\eth_{t+1}+\nu p_{t+1}})x^{n-t-2}+(\overline{\mu\eth_t+\nu p_t+\mu\nu q_t})x^{n-t-1}\\ &\quad+(\overline{\mu\eth_{t-1}+\nu p_{t-2}+\mu\nu q_{t-1}})x^{n-t}+\dots+(\overline{\mu\eth_1+\nu p_1+\mu\nu q_1})x^{n-2}\\ &\quad+(\overline{\mu\eth_0+\nu p_0+\mu\nu q_0})x^{n-1}\\ &=\mu\nu(1+x+\dots x^{n-l-2})+\overline{\mu\eth_l}x^{n-l-1}+\overline{\mu\eth_{l-1}}x^{n-l}+\dots+\overline{\mu\eth_r(x)}x^{n-r-1}\\ &\quad+\nu p_r(x)x^{n-r-1}+\overline{\mu\eth_{r-1}(x)}x^{n-r}+\nu p_{r-1}x^{n-r}+\dots+\overline{\mu\eth_{t+1}}x^{n-t-2}\\ &\quad+\nu p_{t+1}x^{n-t-2}+\overline{\mu\eth_t}x^{n-t-1}+\nu p_tx^{n-t-1}+\mu\nu q_tx^{n-t-1}+\overline{\mu\eth_{t-1}}x^{n-t}+\nu p_{t-1}x^{n-t}\\ &\quad+\mu\nu q_{t-1}x^{n-t}+\dots+\overline{\mu\eth_1}x^{n-2}+\nu p_1x^{n-2}+\mu\nu q_1x^{n-2}+\overline{\mu\eth_0}x^{n-1}+\nu p_0x^{n-1}\\ &\quad+\mu\nu q_0x^{n-1}\end{aligned}$$

Leveraging the fundamental characteristics of $\mathfrak{C}$, then

$$(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\in\mathfrak{C} \tag{4.2}$$

This leads to the conclusion that

$$
\begin{aligned}
&(\mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x))^{rc} + (\mu\nu)(1-x^n)/(1-x) \\
&= \mu\nu(1 + x + \dots x^{n-l-2}) + \overline{\mu\eth_l}x^{n-l-1} + \overline{\mu\eth_{l-1}}x^{n-s} + \dots + \overline{\mu\eth_r}x^{n-r-1} \\
&\quad + \nu p_r x^{n-r-1} + \overline{\mu\eth_{r-1}}x^{n-r} + \nu p_{r-1}x^{n-r} + \dots + \overline{\mu\eth_{t+1}}x^{n-t-2} + \nu p_{t+1}x^{n-t-2} \\
&\quad + \overline{\mu\eth_t}x^{n-t-1} + \nu p_t x^{n-t-1} + \mu\nu q_t x^{n-t-1} + \overline{\mu\eth_{t-1}}x^{n-t} + \nu p_{t-1}x^{n-t} + \mu\nu q_{t-1}x^{n-t} + \dots \\
&\quad + \overline{\mu\eth_1}x^{n-2} + \nu p_1 x^{n-2} + \mu\nu q_1 x^{n-2} + \overline{\mu\eth_0}x^{n-1} + \nu p_0 x^{n-1} + \mu\nu q_0 x^{n-1} \\
&\quad + \mu\nu(1 + x + \dots + x^{n-1}) \\
&= (\overline{\mu\eth_s} + \mu\nu)x^{n-l-1} + (\overline{\mu\eth_{l-1}} + \mu\nu)x^{n-l} + \dots + (\overline{\mu\eth_r} + \mu\nu)x^{n-r-1} + (\overline{\mu\eth_{r-1}} + \mu\nu)x^{n-r} \\
&\quad + \dots + (\overline{\mu\eth_{t+1}} + \mu\nu)x^{n-t-2} + (\overline{\mu\eth_t} + \mu\nu)x^{n-t-1} + (\overline{\mu\eth_{t-1}} + \mu\nu)x^{n-t} + \dots + (\overline{\mu\eth_1} \\
&\quad + \mu\nu)x^{n-2} + (\overline{\mu\eth_0} + \mu\nu)x^{n-1} + \nu p_r x^{n-r-1} + \nu p_{r-1}x^{n-r} + \dots + \nu p_{t+1}x^{n-t-2} \\
&\quad + \nu p_t x^{n-t-1} + \nu p_{t-1}x^{n-t} + \dots + \nu p_1 x^{n-2} + \nu p_0 x^{n-1} + \mu\nu q_t x^{n-t-1} + \mu\nu q_{t-1}x^{n-t} \\
&\quad + \dots + \mu\nu q_1 x^{n-2} + +\mu\nu q_0 x^{n-1} \\
&= \mu\eth_l x^{n-l-1} + \mu\eth_{l-1}x^{n-l} + \dots + \mu\eth_r x^{n-r-1} + \mu\eth_{r-1}x^{n-r} + \dots + \mu\eth_{t+1}x^{n-t-2} \\
&\quad + \mu\eth_t x^{n-t-1} + \mu\eth_{t-1}x^{n-t} + \dots + \mu\eth_1 x^{n-2} + \mu\eth_0 x^{n-1} + \nu p_r x^{n-r-1} + \nu p_{r-1}x^{n-r} \\
&\quad + \dots + \nu p_{t+1}x^{n-t-2} + \nu p_t x^{n-t-1} + \nu p_{t-1}x^{n-t} + \dots + \nu p_1 x^{n-2} \\
&\quad + \nu p_0 x^{n-1} + \mu\nu q_t x^{n-t-1} + \mu\nu q_{t-1}x^{n-t} + \dots + \mu\nu q_1 x^{n-2} + \mu\nu q_0 x^{n-1} \\
&= x^{n-l-1}(\eth_l(x) + \eth_{l-1}(x)x + \dots + \eth_1(x)x^{l-1} + \eth_0(x)x^l) + x^{n-r-1}(\nu p_r(x) + \nu p_{r-1}x \\
&\quad + \dots + \nu p_1 x^{r-1} + \nu p_0 x^r) + x^{n-t-1}(\mu\nu q_t + \mu\nu q_{t-1}x + \dots + \mu\nu q_1 x^{t-1} + \mu\nu q_0 x^t). \\
&= \mu x^{n-l-1}\partial_1^*(x) + \nu x^{n-r-1}\gamma_2^*(x) + \mu\nu x^{n-t-1}\mathfrak{r}_2^*(x) \\
&= x^{n-l-1}(\mu\partial_1^*(x) + \nu\gamma_2^*(x)x^{l-r} + \mu\nu\mathfrak{r}_2^*(x)x^{l-t})
\end{aligned}
$$

Hence, $\mu\partial_1^*(x) + \nu\gamma_2^*(x)x^{l-r} + \mu\nu\mathfrak{r}_2^*(x)x^{l-t} \in \mathfrak{C}$. As a result, we derive $\mu\partial_1^*(x) + \nu\gamma_2^*(x)x^{l-r} + \mu\nu\mathfrak{r}_2^*(x)x^{l-t} = (\mu\partial_1(x) + \nu\gamma_2^*(x)x^{l-r} + \mu\nu\mathfrak{r}_2^*(x)x^{l-t})n(x)$. It is obvious that $n(x) = n_1(x) + \nu n_2(x)$. Then, by Theorem 3.2, $\partial_1(x) = \partial_1^*(x)$ i.e. $\partial_1(x)$ is self-reciprocal. Now suppose that $\nu\partial_2(x) = \nu h_0 + \nu h_1 x + \dots + \nu h_{k-1}x^{k-1} + \nu h_k x^k$ and $\mu\nu\mathfrak{r}_3(x) = \mu\nu h'_0 + \mu\nu h'_1 x + \dots + \mu\nu h'_{l-1}x^{l-1} + \mu\nu h'_l x^l$, where $k > l$. Then

$$
\begin{aligned}
\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x) &= \nu h_0 + \nu h_1 x + \dots + \nu h_{k-1}x^{k-1} + \nu h_k x^k + \mu\nu h'_0 + \mu\nu h'_1 x + \dots \\
&\quad + \mu\nu h'_{l-1}x^{l-1} + \mu\nu h'_l x^l \\
&= (\nu h_0 + \mu\nu h'_0) + (\nu h_1 + \mu\nu h'_1)x + \dots + (\nu h_{l-1} + \mu\nu h'_{l-1})x^{l-1} \\
&\quad + (\nu h_l + \mu\nu h'_l)x^l + \nu h_{l+1}x^{l+1} + \dots + \nu h_{k-1}x^{k-1} + \nu h_k x^k
\end{aligned}
$$

$$
\begin{aligned}
(\nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x))^{rc} &= \mu\nu(1 + x + \dots + x^{n-k-2}) + \overline{\nu h_k}x^{n-k-1} + \overline{\nu h_{k-1}}x^{n-k} \\
&\quad + \dots + \overline{\nu h_{l+1}}x^{n-l-2} + \nu\overline{(\nu h_l + \mu\nu h'_l)}x^{n-l-1} + \overline{(\nu h_{l-1} + \mu\nu h'_{l-1})}x^{n-l} \\
&\quad + \dots + \overline{(\nu h_1 + \mu\nu h'_1)}x^{n-2} + \overline{(\nu h_0 + \mu\nu h'_0)}x^{n-1} \\
&= \mu\nu(1 + x + \dots + x^{n-k-2}) + \overline{\nu h_k}x^{n-k-1} + \overline{\nu h_{k-1}}x^{n-k} + \dots \\
&\quad + \overline{\nu h_{l+1}}x^{n-l-2} + \nu\overline{\nu h_l}x^{n-l-1} + \mu\nu h'_l x^{n-l-1} + \overline{\nu h_{l-1}}x^{n-l} \\
&\quad + \mu\nu h'_{l-1}x^{n-l} + \dots + \overline{\nu h_1}x^{n-2} + \mu\nu h'_1 x^{n-2} + \overline{\nu h_0}x^{n-1} \\
&\quad + \mu\nu h'_0 x^{n-1}
\end{aligned}
$$

Leveraging the fundamental characteristics of $\mathfrak{C}$, then

$$(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\in\mathfrak{C}. \tag{4.3}$$

This leads to the conclusion that

$$\begin{aligned}
&(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\\
&=\mu\nu(1+x+\cdots+x^{n-k-2})+\overline{\nu h_k}x^{n-k-1}+\overline{\nu h_{k-1}}x^{n-k}+\cdots+\overline{\nu h_{l+1}}x^{n-l-2}\\
&\quad+\nu\overline{\nu h_l}x^{n-l-1}+\mu\nu h'_l x^{n-l-1}+\overline{\nu h_{l-1}}x^{n-l}+\mu\nu h'_{l-1}x^{n-l}+\cdots+\overline{\nu h_1}x^{n-2}\\
&\quad+\mu\nu h'_1x^{n-2}+\overline{\nu h_0}x^{n-1}+\mu\nu h'_0x^{n-1}+\mu\nu(1+x+x^2+\cdots+x^{n-1})\\
&=(\overline{\nu h_k}+\mu\nu)x^{n-k-1}+(\overline{\nu h_{k-1}}+\mu\nu)x^{n-k}+\cdots+(\overline{\nu h_{l+1}}\\
&\quad+\mu\nu)x^{n-l-2}+(\overline{\nu h_l}+\mu\nu)x^{n-l-1}+\mu\nu h'_lx^{n-l-1}+(\overline{\nu h_{l-1}}+\mu\nu)x^{n-l}+\mu\nu h'_{l-1}x^{n-l}\\
&\quad+\cdots+(\overline{\nu h_1}+\mu\nu)x^{n-2}+\mu\nu h'_1x^{n-2}+(\overline{\nu h_0}+\mu\nu)x^{n-1}+\mu\nu h'_0x^{n-1}\\
&=x^{n-k-1}(\nu h_k+\nu h_{k-1}x+\cdots+\nu h_1x^{k-1}+\nu h_0x^k)+x^{n-l-1}(\mu\nu h'_l+\mu\nu h'_{l-1}x\\
&\quad+\cdots+\mu\nu h'_1x^{l-1}+\mu\nu h'_0x^l)\\
&=x^{n-k-1}\nu\partial_2^*(x)+\mu\nu x^{n-l-1}\mathfrak{r}_3^*(x)\\
&=x^{n-k-1}(\nu\partial_2^*(x)+\mu\nu x^{k-l}\mathfrak{r}_3^*(x))
\end{aligned}$$

Hence $(\nu\partial_2^*(x)+\mu\nu x^{k-l}\mathfrak{r}_3^*(x))\in\mathfrak{C}$. As a result, we derive $\nu\partial_2^*(x)+\mu\nu x^{k-l}\mathfrak{r}_3^*(x)=(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))n_1(x)$, where $n_1(x)\in\Im_1[x]$. Then by Theorem 3.2, $\partial_2(x)=\partial_2^*(x)$ i.e. $\partial_2(x)$ is self-reciprocal and $\partial_3(x)=e_0+e_1x+\cdots+e_{d-1}x^{d-1}+e_dx^d$, $\mu\nu\partial_3(x)=\mu\nu e_0+\mu\nu e_1x+\cdots+\mu\nu e_{d-1}x^{d-1}+\mu\nu e_dx^d$. Then $(\mu\nu\partial_3(x))^{rc}=\mu\nu(1+x+\cdots+x^{n-d-2})+\overline{\mu\nu e_d}x^{n-d-1}+\overline{\mu\nu e_{d-1}}x^{n-d}+\cdots+\overline{\mu\nu e_1}x^{n-2}+\overline{\mu\nu e_0}x^{n-1}\in\mathfrak{C}$. Since C is a linear code and $(\mu\nu)(1-x^n)/(1-x)\in\mathfrak{C}$, we must have $(\mu\nu\partial_3(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\in\mathfrak{C}$. Hence

$$\begin{aligned}
&(\mu\nu\partial_3(x))^{rc}+(\mu\nu)(1-x^n)/(1-x)\\
&=\mu\nu(1+x+\cdots+x^{n-d-2})+\overline{\mu\nu e_d}x^{n-d-1}+\overline{\mu\nu e_{d-1}}x^{n-d}+\dots\\
&\quad+\overline{\mu\nu e_1}x^{n-2}+\overline{\mu\nu e_0}x^{n-1}+\mu\nu(1+x+x^2+\cdots+x^{n-1}\\
&=(\overline{\mu\nu e_d}+\mu\nu)x^{n-d-1}+(\overline{\mu\nu e_{d-1}}+\mu\nu)x^{n-d}+\cdots+(\overline{\mu\nu e_1}+\mu\nu)x^{n-2}\\
&\quad+(\overline{\mu\nu e_0}+\mu\nu)x^{n-1}\\
&=\mu\nu e_dx^{n-d-1}+\mu\nu e_{d-1}x^{n-d}+\cdots+\mu\nu e_1x^{n-2}+\mu\nu e_0x^{n-1}\\
&=x^{n-d-1}(\mu\nu e_d+\mu\nu e_{d-1}x+\cdots+\mu\nu e_1x^{d-1}+\mu\nu e_0x^d)\\
&=x^{n-d-1}(\mu\nu\partial_3^*(x)).
\end{aligned}$$

Therefore $(\mu\nu\partial_3^*(x))\in\mathfrak{C}$. As a result, we derive $\mu\nu\partial_3^*(x)=\mu\nu\partial_3(x)k_1(x)$, where $k_1(x)\in\Im_1[x]$. Then by Theorem 3.2, $\partial_3(x)=\partial_3^*(x)$ i.e. $\partial_3(x)$ is self-reciprocal.

Conversely, let $c(x)\in\mathfrak{C}$, then $c(x)=(\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))\mathfrak{m}(x)+(\mu\partial_1(x)+\nu\gamma_2(x)+\mu\nu\mathfrak{r}_2(x))n(x)+(\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))l(x)+\mu\nu\partial_3(x)k(x)$. Since $\eth(x)$, $\partial_1(x)$ $\partial_2(x)$ and $\partial_3(x)$ are self-reciprocal, $\partial_2(x)|(x^j\gamma_1^*(x)-\gamma_1(x))$ and $\partial_1(x)|(x^i\rho_1^*(x)-\rho_1(x))$, as a result, we obtain,

$$\begin{aligned}
c^*(x)&=((\eth(x)+\mu\rho_1(x)+\nu\gamma_1(x)+\mu\nu\mathfrak{r}_1(x))\mathfrak{m}(x))^*+((\mu\partial_1(x)+\nu\gamma_2(x)\\
&\quad+\mu\nu\mathfrak{r}_2(x))n(x))^*+((\nu\partial_2(x)+\mu\nu\mathfrak{r}_3(x))l(x))^*+(\mu\nu\partial_3(x)k(x))^*\\
&=(\eth^*(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x))\mathfrak{m}^*(x)+(\mu x^{i'}\partial_1^*(x)+\nu x^{j'}\gamma_2^*(x)\\
&\quad+\mu\nu x^{k'}\mathfrak{r}_2^*(x))n^*(x)+(\nu x^l\partial_2^*(x)+\mu\nu x^{k'}\mathfrak{r}_3^*(x))n^*(x)+\mu\nu x^k\partial_3^*(x))k^*(x)\\
&=(\eth(x)+\mu x^i\rho_1^*(x)+\nu x^j\gamma_1^*(x)+\mu\nu x^k\mathfrak{r}_1^*(x))\mathfrak{m}^*(x)+(\mu x^{i'}\partial_1(x)+\nu x^{j'}\gamma_2^*(x)\\
&\quad+\mu\nu x^{k'}\mathfrak{r}_2^*(x))n^*(x)+(\nu x^l\partial_2(x)+\mu\nu x^{k''}\mathfrak{r}_3^*(x))l^*(x)+\mu\nu x^k\partial_3(x))k^*(x),
\end{aligned}$$

where $i' = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\partial_1(x)n(x)), j' = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\gamma_2(x)n(x))$, $k' = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\mathfrak{r}_2(x)n(x)), l = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\partial_2(x)l(x))$, $k'' = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\mathfrak{r}_3(x)l(x)), k = \deg(\eth(x)\mathfrak{m}(x)) - \deg(\partial_3(x)k(x))$.
So $c^*(x) \in \mathfrak{C}$ and $\mu\nu((1-x^n)/(1-x)) \in \mathfrak{C}$, then

$$\mu\nu\frac{1-x^n}{1-x} = \mu\nu + \mu\nu x + \cdots + \mu\nu x^{n-1} \in \mathfrak{C}.$$

As we know that $\mathfrak{c}(x) = c_0 + c_1 x + \cdots + c_s x^s \in \mathfrak{C}$. Leveraging the fundamental characteristics of $\mathfrak{C}$, then

$$x^{n-s-1}c(x) = c_0 x^{n-s-1} + c_1 x^{n-s} + \cdots + c_s x^{n-1} \in \mathfrak{C}.$$

Hence, $\mu\nu + \mu\nu x + \cdots + \mu\nu x^{n-s-2} + (c_0 + \mu\nu)x^{n-s-1} + (c_1 + \mu\nu)x^{n-s} + \cdots + (c_s + \mu\nu)x^{n-1} = \mu\nu + \mu\nu x + \cdots + \mu\nu x^{n-s-2} + \overline{c_0}x^{n-s-1} + \overline{c_1}x^{n-s} + \cdots + \overline{c_s}x^{n-1} = (\mu\nu + \mu\nu x + \cdots + \mu\nu x^{n-s-2} + \overline{c_0}x^{n-s-1} + \overline{c_1}x^{n-s} + \cdots + \overline{c_s}x^{n-1}) \in \mathfrak{C}$. Therefore, $c^*(x)^{rc} \in \mathfrak{C}$ and $(c^*(x)^{rc})^* = c(x)^{rc} \in \mathfrak{C}$. □

Theorem 4.3. *Assume that $\mathfrak{C}$ is a cyclic code with odd length n over $\Im$. Consequently, $\mathfrak{C}$ is an ideal in $\Im_n$ that may be produced by $\mathfrak{C} = \langle \eta(x) \rangle$. Therefore, $\mathfrak{C}$ possesses the reverse-complement property if and only if*

(a) $\eth(x)$ *and* $\partial_2(x)$ *are self-reciprocal and* $\mu\nu((1-x^n)/(1-x)) \in \mathfrak{C}$*;*

(b) (i) $\partial_1(x) | (x^i \partial_1^*(x) - \partial_1(x))$ *or*

(ii) $\mu x^i \partial_1^*(x) - \mu\partial_1(x) + \mu\nu x^j \mathfrak{r}_1^*(x) - \mu\nu\mathfrak{r}_1(x) = (\eth(x) + \mu\partial_1(x) + \mu\nu\mathfrak{r}_1(x))k(x) + (\nu\partial_2(x) + \mu\nu\partial_3(x))n_1(x)$, *where* $k(x) \in \Im[x]$, *and* $n_1(x) \in \Im_1[x]$

Proof. Its methodology is identical to that of Theorem 4.2. The conclusion is easily deduced by using the findings and procedures demonstrated in Theorem 4.2 , Lemma 4.1, and Theorem 3.3. □

5 GC-Content with Deletion Distance Δ

Here, we concentrate on the analysis of both the GC-content as well as deletion distance Δ. The number of guanine (G) and cytosine (C) that are present in a given codeword is the formal definition of the GC-content.

Definition 5.1. [21] Given a cyclic code $\mathfrak{C} = (\eth(x), (1+\mu)a(x))$, we define the subcode $\mathfrak{C}_{1+\mu}$ as the set of all codewords in $\mathfrak{C}$ that can be expressed as multiples of $(1+\mu)$.

Theorem 5.2. *Let* $\mathfrak{C} = \langle \eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu\nu\mathfrak{r}_1(x), \mu\partial_1(x) + \nu\gamma_2(x) + \mu\nu\mathfrak{r}_2(x), \nu\partial_2(x) + \mu\nu\mathfrak{r}_3(x), \mu\nu\partial_3(x) \rangle$ *be a cyclic code of even length n. Then,* $\mathfrak{C}_{\mu\nu} = ((\mu\nu)\partial_3(x))$.

Proof. It is well established that $((\mu\nu)\partial_3(x)) \subseteq \mathfrak{C}_{\mu\nu}$. Now, we have to show that $\mathfrak{C}_{\mu\nu} \subseteq ((\mu\nu)\partial_3(x))$. Let $c \in \mathfrak{C}$. Then, $c(x) = (\eth(x) + \mu\rho_1(x) + \nu\gamma_1(x) + \mu v\mathfrak{r}_1(x))l_1(x) + (\mu\partial_1(x) + v\gamma_2(x) + \mu v\mathfrak{r}_2(x))l_2(x) + (v\partial_2(x) + \mu v\mathfrak{r}_3(x))l_3(x) + \mu\nu\partial_3(x)l_4(x)$, where $l_1(x), l_2(x), l_3(x), l_4(x) \in \mathbb{F}_2[x]$. If c is multiple of $(\mu\nu)$, hence

$$c(x) = (\mu\nu)\eth(x)l_1(x) + (\mu\nu)\partial_1(x)l_2(x) + (\mu\nu)\partial_2(x)l_3(x) + (\mu\nu)\partial_3(x)l_4(x).$$

Since $\partial_3(x)|\partial_1(x)|\eth(x)$ and $\partial_3(x)|\partial_2(x)|\eth(x)$, then $c(x) \in (1+\mu)\eth_3(x)$, thus, it is clear that $\mathfrak{C}_{\mu\nu} \subseteq ((\mu\nu)\partial_3(x))$. Therefore, $\mathfrak{C}_{\mu\nu} = ((\mu\nu)\partial_3(x))$. □

In the subsequent discussion, we explore and elucidate the GC-content of cyclic codes denoted as C. This examination will be conducted by employing the Hamming weight enumerator as a tool to analyze the distribution of GC-content within these cyclic codes. The Hamming weight enumerator provides insights into the count of non-zero elements or '1's in the code words, and by applying it to cyclic codes, we aim to gain a comprehensive understanding of the GC-content patterns within the code structure.

Theorem 5.3. *Assume that, according to Theorem 5.2, $\mathfrak{C}$ is a cyclic code. Suppose c is an element of $\mathfrak{C}$ such that $\omega(c) = w_H(\overline{c})$, where $\overline{c} \in \mathbb{F}_2[x]$. The Hamming weight enumerator of the ideal $\langle \partial_3(x) \rangle$ is used to determine the GC-content of $\mathfrak{C}$.*

A deletion similarity between X and Y is the greatest length of a sequence that appears as a (not necessarily contiguous) subsequence of both $X, Y \in \mathfrak{C},\ X \neq Y$.

Definition 5.4. [7, 21] Let Δ be a fixed integer, where $1 \leq \Delta \leq n$. A DNA code $\mathfrak{C}$ is designated as an (n, Δ)-code, or a DNA code of distance Δ based on deletion similarity, if the deletion similarity satisfies

$$S(X,Y) \leq n - \Delta - 1,\ X, Y \in \mathfrak{C},\ X \neq Y.$$

Lastly, we focus on the deletion distance of the cyclic code $\mathfrak{C}$ defined over the ring $\Im$.

Theorem 5.5. *Let $\mathfrak{C}$ be a cyclic code as described in Theorem 5.3, and let $\Delta_{\mu\nu}$ represent the deletion similarity distance associated with the subcode $\mathfrak{C}_{\mu\nu}$. In this case, the deletion similarity distance of $\mathfrak{C}$ satisfies $\Delta = \Delta_{\mu\nu}$.*

Proof. Since $\mathfrak{C}_{\mu\nu} \subseteq \mathfrak{C}$, for $X, Y \in \mathfrak{C}$

$S(X,Y) \leq n - \Delta - 1$, which $\Delta_{\mu v} \geq \Delta$. For $A, B \in \mathfrak{C}$ such that

$S(A,B) > n - \Delta_{\mu\nu} - 1$.

By invoking Theorem 6, considering sets A and B within $\mathfrak{C}$, it follows that $(\mu\nu)A$ and $(\mu\nu)B$ reside in $\mathfrak{C}_{\mu\nu}$. Consequently,

$$S((\mu\nu)A, (\mu\nu)B) > S(A,B) > n - \Delta_{\mu\nu} - 1,$$

which leads to a contradiction. As a result, we conclude that $\Delta = \Delta_{\mu\nu}$. □

6 Computational Work

To illustrate the characteristics and results of cyclic DNA codes over the ring $\Im$, examples are provided in the figures. First, I provide an example of cyclic codes that meet the reversible feature in Figure 3, illustrating the results of Theorem 3.3. Figures 4 and 5 illustrate Theorem 4.3, which describes the reverse-complement property of cyclic codes of odd lengths. I further investigate that the reverse-complement property is preserved in Theorem 4.2 for cyclic DNA codes over $\Im$ of even lengths, and confirm that they satisfy the similarity deletion distance requirement as established in Theorems 5.3 and 5.5, as illustrated in Figures 6 and 7.

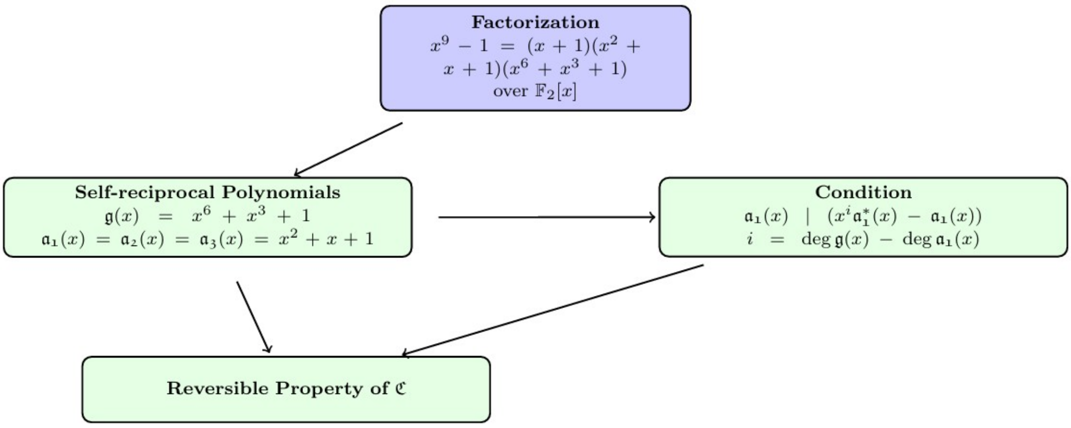

Figure 3. Reversible Property

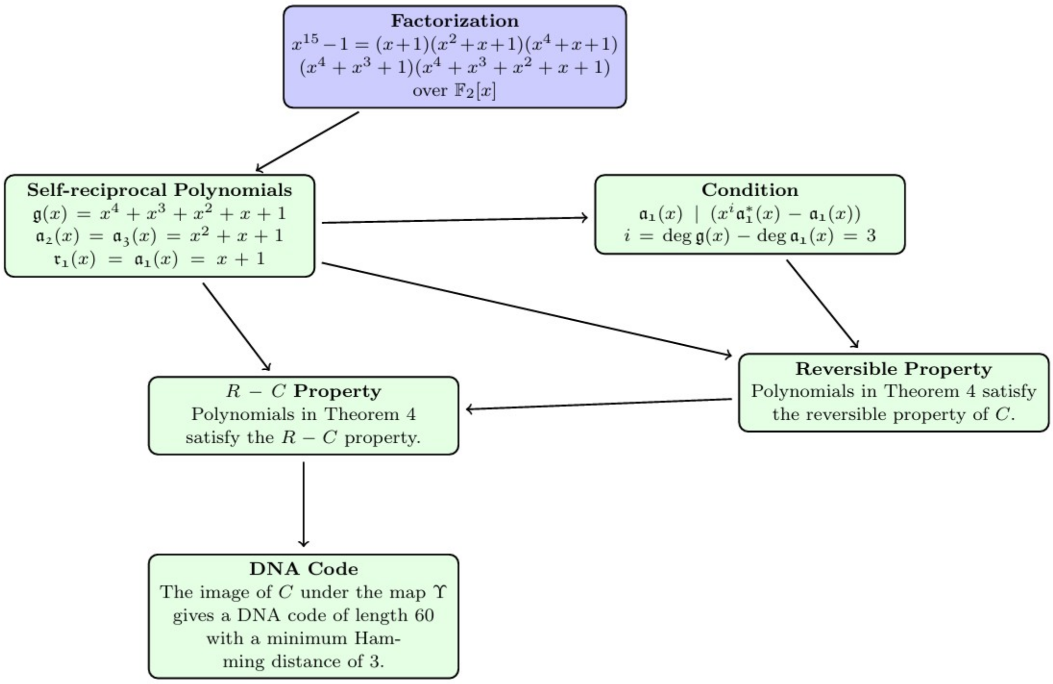

Figure 4. Reverse-Complement Property

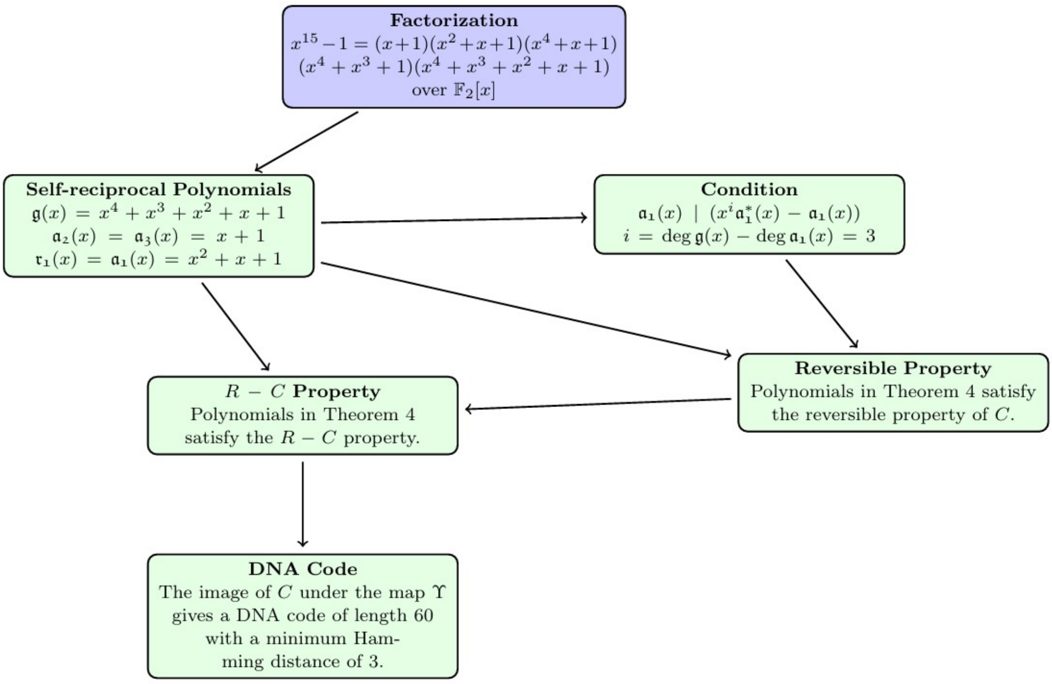

Figure 5. Reverse-Complement Property

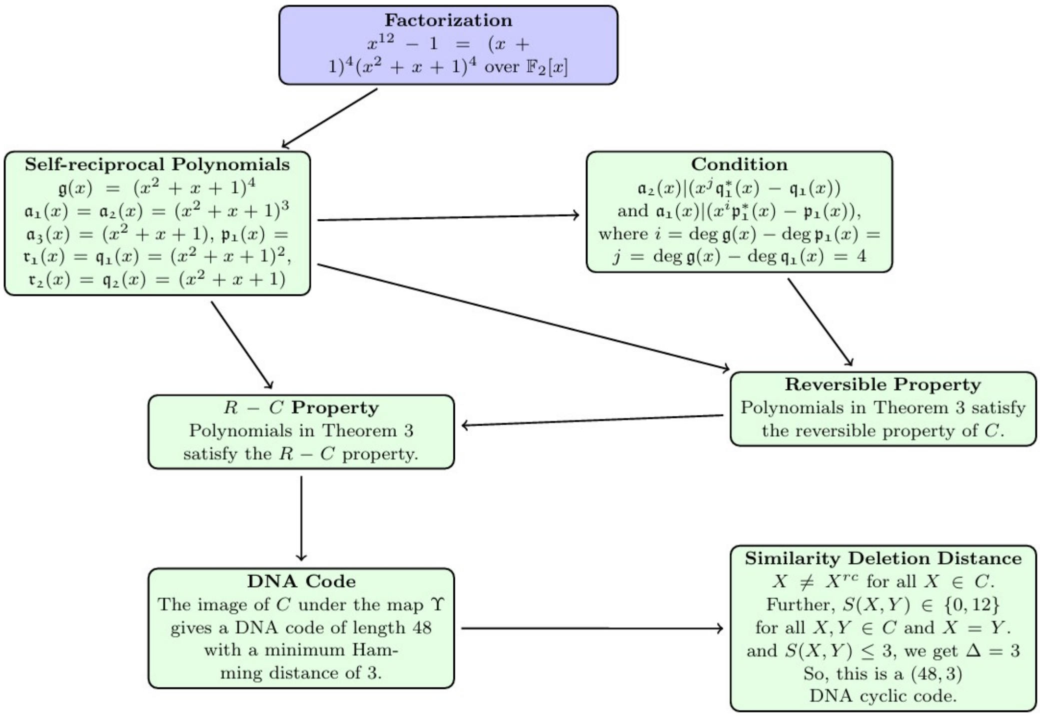

Figure 6. DNA Codes with Deletion Distance

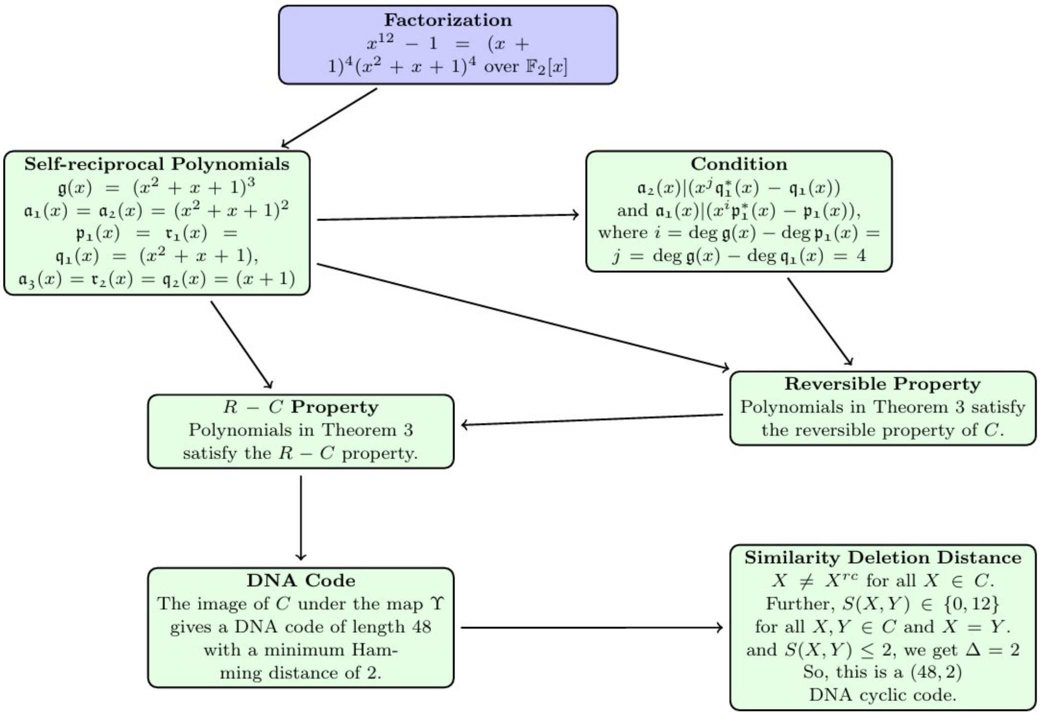

Figure 7. DNA Codes with Deletion Distance

7 Conclusion

The present study has established the structural characteristics of cyclic DNA coding over the ring $\Im$. The discussion clarified the GC-content of these codes over $\Im$ by addressing the reverse and reverse-complement constraints. Notably, our findings demonstrated a significant relationship with deletion distance, and a subcode of code $\mathfrak{C}$. The computed examples of cyclic DNA codes served as illustrative instances of these principles. It would be intriguing to think about building DNA codes across finite rings using mixed alphabets of any length for future research.

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Author information

Narendra Kumar, Department of Mathematics, School of Advanced Sciences, VIT-AP University, Amaravati, Andhra Pradesh, 522237, India.
E-mail: `narendrakumar@vitap.ac.in`

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