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Exploring Fuzzy Structures in JU-Algebras

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Abstract In this study, we explored the fuzzy extension of JU-subalgebras and JU-ideals of JU-algebra and obtained some results. Additionally, we proved that both the homomorphic image and the pre-image of fuzzy JU-subalgebras and fuzzy JU-ideals of a JU-algebra preserve their respective fuzzy properties. We also studied the connections between prime fuzzy sets and fuzzy p-multiplicative in JU-algebra and obtained some relevant results. Moreover, we prove that the intersection of two fuzzy JU-subalgebras and JU-ideals of JU-algebra is also a fuzzy JU-subalgebra and JU-ideals of a JU-algebra respectively.

1 Introduction

The concept of fuzzy sets was first introduced by [19]. Since then, researchers have expanded this concept to various algebraic structures, including groups, rings, modules, vector spaces, and topological spaces. In 1991, [18] extended fuzzy set theory to BCK-algebras, introducing the notions of fuzzy subalgebras and fuzzy ideals within this framework. This idea was further developed in 2011 by [9], who extended fuzzy subalgebras and fuzzy ideals to KU-algebras. In 2024,[7] introduced Lukasiewicz fuzzy ideals in to BCK/BCI-algebras and [3] introduced anti-fuzzy structures in B-algebras.

In 2009, [6] introduced the concept of fuzzy translations for fuzzy BCK/BCI-algebras, and in 2014, [14] extended this concept to fuzzy PS-algebras. To avoid, the membership and nonmembership restriction function in fuzzy sets[1] introduced a linear Diophantine. The exploration of prime fuzzy sets with fuzzy translations in UP-algebras was conducted in 2016 [17] and 2017 [5]. Additionally, the notion of homomorphisms in fuzzy KU-algebras was introduced by [11] in 2009, and [12] further examined the properties of fuzzy PS-algebras under homomorphisms in 2014.

The concept of JU-algebras was first proposed by [2] in 2020. Following [16] explored ideals and subalgebras within JU-algebras. While JU-algebras have been studied primarily in the context of crisp sets, their fuzzy counterparts have not been explored extensively. This gap in the literature motivates us to extend the concept of fuzzy JU-subalgebras and fuzzy JU-ideals and investigate the properties of these fuzzy structures, introducing new definitions and properties that extend beyond existing research. Furthermore, this study investigates the homomorphic images and pre-images of fuzzy JU-subalgebras and fuzzy JU-ideals, proved that these properties are preserved in their fuzzy counterparts. Our work also explores prime fuzzy sets and fuzzy p-multiplicative properties in JU-algebras, making a significant contribution to both algebraic theory and fuzzy set theory. In this study, the meet and join operator instead of minimum and maximum was utilized, and infimum is used since the infimum provides a stable, mathematically rigorous lower bound in fuzzy set operations while preserving consistency.

This study structured as Section 2 focused on a review of the subalgebra and ideals of the JU-algebra and fuzzy set to help us with our explanation. Extend the concept of fuzzy set in to JU-subalgebra covered in Section 3 and the fuzzy ideals of JU- algebra were discussed in Section 4. We then provide a conclusion at the end of the manuscript and suggest additional research.

2 Preliminaries

Definition 2.1. [2] A JU-algebra is an algebra of $(X;\circ,1)$ of type (2, 0) with a binary operation $\circ$ and a fixed element 1, if it holds, ∀ x, y, z ∈ X :
(a) $(x\circ y)\circ[(y\circ z)\circ(x\circ z)]=1$,
(b) $1\circ x=x$,
(c) $x\circ y=1$ and $y\circ x=1\Rightarrow x=y, \forall x,y\in X$.
In a JU-algebra X, we can define a partial ordering " $\leq$ " by Putting $x\leq y$ if and only $y\circ x=1$.

Lemma 2.2. *[2, 16] If X is a JU-algebra, then the following properties satisfied for any* $x,y,z\in X$*:*
(a) $x\leq y$ *implies* $y\circ z\leq x\circ z$
(b) $x\leq y$ *and* $y\leq x\Rightarrow x=y$
(c) $x\circ x=1$
(d) $z\circ(y\circ x)=y\circ(z\circ x)$
(e) $(x\circ[(x\circ y)\circ y])=1$

Definition 2.3. [16] A nonempty subset S of a JU-algebra X is called sub-algebra of X, if $x\circ y\in S,\forall x,y\in S$.

Definition 2.4. [16] A nonempty subset J of a JU-algebra X is said to be a JU-ideals of X if it satisfies:
(a) $1\in J$
(b) $x,x\circ y\in J\Rightarrow y\in J,\ \forall x,y\in X$.

Definition 2.5. [19] A fuzzy set in a non empty set X (or fuzzy subset of X) is an arbitrarily function $\vartheta: X\rightarrow[0,1]$.

Definition 2.6. [4, 10] Let ϑ be a fuzzy set in a JU-algebra X. Then the set $\vartheta_t=\{x\in X:\vartheta(x)\geq t\}$ is called a level subset of ϑ, where $t\in[0,1]$. Clearly, $\vartheta_t\subseteq\vartheta_s$ whenever $t>s$

Definition 2.7. [5, 17] A nonempty subset A of a set X is called a prime subset of X, if for any $x,y\in X$ then $x\circ y\in A\implies x\in A$ or $y\in A$.

Definition 2.8. [5, 17] A fuzzy set ϑ in a set A is called a prime fuzzy set in A, if $f(x\circ y)\leq max\{f(x),f(y)\},\forall x,y\in A$.

Definition 2.9. [8] Let f be any function from a set X to a set Y; ϑ be any fuzzy subset of X and λ be any fuzzy subset Y. The image of ϑ under f, denoted by $f(\vartheta)$, is a fuzzy subset of Y defined by

$$f(\vartheta)(y)=\begin{cases} sup\{\vartheta(x)|x\in f^{-1}(y), if f^{-1}(y)\neq\emptyset \\ 0, otherwise\end{cases}$$

where $y\in Y$. The Pre - image of λ under f, symbolized by $f^{-1}(\lambda)$, is a fuzzy subset of X defined by $f^{-1}(\lambda)(x)=\lambda(f(x)),\forall x\in X$

3 Fuzzy subalgebras of a JU-algebra

In this section we discuss and investigate a new notion of fuzzy JU- subalgebras in JU-algebra and study their related properties.

Definition 3.1. Let X be a JU-algebra, a fuzzy set ϑ in X is called fuzzy JU-subalgebra of X if it satisfies:
$\vartheta(x\circ y)\geq\{\vartheta(x)\wedge\vartheta(y)\},\forall x,y\in X$.

Example 3.2. Consider X = $\{1,2,3,4,5\}$, we construct the following table.

Clearly, $(X,\circ,1)$ is a JU-algebra. Define a fuzzy set $\vartheta: X\rightarrow[0,1]$ by

$$\vartheta(x)=\begin{cases}0.7, & if\ \ x=1\\ 0.5, & if\ \ x=2\\ 0.3, & otherwise\end{cases}$$

Routine calculation gives that ϑ is a fuzzy JU-subalgebra of X.
(For instance, $\vartheta(4\circ 2)=\vartheta(1)=0.7\geq\{\vartheta(4)\wedge\vartheta(2)\}=\{0.3\wedge 0.5\}=0.3$)

∘	**1**	**2**	**3**	**4**	**5**
1	1	2	3	4	5
2	1	1	3	4	5
3	1	2	1	4	4
4	1	1	3	1	3
5	1	1	1	1	1

Table 3.1 [2]

Lemma 3.3. *If ϑ is a fuzzy JU- subalgebra of a JU- algebra X. Then $\vartheta(1) \geq \vartheta(x),\ \forall x \in X$.*

Proof. Suppose ϑ be a fuzzy JU- subalgebra of a JU- algebra X. Since, $\vartheta(1) = \vartheta(x \circ x)$ ... (Lemma no. 2.2 c)

$\geq \{\vartheta(x) \wedge \vartheta(x)\}$

$= \vartheta(x)$ □

Theorem 3.4. *A fuzzy set ϑ of a JU- algebra X is a fuzzy JU- subalgebra if and only if for every $t \in [0, 1]$, ϑ_t is a subalgebra of X.*

Proof. Suppose that ϑ is a fuzzy JU-subalgebra of a JU-algebra X and $\vartheta_t \neq \emptyset$.
For any $x, y \in \vartheta_t$ such that $\vartheta(x) \geq t$ and $\vartheta(y) \geq t$

Then $\vartheta(x \circ y) \geq \{\vartheta(x) \wedge \vartheta(y)\}$

$\geq \{t \wedge t\} = t$

Therefore, $x \circ y \in \vartheta_t$. Hence, ϑ_t is a subalgebra of X.

Conversely, assume that ϑ_t is a subalgebra of X. Let $x, y \in X$ and there exist $t_1, t_2 \in [0, 1]$ such that $\vartheta(x) = t_1$ and $\vartheta(y) = t_2$.Take, $t = \{t_1 \wedge t_2\}$.

By assumption ϑ_t is a subalgebra of X, implies that $x \circ y \in \vartheta_t$. Then $\vartheta(x \circ y) \geq t = \{t_1 \wedge t_2\} = \{\vartheta(x) \wedge \vartheta(y)\}$.

Therefore ϑ is a fuzzy JU-subalgebra of X. □

Theorem 3.5. *Two level subalgebras $\vartheta_\beta, \vartheta_\gamma (\beta < \gamma)$ of fuzzy JU- algebras are equal if and only if there is no $x \in X$ such that $\beta \leq \vartheta(x) < \gamma$*

Proof. Suppose $\vartheta_\beta = \vartheta_\gamma$ for some $\beta < \gamma$. Assume that if there exist $x \in X$ such that $\beta \leq \vartheta(x) < \gamma$

$\Rightarrow x \in \vartheta_\beta$ and $x \notin \vartheta_\gamma$

$\Rightarrow \vartheta_\beta \neq \vartheta_\gamma$, which is contradict to our assumption.

Conversely, assume that there is no $x \in X$ such that $\beta \leq \vartheta(x) < \gamma$. For $\beta < \gamma$, we have $\vartheta_\gamma \subseteq \vartheta_\beta$ ....(*)

But there is no $x \in X$ such that $\beta \leq \vartheta(x) < \gamma$

Hence $\vartheta(x) \geq \gamma$, for all $x \in \vartheta_\beta$

$\Rightarrow x \in \vartheta_\gamma$

$\Rightarrow \vartheta_\beta \subseteq \vartheta_\gamma$..(**)

From (*) and (**) we get $\vartheta_\beta = \vartheta_\gamma$ □

Theorem 3.6. *If ϑ and λ are fuzzy JU- subalgebras of JU- algebra X. Then $\vartheta \cap \lambda$ is also a fuzzy JU-subalgebra of X.*

Proof. Suppose ϑ and λ are fuzzy JU- subalgebras of a JU- algebra X. Let $x, y \in X$. Then

$(\vartheta \cap \lambda)(x \circ y) = \{\vartheta(x \circ y) \wedge \lambda(x \circ y)\}$

$\geq \{\{\vartheta(x) \wedge \vartheta(y)\} \wedge \{\lambda(x) \wedge \lambda(y)\}\}$

$= \{\{\vartheta(x) \wedge \lambda(x)\} \wedge \{\vartheta(y) \wedge \lambda(y)\}\}$

$= \{(\vartheta \cap \lambda)(x) \wedge (\vartheta \cap \lambda)(y)\}$ □

Corollary 3.7. *Let $\{\vartheta_i | i \in \Omega\}$ be a family of fuzzy JU- subalgebras of a JU- algebra X. Then $\cap_{i\in\Omega}\vartheta_i$ is also fuzzy JU-sub-algebra of X.*

Remark 3.8. The union of any two fuzzy JU-subalgebras of a JU- algebra X may not be a fuzzy JU-sub-algebra of X.

$\circ$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	1	1	4	1
3	1	1	1	1
4	1	4	4	1

Table 3.2 [2]

Example 3.9. Let X = $\{1, 2, 3, 4\}$ in which $\circ$ is defined by the following table.

Clearly, $(X; \circ, 1)$ is a JU-algebra. We define $\vartheta : X \to [0, 1]$ as follows:

$$\vartheta(x) = \begin{cases} 0.8, if\ x = 1 \\ 0.4,\ if\ x = 2 \\ 0.2,\ otherwise \end{cases}$$

And define $\lambda : X \to [0, 1]$ as follows:

$$\lambda(x) = \begin{cases} 0.9, if\ x = 1 \\ 0.6, if\ x = \{2, 3\} \\ 0.3, if\ x = 4 \end{cases}$$

Then, $(\vartheta \cup \lambda)(2 \circ 3) = \{\vartheta(2 \circ 3) \vee \lambda(2 \circ 3)\} = 0.3$....................................$(***)$
And, $(\vartheta \cup \lambda)(2 \circ 3) = \{\vartheta(2 \circ 3) \vee \lambda(2 \circ 3)\}$
$\geq \{\{\vartheta(2) \wedge \vartheta(3)\} \vee \{\lambda(2) \wedge \lambda(3)\}\} = 0.6$.................$(****)$
From $(***)$ and $(****)$ we get $0.3 \geq 0.6$ which is false.

Proposition 3.10. *Let S be a subset of a JU-algebra X. Then the characteristics function χ_S is a fuzzy JU- subalgebra of X if and only if S is a subalgebra of X.*

Proof. Let $\chi_S : X \to [0, 1]$ be a fuzzy JU-subalgebra of a JU- algebra X and let $x, y \in S$ implies that $\chi_S(x) = \chi_S(y) = 1$. Then,
$\chi_S(x \circ y) \geq \{\chi_S(x) \wedge \chi_S(y)\} = \{1 \wedge 1\} = 1$
$\Rightarrow \chi_S(x \circ y) \geq 1$, but $\chi_S(x \circ y) \leq 1$
$\Rightarrow \chi_S(x \circ y) = 1$
$\Rightarrow x \circ y \in S$, implies that S is a sub-algebra of X.
Conversely, let S be a sub-algebra of X and $x, y \in X$. Consider the following cases:
Case 1, if $x, y \in S$, then $x \circ y \in S$, we get that
$\chi_S(x \circ y) = 1 = \{\chi_S(x) \wedge \chi_S(y)\}$
Case 2, if $x, y \notin S$, then $\chi_S(x) = 0 = \chi_S(x)$
Thus, $\chi_S(x \circ y) = 0 = \{\chi_S(x) \wedge \chi_S(y)\}$
Case 3, if x or y not in S, then $\{\chi_S(x) \wedge \chi_S(y)\} = 0$.
Then, $\chi_S(x \circ y) = 0 = \{\chi_S(x) \wedge \chi_S(y)\}$
Hence, χ_S is a fuzzy JU- subalgebra of X. □

Definition 3.11. For any fuzzy set ϑ in a JU- algebra X, we let $\Delta = inf\{1 - \vartheta(x) | x \in X\}$ and $p \in [0, \Delta]$. A mapping $\vartheta^p_\Delta : X \to [0, 1]$ defined by $\vartheta^p_\Delta(x) = (p)(\vartheta(x))$ for all $x \in X$, is said to be a fuzzy p-multiplicative of ϑ.

Example 3.12. Let X = $\{1, 2, 3, 4\}$ in which $\circ$ is defined by the following table.

$\circ$	**1**	**2**	**3**	**4**
1	1	2	3	4
2	2	1	2	2
3	1	2	1	3
4	1	2	1	1

Table 3.3 [2]

Clearly $(X; \circ, 1)$ is a JU-algebra. Define a mapping $\vartheta : X \to [0, 1]$ as follows;
$\vartheta(1) = 0.9, \vartheta(2) = 0.7, \vartheta(3) = 0.6, \vartheta(4) = 0.4$

Then $\Delta = inf\{1-0.9, 1-0.7, 1-0.6, 1-0.4\} = 0.1$
Let $p = 0.1 \in [0, 0.1]$, then a mapping $\vartheta^p_\Delta : X \to [0,1]$ defined by
$\vartheta^p_\Delta(1) = 0.09, \vartheta^p_\Delta(2) = 0.07, \vartheta^p_\Delta(3) = 0.06, \vartheta^p_\Delta(4) = 0.04$

Theorem 3.13. *Let ϑ be a fuzzy JU- sub-algebra of a JU -algebra X and for every $p \in [0, \Delta]$. Then the fuzzy p-multiplicative ϑ^p_Δ of ϑ is a fuzzy JU- sub-algebra of X.*

Proof. Suppose ϑ be a fuzzy sub-algebra of a JU -algebra X and $\forall p \in [0, \Delta]$. Then $\forall x, y \in X$,
$\vartheta^p_\Delta(x \circ y) = (p)(\vartheta(x \circ y)) \geq (p)(\{\vartheta(x) \wedge \vartheta(y)\})$
$= \{(p)(\vartheta(x)) \wedge (p)(\vartheta(y))\}$
$= \{\vartheta^p_\Delta(x) \wedge \vartheta^p_\Delta(y)\}$. □

Theorem 3.14. *If there exist $p \in [0, \Delta]$ such that the fuzzy p-multiplicative ϑ^p_Δ of ϑ is a fuzzy JU-sub-algebra of a JU- algebra X, then ϑ is a fuzzy JU- sub-algebra of X.*

Proof. Suppose the fuzzy p-multiplicative ϑ^p_Δ of ϑ be a fuzzy JU-subalgebra of a JU-algebra X for some $p \in [0, \Delta]$. Then $\forall x, y \in X$,
$(p)(\vartheta(x \circ y)) = \vartheta^p_\Delta(x \circ y)$
$\geq \{\vartheta^p_\Delta(x) \wedge \vartheta^p_\Delta(y)\}$
$= \{(p)(\vartheta(x)) \wedge (p)(\vartheta(y))\}$
$= (p)(\{\vartheta(x) \wedge \vartheta(y)\})$
Therefore, $\vartheta(x \circ y) \geq \{\vartheta(x) \wedge \vartheta(y)\}$ □

Definition 3.15. Let $(X, \circ, 1)$ and $(Y, \circ', 1')$ be JU-algebras. A homomorphism is a map $f : X \to Y$ satisfying;
$f(x \circ y) = f(x) \circ' f(y), \forall x, y \in X$

Theorem 3.16. *Let $(X, \circ, 1)$ and $(Y, \circ', 1')$ be JU- algebras and a map $f : X \to Y$ be an onto homomorphism and ϑ be a fuzzy JU-subalgebra of a JU- algebra X. Then the image $f(\vartheta)$, is a fuzzy JU-subalgebra of Y with sup property.*

Proof. Let $f : X \to Y$ be an onto homomorphism and ϑ be a fuzzy JU- subalgebra of X.
Assume that $m, n \in Y$, and $p \in f^{-1}(m), q \in f^{-1}(n)$ be such that
$\vartheta(p) = sup\{\vartheta(a) : a \in f^{-1}(m)\}$ and
$\vartheta(q) = sup\{\vartheta(b) : b \in f^{-1}(n)\}$
Then, $f(\vartheta)(m \circ n) = sup\{\vartheta(c) : c \in f^{-1}(m \circ n)\}$
$\geq \{\vartheta(p) \wedge \vartheta(q)\}$
$= \{sup\{\vartheta(a) : a \in f^{-1}(m)\} \wedge sup\{\vartheta(b) : b \in f^{-1}(n)\}\}$
$= \{f(\vartheta)(m) \wedge f(\vartheta)(n)\}$
Hence, $f(\vartheta)$ is a fuzzy subalgebra of X. □

Theorem 3.17. *Let $(X, \circ, 1)$ and $(Y, \circ', 1')$ be JU-algebras and a map $f : X \to Y$ be homomorphism and λ be a fuzzy JU-subalgebra of Y. Then the Per-image $f^{-1}(\lambda)$, is a fuzzy JU-subalgebra of X.*

Proof. Let λ be a fuzzy JU-subalgebra of Y and for all $x, y \in X$, then
$f^{-1}(\lambda)(x \circ y) = \lambda(f(x \circ y))$
$= \lambda(f(x) \circ f(y))$
$\geq \{\lambda(f(x)) \wedge \lambda(f(y))\}$
$= \{f^{-1}(\lambda)(x) \wedge f^{-1}(\lambda)(y)\}$ □

Theorem 3.18. *Let X, Y, Z be JU-algebras. Then the function $f : X \to Y$ and $g : Y \to Z$ be JU-homomorphism then the following holds:*
(a) If f, g are surjective then $(gof)(\vartheta)$ a fuzzy JU-sub-algebra Z, for any fuzzy JU- sub-algebra of ϑ of X, provided that sup property holds.
(b) $(gof)^{-1}(\lambda)$ is a fuzzy JU-sub-algebra of X for any fuzzy JU- sub-algebra λ of Z.

Proof. (a) let ϑ be a fuzzy JU-sub-algebra of X,for $m, n \in Z$,there exist $p, q \in X$ such that, $(gof)(p) = m$ and $(gof)(q) = n$. Then $\vartheta(s) = sup\{\vartheta(r) | r \in (gof)^{-1}(m)\}$ and $\vartheta(t) = sup\{\vartheta(r) | r \in (gof)^{-1}(n)\}$ for some $s, t \in X$. Since f,g are surjective, so is gof, and hence $s \circ t \in (gof)^{-1}(m \circ n)$

$(gof)(\vartheta)(m \circ n) \geq \vartheta(s \circ t)$
$\geq \{\vartheta(s) \wedge \vartheta(t)\}$
$= \{sup\{\vartheta(r)|r \in (gof)^{-1}(m) \wedge sup\{\vartheta(r)|r \in (gof)^{-1}(n)\}$
$= \{(gof)(\vartheta)(m) \wedge (gof)(\vartheta)(n)\}$.

(b). Let λ be a fuzzy JU-subalgebra of Z, for $x, y \in X$. Then,
$(gof)^{-1}(\lambda)(m \circ n) = \lambda(gof)(m \circ n)$
$= \lambda((gof)(m) \circ (gof)(n))$
$\geq \{\lambda(gof)(m) \wedge \lambda(gof)(n)\}$
$= \{(gof)^{-1}(\lambda)(m) \wedge (gof)^{-1}(\lambda)(n)\}$

□

4 Fuzzy Ideal of a JU-algebra

In this section we discuss and investigate new notion of fuzzy JU- ideals in JU-algebra and study their related properties.

Definition 4.1. Let X be a JU-algebra, a fuzzy set ϑ in X is called fuzzy JU-ideal of X if it satisfies the following conditions:
(a) $\vartheta(1) \geq \vartheta(x)$
(b) $\vartheta(y) \geq \{\vartheta(x) \wedge \vartheta(x \circ y)\}, \forall x, y \in X$.

Example 4.2. (See table 3.1) consider X = $\{1, 2, 3, 4, 5\}$, and define a fuzzy set $\vartheta : X \to [0, 1]$ by

$$\vartheta(x) = \begin{cases} 0.6, & if\ x = 1 \\ 0.3, & if\ x = \{2, 3\} \\ 0.2, & otherwise \end{cases}$$

Routine calculation gives that ϑ is a fuzzy JU-ideal of X.

Theorem 4.3. *Every fuzzy JU-ideals of a JU-algebra X is order reversing.*

Proof. Let ϑ be a fuzzy JU- ideals of a JU-algebra X and for any $x, y \in X$ be such that $x \leq y$, then $y \circ x = 1$
Then $\vartheta(x) \geq \{\vartheta(y \circ x) \wedge \vartheta(y)\}$
$= \{\vartheta(1) \wedge \vartheta(y)\} = \vartheta(y)$ □

Theorem 4.4. *If ϑ is a fuzzy JU-ideals of X, then $\vartheta((x \circ y) \circ y) \geq \vartheta(x)$.*

Proof. Since $(x \circ [(x \circ y) \circ y]) = 1$..........................(Lemma no. 2.2 e)
Now, $\vartheta((x \circ y) \circ y) \geq \{\vartheta(x) \wedge \vartheta(x \circ [(x \circ y) \circ y])\}$
$= \{\vartheta(x) \wedge \vartheta(1)\} = \vartheta(x)$ □

Theorem 4.5. *A fuzzy ideal of a JU- algebra X is fuzzy JU-subalgebra when $\vartheta(x \circ (x \circ y)) \geq \vartheta(y)$, for any $x, y \in X$.*

Proof. Suppose ϑ is a fuzzy ideal of a JU- algebra X and $\vartheta(x \circ (x \circ y)) \geq \vartheta(y)$. Then for any $x, y \in X$,
$\vartheta(x \circ y) \geq \{\vartheta(x) \wedge \vartheta(x \circ (x \circ y))\}$
$\geq \{\vartheta(x) \wedge \vartheta(y)\}$ □

Theorem 4.6. *Let ϑ be a fuzzy set in a JU-algebra X and let $t \in [0, 1]$. Then ϑ is a fuzzy JU-ideal of X,if and only if the level subset ϑ_t is a JU-ideal of X.*

Proof. Suppose ϑ is a fuzzy JU -ideal in a JU-algebra X and $\vartheta_t \neq \emptyset$
Let $x, x \circ y \in \vartheta_t$ implies that $\vartheta(x) \geq t$ and $\vartheta(x \circ y) \geq t$
Then $\vartheta(y) \geq \{\vartheta(x) \wedge \vartheta(x \circ y)\}$
$\geq \{t \wedge t\} = t$
$\Rightarrow \vartheta(y) \geq t$
This implies that $y \in \vartheta_t$ and hence ϑ_t is a JU-ideal of X.
Conversely, assume that the level subset ϑ_t is a JU-ideal of X. Further, let we assume that there exist some $x_0 \in X$ such that $\vartheta(1) < \vartheta(x_0)$

Take $s = \frac{1}{2}[\vartheta(1) + \vartheta(x_0)]$
$\Rightarrow \vartheta(1) < s < \vartheta(x_0)$
$x_0 \in \vartheta_s$ and $1 \notin \vartheta_s$, which is contradict to our assumption that ϑ_s is a JU-ideal of X.
Therefore, $\vartheta(1) \geq \vartheta(x), \forall x \in X$.
And
assume that $x_0, y_0 \in X$ such that
$\vartheta(y_0) < \{\vartheta(x_0) \wedge \vartheta(x_0 \circ y_0)\}$
Take $s = \frac{1}{2}[\vartheta(y_0) + \{\vartheta(x_0) \wedge \vartheta(x_0 \circ y_0)\}]$
$\Rightarrow \vartheta(y_0) < s < \{\vartheta(x_0) \wedge \vartheta(x_0 \circ y_0)\}$
$\Rightarrow s > \vartheta(y_0)$ and $s < \{\vartheta(x_0) \wedge \vartheta(x_0 \circ y_0)\}$
$\Rightarrow s > \vartheta(y_0)$, $s < \vartheta(x_0)$, and $s < \vartheta(x_0 \circ y_0)$
$y_0 \notin \vartheta_s$, a contradiction, since ϑ_s is a JU-ideal of X.
Therefore, $\vartheta(y_0) \geq \{\vartheta(x_0) \wedge \vartheta(x_0 \circ y_0)\}$ for any $x, y \in X$. □

Theorem 4.7. *Let $\{\vartheta_i | i \in \Omega\}$ be the family of fuzzy JU- ideals in a JU- algebra X. Then $\cap_{i\in\Omega}\vartheta_i$ is also fuzzy JU-ideal of X.*

Proof. Let $\{\vartheta_i | i \in \Omega\}$ be the family of JU-ideals in a JU-algebra X, then for any $x, y \in X$.
$(\cap\vartheta_i)(1) = inf(\vartheta_i(1)) \geq inf(\vartheta_i(x)) = (\cap\vartheta_i)(x)$ And
$(\cap\vartheta_i)(y) = inf(\vartheta_i(y)) \geq inf(\{\vartheta_i(x) \wedge \vartheta_i(x \circ y)\})$
$= \{inf(\vartheta_i(x)) \wedge inf(\vartheta_i(x \circ y))\}$
$= \{(\cap\vartheta_i)(x) \wedge (\cap\vartheta_i)(x \circ y)\}$ □

Proposition 4.8. *Let J be a nonempty subset of a JU-algebra X. Then the characteristics function χ_J is a fuzzy JU- ideal of X if and only if J is an ideal of X.*

Proof. Let $\chi_J : X \to [0, 1]$ be a fuzzy JU- ideal of X and $J \neq \emptyset$.
Let $x, x \circ y \in J$ implies that $\chi_J(x) = \chi_J(x \circ y) = 1$
Thus, $\chi_J(y) \geq \{\chi_J(x) \wedge \chi_J(x \circ y)\} = \{1 \wedge 1\} = 1$
$\Rightarrow \chi_J(y) = 1$
$\Rightarrow y \in J$
Hence, J is an ideal of X.
Conversely, let J be an ideal of X.
Since $1 \in J$. Then
$\chi_J(1) = \chi_J(x \circ x)$...(Lemma no. 2.2 c)
$\geq \{\chi_J(x) \wedge \chi_J(x)\} = \chi_J(x)$
And
Consider the following cases:
Case 1, if $x, x \circ y \in J$, then $y \in J$, we get that
$\chi_J(y) = 1 = \{\chi_J(x) \wedge \chi_J(x \circ y)\}$
Case 2, if $x, x \circ y \notin J$, then $\chi_J(x) = 0 = \chi_J(x \circ y)$
Thus, $\chi_J(y) = 0 = \{\chi_J(x) \wedge \chi_J(x \circ y)\}$
Case 3, if x or x $\circ$ y not in J, then $\{\chi_J(x) \wedge \chi_J(x \circ y)\} = 0$.
Then, $\chi_J(y) = 0 = \{\chi_J(x) \wedge \chi_J(x \circ y)\}$
Hence χ_J is a fuzzy ideal of X. □

Definition 4.9. An ideal J of a JU- algebra X is called prime if, $x \circ y \in J$, implies that either $x \in J$ or $y \in J$.

Definition 4.10. A fuzzy JU ideal ϑ of a JU- algebra X is called fuzzy prime ideal of X if, $\vartheta(x \circ y) \leq \{\vartheta(x) \vee \vartheta(y)\}$

Theorem 4.11. *Let J be nonempty subset of a JU- algebra X. Then J is a prime JU- ideal of X if and only if the characteristic function χ_J is a prime fuzzy JU- ideal in X.*

Proof. Suppose that J is a prime JU- ideal of X and let $x, y \in X$.
Case 1: if $x \circ y \in J$. Since J is a prime JU- ideal of X, we have
$x \in J$ or $y \in J$, then $\chi_J(x) = 1$ or $\chi_J(y) = 1$.
Thus $\{\chi_J(x) \vee \chi_J(y)\} = 1$.
$\Rightarrow \chi_J(x \circ y) \leq 1 = \{\chi_J(x) \vee \chi_J(y)\}$.

Case 2: if $x \circ y \notin J$. Then $\chi_J(x \circ y) = 0 \leq \{\chi_J(x) \vee \chi_J(y)\}$.
Thus χ_J is a prime fuzzy JU- ideal in X.
Conversely, suppose that χ_J is a prime fuzzy JU- ideal in X. Let $x, y \in X$ be such that $x \circ y \in J$.
Then $\chi_J(x \circ y) = 1$. Since 1 = $\chi_J(x \circ y) \leq \{\chi_J(x) \vee \chi_J(y)\}$.
Thus $\{\chi_J(x) \vee \chi_J(y)\} = 1$.
$\Rightarrow \chi_J(x) = 1$ or $\chi_J(y) = 1$.
$\Rightarrow x \in J$ or $y \in J$
Therefore J is a prime ideal of X. □

Proposition 4.12. *If ϑ and λ are any two fuzzy prime ideal of a JU- algebra X, then $\vartheta \cap \lambda$ be a prime ideal of X if and only if $\vartheta \subseteq \lambda$ or $\lambda \subseteq \vartheta$.*

Proof. let ϑ and λ be any fuzzy prime ideals of a JU- algebra X and $\vartheta \cap \lambda$ be a prime ideal of X.
Since $\vartheta \cap \lambda \subseteq \vartheta$ and $\vartheta \cap \lambda \subseteq \lambda$
$\implies \lambda \subseteq \vartheta$ or $\vartheta \subseteq \lambda$, since ϑ and λ are fuzzy prime ideals.
Conversely, Let ϑ and λ be any fuzzy prime ideal of X.
Let $\vartheta \subseteq \lambda$ or $\lambda \subseteq \vartheta$
Since $\vartheta \subseteq \lambda \implies \vartheta \cap \lambda \subseteq \lambda \implies \vartheta \cap \lambda$ is a prime fuzzy ideals as λ is a fuzzy prime ideal of X and
$\lambda \subseteq \vartheta \implies \vartheta \cap \lambda \subseteq \vartheta \implies \vartheta \cap \lambda$ is a fuzzy prime ideal as ϑ is a fuzzy prime ideal of X. □

Theorem 4.13. *Let ϑ be a fuzzy JU- ideal of a JU -algebra X, then the fuzzy p-multiplicative ϑ^p_Δ of ϑ is a fuzzy JU-ideal of X for every $p \in [0, \Delta]$.*

Proof. Let ϑ be a fuzzy JU- ideal of a JU -algebra X and $\forall p \in [0, \Delta]$. Then $\forall x, y \in X$.
$\vartheta^p_\Delta(1) = (p)(\vartheta(1)) \geq (p)(\vartheta(x)) = \vartheta^p_\Delta(x)$
And, $\vartheta^p_\Delta(x) = (p)(\vartheta(x))$
$\geq (p)(\{\vartheta(y \circ x) \wedge \vartheta(y)\})$
$= \{(p)(\vartheta(y \circ x)) \wedge (P)(\vartheta(y))\}$
$= \{\vartheta^p_\Delta(y \circ x) \wedge \vartheta^p_\Delta(y)\}$ □

Theorem 4.14. *If there exist $p \in [0, \Delta]$ such that the fuzzy p-multiplicative ϑ^p_Δ of ϑ is a fuzzy JU-ideal of a JU- algebra X, then ϑ is a fuzzy JU-ideal of X.*

Proof. Suppose the fuzzy p-multiplicative ϑ^p_Δ of ϑ is a fuzzy JU- ideal of a JU- algebra X and for some $p \in [0, \Delta]$. Then $\forall x, y \in X$.
$(p)(\vartheta(1)) = \vartheta^p_\Delta(1)$
$\geq \vartheta^p_\Delta(x) = (p)(\vartheta(x))$
So $\vartheta(1) \geq \vartheta(x)$
And, $(p)(\vartheta(x)) = \vartheta^p_\Delta(x)$
$\geq \{\vartheta^p_\Delta(y \circ x) \wedge \vartheta^p_\Delta(y)\}$
$= \{(p)(\vartheta(y \circ x)) \wedge (p)(\vartheta(y))\}$
$= (p)(\{\vartheta(y \circ x) \wedge \vartheta(y)\})$
$\Rightarrow \vartheta(x) \geq \{\vartheta(y \circ x) \wedge \vartheta(y)\}$ □

Theorem 4.15. *If ϑ is a prime fuzzy ideal of a JU- algebra X, then the fuzzy p-multiplicative ϑ^Δ_p of ϑ is a prime fuzzy JU-ideal of X for all $p \in [0, \Delta]$.*

Proof. Let ϑ be a prime fuzzy JU- ideal of a JU -algebra X and $p \in [0, \Delta]$.Then
$\vartheta^\Delta_p(x \circ y) = (p)(\vartheta(x \circ y))$
$\leq (p)(\{\vartheta(x) \vee \vartheta(y)\})$
$= \{(p)(\vartheta(x)) \vee (p)(\vartheta(y))\}$
$= \{\vartheta^\Delta_p(x) \vee \vartheta^\Delta_p(y)\}$ □

Theorem 4.16. *If there exist $p \in [0, \Delta]$ such that the fuzzy p- multiplicative ϑ^Δ_p of ϑ is a prime fuzzy JU-ideal of a JU-algebra X, then ϑ is a prime fuzzy JU-ideal of X.*

Proof. Assume that the fuzzy p- multiplicative ϑ^Δ_p of ϑ is a prime fuzzy ideal of a JU-algebra X, for some $p \in [0, \Delta]$ and for any $x, y \in X$. Then
$(p)(\vartheta(x \circ y)) = \vartheta^\Delta_p(x \circ y)$

$\leq \{\vartheta_p^{\Delta}(x) \vee \vartheta_p^{\Delta}(y)\}$
$= \{(p)(\vartheta(x)) \vee (p)(\vartheta(y))\}$
$= (p)(\{\vartheta(x) \vee \vartheta(y)\})$
Therefore, $\vartheta(x \circ y) \leq \{\vartheta(x) \vee \vartheta(y)\}$

□

Corollary 4.17. *The intersection of any two fuzzy* p*- multiplicative of fuzzy JU-subalgebras and fuzzy JU- ideals of JU-algebras are also fuzzy JU-subalgebra and fuzzy JU-ideal respectively.*

Proof. it is straight forward by theorem 3.6 □

Theorem 4.18. *Let f be endomorphism of a JU-algebra X. If* ϑ *is a fuzzy JU-ideal of X,then so is* ϑ_f.

Proof. Let ϑ be a fuzzy JU-ideal of X.
Now,$\vartheta_f(x) = \vartheta(f(x)) \leq \vartheta(f(1)) = \vartheta_f(1), \forall x \in X.$
Therefore, $\vartheta_f(1) \geq \vartheta_f(x)$
Let $x, y \in X$, then $\vartheta_f(y) = \vartheta(f(y))$
$\geq \{\vartheta(f(x)) \wedge \vartheta(f(x) \circ f(y))\}$
$= \{\vartheta(f(x)) \wedge \vartheta(f(x \circ y))$
$= \{\vartheta_f(x) \wedge \vartheta_f(x \circ y)\}$
Therefore, $\vartheta_f(y) \geq \{\vartheta_f(x) \wedge \vartheta_f(x \circ y)\}$
Hence, ϑ_f is a fuzzy JU-ideal of X. □

Theorem 4.19. *Let* $f : X \to Y$ *be an epimorphism of JU- algebra. If* ϑ_f *is fuzzy JU - ideal of X, then* ϑ *is a fuzzy JU - ideal of Y.*

Proof. Let ϑ_f be a fuzzy JU- ideal of X and let $y \in Y$. Then there exist $x \in X$ such that $f(x) = y$
$\vartheta(1) = \vartheta(f(1)) = \vartheta_f(1) \geq \vartheta_f(x) = \vartheta(f(x)) = \vartheta(y)$
Hence $\vartheta(1) \geq \vartheta(y)$.
Let $y_1, y_2 \in Y$, then $\vartheta(y_1) = \vartheta(f(x_1)) = \vartheta_f(x_1)$
$\geq \{\vartheta_f(x_2 \circ x_1) \wedge \vartheta_f(x_2)\}$
$= \{\vartheta(f(x_2 \circ x_1) \wedge \vartheta(f(x_2)\}$
$= \{\vartheta(f(x_2) \circ f(x_1)) \wedge \vartheta(f(x_2)\}$
$= \{\vartheta(y_2 \circ y_1) \wedge \vartheta(y_2\}$
Thus $\vartheta(y_1) \geq \{\vartheta(y_2 \circ y_1) \wedge \vartheta(y_2\}$
Therefore ϑ is a fuzzy JU- ideal of Y. □

Theorem 4.20. *Let* $f : X \to Y$ *be a homomorphism of JU-algebra. If* ϑ *is fuzzy JU - ideal of Y, then* ϑ_f *is a fuzzy JU - ideal of X.*

Proof. Let ϑ be a fuzzy JU- ideal of Y and let $x, y \in X$, then
$\vartheta_f(1) = \vartheta(f(1))$
$\geq \vartheta(f(x)) = \vartheta_f(x)$
Hence $\vartheta_f(1) \geq \vartheta_f(x)$
Also $\vartheta_f(x) = \vartheta(f(x))$
$\geq \{\vartheta(f(y) \circ \vartheta(f(x)) \wedge \vartheta(f(y))\}$
$= \{\vartheta(f(y \circ x)) \wedge \vartheta(f(y))\}$
$= \{\vartheta_f(y \circ x) \wedge \vartheta_f(y)\}$
Therefore, $\vartheta_f(x) \geq \{\vartheta_f(y \circ x) \wedge \vartheta_f(y)\}$
Hence ϑ_f is a fuzzy JU- ideals of X. □

5 Conclusion

In this study, we fuzzify the JU-subalgebras and the JU-ideals and obtained some results. The homomorphic image and pre-image of fuzzy JU-subalgebras and fuzzy JU-ideals were discussed. We also studied prime fuzzy sets and fuzzy p-multiplicative in JU-algebra and their relations. This fuzzification of JU-algebra will help to deal with uncertainness precisely and potential topic to develop its study in the future like a derivative, bipolar form, Q-fuzzy set, e.t.c.

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