Weighted $\mathcal{O}$-operators on Lie Superalgebras and their classifications on Witt Superalgebras

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Abstract *In this paper, we study $\mathcal{O}$-operators of any weight on a Lie superalgebra as a generalization of Rota-Baxter operators with respect to a given representation. These operators can be characterized by graphs and the Nijenhuis operators of the semi-direct product of specified Lie superalgebras. As applications we use them will be used to split a Lie superalgebra operation and get a post-Lie superalgebr a structure. Then, we classify the homogeneous Rota-Baxter operator of weight 1 on the Witt superalgebra. To apply our approach, we construct post-Lie superalgebras from the homogeneous Rota-Baxter operator of weight 1 and their subadjacent Lie superalgebras. Finally, we define the cohomology of an $\mathcal{O}$-operator as the Chevalley-Eilenberg cohomology of a Lie superalgebra with coefficients in a suitable representation.*

1 Introduction

Rota-Baxter operators in associative algebras first came from G. Baxter and play a crucial role in the algebraic side of Connes-Kreimer's work on quantum field theory [2, 21, 30]. In a similar vein, relative Rota-Baxter operators, also known as $\mathcal{O}$-operators, arise in Lie algebras and are connected to the classic Yang-Baxter equations [2, 39, 9]. This different algebraic approach gives rise to the expansion of pre-Lie algebra or post-Lie algebras, which are meaningful in mathematical physics [8, 16]. The concept has attracted the attention of various areas of mathematics, especially due to G.-C. Rota's insight on its importance in combinatorics [43, 44, 45]. Mainly the Rota-Baxter relation generalizes the standard shuffle relation and is intertwined with mathematical physics through its connections to classic Yang-Baxter [33, 43, 44]. The interplay between this relation and mathematical physics is exemplified by the Rota-Baxter operator's role in the classical Yang-Baxter equation within its underlying Lie algebra [1, 25, 26], contributing to the study of integrable systems [10, 47]. Recent works have demonstrated its relevance in the domain of quantum field theory [20, 23]. In simpler terms, a Rota-Baxter operator (with weight λ) on an algebra $\mathfrak{g}$ (which may not be associative) over a field $\mathbb{K}$ is defined by a linear map $R : \mathfrak{g} \rightarrow \mathfrak{g}$ that meets the following integrability condition: $\forall x, y \in \mathfrak{g}, \lambda \in \mathbb{K}$

$$R(x)R(y) = R\big(R(x)y + xR(y) + \lambda xy\big).$$

Rota-Baxter structures have been looked at from various perspectives, including free Rota-Baxter structures [31], dendriform algebras [25], the classical Yang-Baxter equation [1], and quantum field theory [24]. Pre-Lie algebras have also been studied in earlier works [5, 36, 40]. Moreover, Lie superalgebras play a crucial role in theoretical physics, particularly in conformal field theory and supersymmetry [15, 28, 35, 41]. There is growing interest in group gradings of various algebraic structures, which are linked to physics, geometry, and topology, serving as foundations for algebraic modeling [6, 7, 17, 18].

This setup establishes the premise for a study of $\mathcal{O}$-operators as a natural extension of Rota-Baxter operators, whose relevance clearly emerges when viewed in a particular representation. These operators are central to algebraic decompositions within Lie superalgebra action, resulting in a post-Lie superalgebra structure. Utilizing the flexibility in $\mathcal{O}$-operators over varying weights λ, especially weight zero as discussed in [4], a unifying template arises where such operators are critical for understanding the complex algebraic interplay found within Lie superalgebras. Making use of them within a particular representation not only affords us more understanding of embedded structures, it creates new mathematical avenues to explore. Essentially, applying $\mathcal{O}$-operators in this fashion creates a robust means for exploring and determining algebraic properties in Lie superalgebra, advancing our understanding further while paving the path for future investigation.

The authors introduced in [37] a new cohomological approach to relative Rota-Baxter operators. In particular the ones with a weight 1 on Lie algebras, with a description of their linear deformations. This cohomological study, based on the Chevalley-Eilenberg cohomology of a pertinent Lie algebra, with coefficients taken from a pertinent representation, provides useful information. It cannot provide the detail necessary to comprehend the structure of the cohomology ring to a satisfactory level and it never discusses formal deformations, or changes of a finite scope. The cohomology plays a central role in the understanding of the mathematical structures, particularly in the relation of deformations and extensions of the corresponding constructs. The cohomology groups enable the investigation of these concepts in various algebraic structures in this context and allow to investigate the cohomological aspects and deformations. To give some samples of the applications, we refer [19, 42] in the context of the Lie algebras and we cite [13, 46] in the setting of the Lie superalgebras.

Specifically, the study of derivations inside the Lie algebra of the Laurent polynomial ring

$$A_n = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}].$$

requires the consideration of the Witt algebra of vector fields on an n-dimensional torus structure. The Witt algebra is one of the four types of Cartan-Lie algebras, first introduced in 1909 via the study of infinite-dimensional simple Lie algebras by Cartan. Many mathematicians and physicists have studied the representation theory, specifically for the Witt algebras, as seen in references such as [11, 12, 32]. The role of the Witt algebra finds a major position in classical integral and differential calculus as well as creating a bridge to topology and geometry. It also plays a central role to calculate most of its major algebraic properties (for further details, see [34]). The notion of Witt superalgebras was originally introduced in [14] and has evolved significantly since then, as demonstrated in works such as [3, 27, 48]. More recent studies are now exploring Rota-Baxter operators in relation to Witt and Virasoro algebras. Consequently, there's a need for a thorough exploration of Witt superalgebras, which expand on the basic notion of Witt algebras while also looking into Rota-Baxter superalgebras concurrently. This will contribute to a clear definition of homogeneous Rota-Baxter operators within Witt superalgebras, aiming to classify these specialized operators thoroughly [29]. For this reason, this juncture calls for a comprehensive investigation into Witt superalgebra, which generalizes the core concept of Witt algebras. Simultaneously, an investigation of Rota-Baxter superalgebras is of critical importance. This combined effort attempts to formulate a coherent definition for homogeneous Rota-Baxter operators situated within the framework of a Witt superalgebra, thereby paving the way for an elucidating classification. This investigation aims to systemically reveal the inherent structure and properties of these specialized operators.

The present paper extends the framework developed in [4], while maintaining its introductory elements. Simultaneously, we adapt these foundational concepts to the domain of $\mathcal{O}$-operator cohomology. In the current work, we present findings that significantly contribute to the exploration of $\mathcal{O}$-operator cohomology in the context of superalgebras. These advancements play a crucial role in characterizing $\mathcal{O}$-operators, including those the Rota-Baxter operators of weight 1, which we will discuss in the current study, as well as those of weight 0, discussed in [4], all within the context of the Witt superalgebra. As their canonically resulting structure, we build Post-Lie superalgebras, emphasizing the interrelation between theoretical construction and ap-

plicability. These algebras exhibit profound interrelations among ideas and can have repercussions across other areas. In [4], we examine the cohomology of Rota-Baxter operators of weight zero in the setting of Lie superalgebras. This work aims to develop an extended cohomological framework for relative Rota-Baxter operators and their homomorphisms within the context of representations.

This manuscript is organized as follows: Section 2 recaps elementary definitions regarding Lie superalgebras and Rota-Baxter operators and presents the corresponding constructions. In Section 3, we present a detailed classification of homogeneous Rota-Baxter operators with a weight of 1 in the setting of Witt superalgebras. Section 4 describes step by step the method of building post-Lie superalgebras. It constructs this via a careful study of homogeneous Rota-Baxter operators with weight 1, thereby illustrating the interplay between theory and practice in this area. Section 5 discusses cohomology theory for Lie superalgebras and its relation to Rota-Baxter operators via Chevalley-Eilenberg cohomology of certain Lie superalgebras.

Conventions and notations.

(i) *In the paper, $\mathbb{K}$ denotes an algebraically closed field with characteristic 0, and all vector spaces are defined over $\mathbb{K}$.*

(ii) *We refer to the standard one-to-one correspondence between linear maps $F : V_1 \otimes \cdots \otimes V_n \to W$ and multilinear maps $F : V_1 \times \cdots \times V_n \to W$, which is established by the relationship $F(v_1, \ldots, v_n) := F(v_1 \otimes \cdots \otimes v_n)$ whenever the same notation is employed for both types of maps.*

(iii) *A vector space V is said to be $\mathbb{Z}_2$-graded if we are given a family $(V_i)_{i \in \mathbb{Z}_2}$ of vector subspace of V such that $V := V_{\bar{0}} \oplus V_{\bar{1}}$.*

(iv) *The symbol $|x|$ always implies that x is a $\mathbb{Z}_2$-homogeneous element, and $\bar{x}$ represents its $\mathbb{Z}_2$-degree. We denote $\mathcal{H}(V)$ as the set containing all homogeneous elements of V, and $\mathcal{H}(V^n)$ refers to the set of tuples composed of homogeneous elements.*

(v) *Let $End(V)$ be the $\mathbb{Z}_2$-graded vector space of endomorphisms of a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. The graded binary commutator $[f, g] := f \circ g - (-1)^{\bar{f}\bar{g}} g \circ f$ induces the structure of Lie superalgebra in $End(V)$.*

2 Background on Lie superalgebras

In this section, we recall and we give some basic definitions of Lie superalgebras and their representations. We introduce the notion of module Lie superalgebra of a given Lie superalgebra and give a characterization via semi-direct product. A Lie superalgebra is a pair $(\mathfrak{g}, [\cdot, \cdot])$ consisting of a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ and an even bilinear map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the following identities:

$$[x, y] := -(-1)^{\bar{x}\bar{y}}[y, x], \tag{2.1}$$

$$(-1)^{\bar{x}\bar{z}}[x, [y, z]] + (-1)^{\bar{y}\bar{x}}[y, [z, x]] + (-1)^{\bar{z}\bar{y}}[z, [x, y]] = 0, \tag{2.2}$$

for all x, y and z are homogeneous elements in $\mathfrak{g}$. Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{g}', [\cdot, \cdot]')$ two Lie superalgebras. A linear map $f : \mathfrak{g} \to \mathfrak{g}'$ is called a morphism of Lie superalgebras if

$$f([x, y]) = [f(x), f(y)]', \ \forall \, x, y \in \mathfrak{g}. \tag{2.3}$$

Definition 2.1. *A representation of a Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot])$ on a $\mathbb{Z}_2$-graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is an even linear map $\rho : \mathfrak{g} \to gl(V)$, such that for all $x, y \in \mathfrak{g}$, the following identity is satisfied:*

$$\rho([x, y]) := \rho(x)\rho(y) - (-1)^{\bar{x}\bar{y}}\rho(y)\rho(x) \tag{2.4}$$

for all x, y two homogeneous elements in $\mathfrak{g}$. We denote it by (V, ρ).

Example 2.2. (i) Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra and V be a super vector space. Then, consider the map $\rho : \mathfrak{g} \to gl(V)$ defined as $\rho(x)(y) = 0$ for all $y \in V$. This map defines a representation of $\mathfrak{g}$ in V, which is referred to as the trivial representation. Therefore, we also say that (V, ρ) is a trivial $\mathfrak{g}$-module.

(ii) If $V = \mathfrak{g}$ in this case we get the representation ad defined by

$$ad_x(y) := [x, y], \ \forall x, y \in \mathfrak{g}. \tag{2.5}$$

It is called the adjoint representation.

Definition 2.3. Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(V, [\cdot, \cdot]_V)$ be two Lie superalgebras. Let $\rho : \mathfrak{g} \to End(V)$ be a linear map such that (V, ρ) is a representation of $(\mathfrak{g}, [\cdot, \cdot])$ and the following compatibility condition holds for all homogeneous elements $x \in \mathfrak{g}$ and $a, b \in V$,

$$\rho(x)[a, b]_V = [\rho(x)a, b]_V - (-1)^{\bar{x}\bar{a}}[a, \rho(x)b]_V. \tag{2.6}$$

Then $(V, [\cdot, \cdot]_V, \rho)$ is called a $\mathfrak{g}$-module Lie superalgebra.

Example 2.4. It is known that $(\mathfrak{g}, ad)$ is a representation of $\mathfrak{g}$ called the adjoint representation. Then $(\mathfrak{g}, [\cdot, \cdot], ad)$ is an $\mathfrak{g}$-module Lie superalgebra.

Proposition 2.5. Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(V, [\cdot, \cdot]_V)$ be two Lie superalgebras and $\rho : \mathfrak{g} \to End(V)$ be an even linear map. Then $(V, [\cdot, \cdot]_V, \rho)$ be an $\mathfrak{g}$-module Lie superalgebra if and only if $(\mathfrak{g} \oplus V, [\cdot, \cdot]_\rho)$ carries a new Lie superalgebra structure with the bracket $[\cdot, \cdot]_\rho$, given by

$$[x + a, y + b]_\rho := [x, y] + \rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(y)a + [a, b]_V, \ for \ all \ x, y \in \mathfrak{g} \ and \ a, b \in V. \tag{2.7}$$

This is called the semi-direct product and denoted by $\mathfrak{g} \ltimes_\rho V$ or simply $\mathfrak{g} \ltimes V$.

Proof. For $x, y, z \in \mathfrak{g}$ and $a, b, c \in V$, by Equation (2.7), we have

$$
\begin{aligned}
[x + a, [y + b, z + c]_\rho]_\rho &= [x + a, [y, z] + \rho(y)c - (-1)^{\bar{z}\bar{y}}\rho(z)b + [b, c]_V]_\rho \\
&= [x, [y, z]] + \rho(x)(\rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(z)b + [b, c]_V) \\
&\quad - (-1)^{\bar{x}(\bar{y}+\bar{z})}\rho([y, z])a + [a, \rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(z)b + [b, c]_V]_V \\
&= [x, [y, z]] + \rho(x)\rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(x)\rho(z)b + \rho(x)[b, c]_V \\
&\quad - (-1)^{\bar{x}(\bar{z}+\bar{y})}\rho([y, z])a + [a, \rho(y)c]_V - (-1)^{\bar{y}\bar{z}}[a, \rho(z)b]_V + [a, [b, c]_V]_V,
\end{aligned}
$$

$$
\begin{aligned}
[y + b, [z + c, x + a]_\rho]_\rho &= [y + b, [z, x] + \rho(z)a - (-1)^{\bar{x}\bar{z}}\rho(x)c + [c, a]_V]_\rho \\
&= [y, [z, x]] + \rho(y)(\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(x)c + [c, a]_V) \\
&\quad - (-1)^{\bar{y}(\bar{z}+\bar{x})}\rho([z, x])b + [b, \rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(x)c + [c, a]_V] \\
&= [y, [z, x]] + \rho(y)\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(y)\rho(x)c + \rho(y)[c, a]_V \\
&\quad - (-1)^{\bar{y}(\bar{z}+\bar{x})}\rho([z, x])b + [b, \rho(z)a]_V - (-1)^{\bar{z}\bar{x}}[b, \rho(x)c]_V + [b, [c, a]_V]_V
\end{aligned}
$$

and

$$
\begin{aligned}
[z + c, [x + a, y + b]_\rho]_\rho &= [z + c, [x, y] + \rho(x)b - (-1)^{\bar{y}\bar{x}}\rho(y)a + [a, b]_V] \\
&= [z, [x, y]] + \rho(z)(\rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(y)a + [a, b]_V) \\
&\quad - (-1)^{\bar{z}(\bar{x}+\bar{y})}\rho([x, y])c + [c, \rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(y)a + [a, b]_V]_V \\
&= [z, [x, y]] + \rho(z)\rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(z)\rho(y)a + \rho(z)([a, b]_V) \\
&\quad - (-1)^{\bar{z}(\bar{x}+\bar{y})}\rho([x, y])c + [c, \rho(x)b]_V - (-1)^{\bar{x}\bar{y}}[c, \rho(y)a]_V \\
&\quad + [c, [a, b]_V]_V.
\end{aligned}
$$

Since $(\mathfrak{g}, [\cdot, \cdot]_\mathfrak{g})$ and $(V, [\cdot, \cdot]_V)$ be a tow Lie superalgebras. Then we have

$$(-1)^{\bar{x}\bar{z}}[x + a, [y + b, z + c]_\rho]_\rho + (-1)^{\bar{y}\bar{x}}[y + b, [z + c, x + a]_\rho]_\rho + (-1)^{\bar{z}\bar{y}}[z + c, [x + a, y + b]_\rho]_\rho$$

$$= [x, [y, z]_\mathfrak{g}]_\mathfrak{g} + \rho(x)\rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(x)\rho(z)b + \rho(x)[b, c]_V - (-1)^{\bar{x}(\bar{z}+\bar{y})}\rho([y, z]_V a + [a, \rho(y)c$$

$- (-1)^{\bar{y}\bar{z}}\rho(z)b + [b,c]_V]_V + [y,[z,x]_{\mathfrak{g}}]_{\mathfrak{g}} + \rho(y)\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(y)\rho(x)c + \rho(y)[c,a]_V$

$- (-1)^{\bar{y}(\bar{z}+\bar{x})}\rho([z,x]_V)b + [b,\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(x)c + [c,a]_V]_V$

$+ [y,[z,x]_{\mathfrak{g}}]_{\mathfrak{g}} + \rho(y)\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(y)\rho(x)c + \rho(y)[c,a]_V$

$- (-1)^{\bar{y}(\bar{z}+\bar{x})}\rho([z,x]_V)b + [b,\rho(z)a - (-1)^{\bar{z}\bar{x}}\rho(x)c + [c,a]_V]_V$

$= (-1)^{\bar{x}\bar{z}}(\rho(x)\rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(x)\rho(z)b + \rho(x)([b,c]_V)$

$- (-1)^{\bar{x}(\bar{z}+\bar{y})}\rho([y,z]_V)a + [a,\rho(y)c$

$- (-1)^{\bar{y}\bar{z}}\rho(z)b + [b,c]_V]_V) + (-1)^{\bar{x}\bar{y}}(\rho(z)\rho(y)a - (-1)^{\bar{x}\bar{z}}\rho(x)\rho(y)c + \rho(y)([c,a]_V)$

$- (-1)^{\bar{y}(\bar{z}+\bar{x})}\rho([z,x]_V)b + [b,\rho(z)a - (-1)^{\bar{x}\bar{z}}\rho(x)c + [c,a]_V]_V)$

$+ (-1)^{\bar{z}\bar{y}}(\rho(z)\rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(z)\rho(y)a + \rho(z)([a,b]_V)$

$- (-1)^{\bar{z}(\bar{y}+\bar{x})}\rho([x,y]_V)c + [c,\rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(y)a + [a,b]_V]_V)$

$= \rho(x)\rho(y)c - (-1)^{\bar{y}\bar{z}}\rho(x)\rho(z)b + \rho(x)[b,c]_V - (-1)^{\bar{x}(\bar{z}+\bar{y})}\rho([y,z])a$

$+ [a,\rho(y)c]_V - (-1)^{\bar{y}\bar{z}}[a,\rho(z)b]_V + \rho(z)\rho(x)b - (-1)^{\bar{x}\bar{y}}\rho(z)\rho(y)a + \rho(z)([a,b]_V)$

$- (-1)^{\bar{z}(\bar{x}+\bar{y})}\rho([x,y])c + [c,\rho(x)b]_V - (-1)^{\bar{x}\bar{y}}[c,\rho(y)a]_V + \rho(z)\rho(x)b$

$- (-1)^{\bar{x}\bar{y}}\rho(z)\rho(y)a + \rho(z)([a,b]_V) - (-1)^{\bar{z}(\bar{x}+\bar{y})}\rho([x,y])c + [c,\rho(x)b]_V - (-1)^{\bar{x}\bar{y}}[c,\rho(y)a]_V.$

Therefore $(V,[\cdot,\cdot]_V,\rho)$ is an $\mathfrak{g}$-module Lie superalgebra if and only if $(\mathfrak{g} \oplus V, [\cdot,\cdot]_\rho)$ is a Lie superalgebra. ∎

Remark 2.6. More generally, if we define a λ-semi-direct product denoted by $\mathfrak{g} \ltimes_\rho^\lambda V$ as follows, for all $x,y \in \mathfrak{g}$ and $a,b \in V$:

$$[x+a, y+b]_\rho^\lambda = [x,y] + \rho(x)b - (-1)^{\bar{a}\bar{b}}\rho(y)a + \lambda[a,b]_V, \qquad (2.8)$$

we obtain the same characterization as given in the above Proposition.

3 Weighted $\mathcal{O}$-operators on Lie superalgebras

In this section, we look into into the concept of weighted $\mathcal{O}$-operators within the context of Lie superalgebras. We explore their fundamental properties and characterize them using graph-based representations and Nijenhuis operators. As an application of these investigations, we provide a comprehensive classification of homogeneous Rota-Baxter operators with a weight of 1, specifically focusing on the Witt superalgebra.

3.1 Definition and characterizations

Definition 3.1. Let $(\mathfrak{g},[\cdot,\cdot])$ be a Lie superalgebra and $(V,[\cdot,\cdot]_V,\rho)$ be a $\mathbb{K}$-module Lie superalgebra. An even linear map $\mathcal{T} : V \to \mathfrak{g}$ is said to be a λ-weighted $\mathcal{O}$-operator associated to $(V,[\cdot,\cdot]_V,\rho)$, where $\lambda \in \mathbb{K}$, if it's satisfying the following condition

$$[\mathcal{T}a, \mathcal{T}b]_{\mathfrak{g}} := \mathcal{T}(\rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a,b]_V) \qquad (3.1)$$

for all $a,b \in V$.

Example 3.2.

 (i) The identity map $Id : \mathfrak{g} \to \mathfrak{g}$ is a (-1)-weighted $\mathcal{O}$-operator.

 (ii) If $\mathcal{T}$ is a λ-weighted $\mathcal{O}$-operator, then $\lambda^{-1}\mathcal{T}$ is a 1-weighted $\mathcal{O}$-operator.

 (iii) If $\mathcal{T}$ is a λ-weighted $\mathcal{O}$-operator, then $-\lambda Id - \mathcal{T}$ is a λ-weighted $\mathcal{O}$-operator.

Recall that, a Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on a Lie superalgebra $\mathfrak{g}$ over $\mathbb{K}$ is an even linear map $R : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[R(x), R(y)] = R([R(x),y] + [x,R(y)] + \lambda[x,y]), \ \forall x,y \in \mathfrak{g}. \qquad (3.2)$$

Remark 3.3. A Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on a Lie superalgebra $\mathfrak{g}$ is exactly a λ-weighted $\mathcal{O}$-operator associated to the adjoint representation $(\mathfrak{g}, [\cdot, \cdot], \mathrm{ad})$.

In the following, we characterize λ-weighted $\mathcal{O}$-operators in terms of graphs and Nijenhuis operators.

Proposition 3.4. Let $(V, [\cdot, \cdot]_V, \rho)$ be a $\mathbb{K}$-module Lie superalgebra. Then an even linear map $\mathcal{T} : V \to \mathfrak{g}$ is a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot, \cdot]_\rho, \rho)$ if and only if the graph $Gr(\mathcal{T}) = \{\mathcal{T}(a) + a | a \in V\}$ of the map $\mathcal{T}$ is a superalgebra of the λ-semi-direct product $g \ltimes_\rho^\lambda V$.

Proof. Let $\mathcal{T} : V \to \mathfrak{g}$ be an even linear map. For all $a, b \in V$, we have:

$$[\mathcal{T}a + a, \mathcal{T}b + b]_\rho^\lambda = [\mathcal{T}a, \mathcal{T}b] + \rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a, b]_V,$$

which implies that the graph $Gr(\mathcal{T}) = \{\mathcal{T}(a) + a | a \in V\}$ is a subalgebra of the Lie superalgebra $g \ltimes_\rho^\lambda V$ if and only if $\mathcal{T}$ satisfies

$$[\mathcal{T}a, \mathcal{T}b] = \mathcal{T}(\rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a, b]_V),$$

which means that $\mathcal{T}$ is a λ-weighted $\mathcal{O}$-operator. ∎

In the sequel, we characterize λ-weighted $\mathcal{O}$-operators associated to $(V, [\cdot, \cdot]_V, \rho)$ in terms of the Nijenhuis operators. Recall that a Nijenhuis operator on a Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot])$ is an even linear map $N : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$[Nx, Ny] = N([Nx, y] + [x, Ny] - N[x, y]) \ \forall \, x, y \in \mathfrak{g}. \tag{3.3}$$

Lemma 3.5. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra and N be a Nijenhuis operator. Then $(\mathfrak{g}, [\cdot, \cdot]_N)$ is a Lie superalgebra, where the bracket $[\cdot, \cdot]_N$, is given by

$$[x, y]_N = [Nx, y] + [x, Ny] - N[x, y] \ \forall \, x, y \in \mathfrak{g}. \tag{3.4}$$

Proof. We begin by observing that the bracket $[\cdot, \cdot]_N$ is super-skewsymmetric. Let $x, y, z \in \mathfrak{g}$, by Eqs. (3.3) and (3.4), we have $N[x, y]_N = [Nx, Ny]$. Then we get

$(-1)^{\bar{x}\bar{z}}[x, [y, z]_N]_N + (-1)^{\bar{x}\bar{y}}[y, [z, x]_N]_N + (-1)^{\bar{z}\bar{y}}[z, [x, y]_N]_N$

$= (-1)^{\bar{x}\bar{z}}([Nx, [y, z]_N] + [x, N[y, z]_N] - N[x, [y, z]_N]) + (-1)^{\bar{x}\bar{y}}([Ny, [z, x]_N]$

$+ [y, N[z, x]_N] - N[y, [z, x]_N]) + (-1)^{\bar{z}\bar{y}}([Nz, [x, y]_N] + [z, N[x, y]_N] - N[z, [x, y]_N])$

$= (-1)^{\bar{x}\bar{z}}([Nx, [Ny, z] + [y, Nz] - N[y, z]] + [x, [Ny, Nz]] - N[x, [Ny, z] + [y, Nz] - N[y, z]])$

$+ (-1)^{\bar{x}\bar{y}}([Ny, [Nz, x] + [z, Nx] - N[z, x]] + [y, [Nz, Nx]] - N[y, [Nz, x] + [z, Nx] - N[z, x]])$

$+ (-1)^{\bar{z}\bar{y}}([Nz, [Nx, y] + [x, Ny] - N[x, y]] + [z, [Nx, Ny]] - N[z, [Nx, y] + [x, Ny] - N[x, y]])$

$= (-1)^{\bar{x}\bar{z}}([Nx, [Ny, z] + [Nx, [y, Nz]] - [Nx, N[y, z]] + [x, [Ny, Nz]] - N[x, [Ny, z]]$

$- N[x, [y, Nz]] + N[x, N[y, z]]) + (-1)^{\bar{x}\bar{y}}([Ny, [Nz, x]] + [Ny, [z, Nx] - [Ny, N[z, x]]$

$+ [y, [Nz, Nx]] - N[y, [Nz, x]] - N[y, [z, Nx]] + N[y, N[z, x]]) + (-1)^{\bar{z}\bar{y}}([Nz, [Nx, y]]$

$+ [Nz, [x, Ny] - [Nz, N[x, y]] + [z, [Nx, Ny]] - N[z, [Nx, y]] - N[z, [x, Ny]] + N[z, N[x, y]])$

$= 0.$

Then $(\mathfrak{g}, [\cdot, \cdot]_N)$ is a Lie superalgebra. ∎

Remark 3.6. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra, and let $N : \mathfrak{g} \to \mathfrak{g}$ be a linear map. If $N^2 = N$, then N is a Nijenhuis operator if and only if N is a Rota-Baxter operator with the weight $\lambda = -1$.

Proposition 3.7. Let $(V, [\cdot, \cdot]_V, \rho)$ be a $\mathfrak{g}$-module Lie superalgebra. Then a linear map $\mathcal{T} : V \to \mathfrak{g}$ is a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot, \cdot]_V, \rho)$ if and only if

$$N_{\mathcal{T}} := \begin{bmatrix} \lambda id & -\mathcal{T} \\ 0 & 0 \end{bmatrix} : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$$

is a Nijenhuis operator on the semi-direct product Lie superalgebra $\mathfrak{g} \ltimes V$.

Proof. For all $x, y \in \mathfrak{g}$, $a, b \in V$, on the one hand, we have

$$[N_T(x+a), N_T(y+b)]_\rho = [\lambda x - T(a), \lambda y - T(b)]_\rho$$
$$= \lambda^2[x,y] - \lambda[x, T(b)] - \lambda[T(a), y] + [T(a), T(b)].$$

On the other hand, since $N_T^2 = N_T$, we have:

$$N_T\big([N_T(x+a), y+b]_\rho - [N_T(y+b), x+a]_\rho - N_T([x+a, y+b]_\rho)\big)$$
$$= N_T\big([\lambda x - T(a), y+b]_\rho - [\lambda y - T(b), x+a]_\rho - N_T([x,y] + \rho(x)b - \rho(y)a + [a,b]_V)\big)$$
$$= \lambda^2[x,y] - \lambda[x, T(b)] - \lambda[T(a), y] + T\big(\rho(T(a))b - \rho(T(b))a + \lambda[a,b]_V\big).$$

Therefore, N_T is a Nijenhuis operator on the semi-direct product Lie superalgebra $\mathfrak{g} \ltimes V$ if and only if Eq.(3.1) is satisfied. $\blacksquare$

Corollary 3.8. Let $\mathcal{T} : V \to \mathfrak{g}$ be a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot, \cdot]_V, \rho)$ if and only if the operator

$$N_{\mathcal{T}} = \begin{bmatrix} Id & \mathcal{T} \\ 0 & 0 \end{bmatrix} : \mathfrak{g} \oplus V \to \mathfrak{g} \oplus V$$

is a Nijenhuis operator on the λ-semi-direct product Lie superalgebra $(\mathfrak{g} \ltimes_\rho^\lambda V, [\cdot, \cdot]_\rho^\lambda)$.

Proposition 3.9. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra, $(V, [\cdot, \cdot]_V, \rho)$ be a $\mathfrak{g}$-module Lie superalgebra and $\mathcal{T} : V \to \mathfrak{g}$ be a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot, \cdot]_V, \rho)$. Then $(V, [\cdot, \cdot]_{\mathcal{T}})$ is a Lie superalgebra, where the bracket $[\cdot, \cdot]_{\mathcal{T}}$ is given by

$$[a,b]_{\mathcal{T}} := \rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a,b]_V, \quad \forall a, b \in V. \tag{3.5}$$

Proof. For $a, b, c \in V$, we have

$$(-1)^{\bar{c}\bar{a}}[a, [b,c]_{\mathcal{T}}]_{\mathcal{T}} = (-1)^{\bar{c}\bar{a}}\big(\rho(\mathcal{T}a)(\rho(\mathcal{T}b)c - (-1)^{\bar{c}\bar{b}}\rho(\mathcal{T}c)b + \lambda[b,c]_V)$$
$$- (-1)^{\bar{a}(\bar{b}+\bar{c})}\rho\mathcal{T}(\rho(\mathcal{T}b)c - (-1)^{\bar{c}\bar{b}}\rho(\mathcal{T}c)b + \lambda[b,c]_V)a$$
$$+ \lambda[a, \rho(\mathcal{T}b)c - (-1)^{\bar{b}\bar{c}}\rho(\mathcal{T}c)b + \lambda[b,c]_V]_V)$$
$$= (-1)^{\bar{c}\bar{a}}\rho(\mathcal{T}a)\rho(\mathcal{T}b)c - (-1)^{(\bar{b}+\bar{a})\bar{c}}\rho(\mathcal{T}a)\rho(\mathcal{T}c)b$$
$$+ (-1)^{\bar{c}\bar{a}}\lambda\rho(\mathcal{T}a)[b,c]_V - (-1)^{\bar{a}\bar{b}}\rho\mathcal{T}(\rho(\mathcal{T}b)c - (-1)^{\bar{c}\bar{b}}\rho(\mathcal{T}c)b$$
$$+ \lambda[b,c]_V)a + (-1)^{\bar{c}\bar{a}}\lambda[a, \rho(\mathcal{T}b)c]_V - (-1)^{(\bar{b}+\bar{a})\bar{c}}\lambda[a, \rho(\mathcal{T}c)b]_V$$
$$+ \lambda^2(-1)^{\bar{c}\bar{a}}[a, [b,c]_V]_V,$$

$$(-1)^{\bar{a}\bar{b}}[b, [c,a]_{\mathcal{T}}]_{\mathcal{T}} = (-1)^{\bar{a}\bar{b}}\big(\rho(\mathcal{T}b)(\rho(\mathcal{T}c)a - (-1)^{\bar{a}\bar{c}}\rho(\mathcal{T}a)c + \lambda[c,a]_V)$$
$$- (-1)^{\bar{b}(\bar{c}+\bar{a})}\rho\mathcal{T}(\rho(\mathcal{T}c)a - (-1)^{\bar{a}\bar{c}}\rho(\mathcal{T}a)c + \lambda[c,a]_V)b$$
$$+ \lambda[b, \rho(\mathcal{T}c)a - (-1)^{\bar{c}\bar{a}}\rho(\mathcal{T}a)c + \lambda[c,a]_V]_V)$$
$$= (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)\rho(\mathcal{T}c)a - (-1)^{(\bar{c}+\bar{b})\bar{a}}\rho(\mathcal{T}b)\rho(\mathcal{T}a)c$$
$$+ (-1)^{\bar{a}\bar{b}}\lambda\rho(\mathcal{T}b)[c,a]_V - (-1)^{\bar{b}\bar{c}}\rho\mathcal{T}(\rho(\mathcal{T}c)a - (-1)^{\bar{a}\bar{c}}\rho(\mathcal{T}a)c$$
$$+ \lambda[c,a]_V)b + (-1)^{\bar{a}\bar{b}}\lambda[b, \rho(\mathcal{T}c)a]_V$$
$$- (-1)^{(\bar{c}+\bar{b})\bar{a}}\lambda[b, \rho(\mathcal{T}a)c]_V + \lambda^2(-1)^{\bar{a}\bar{b}}[b, [c,a]_V]_V$$

and

$$(-1)^{\bar{b}\bar{c}}[c, [a,b]_{\mathcal{T}}]_{\mathcal{T}} = (-1)^{\bar{b}\bar{c}}\big(\rho(\mathcal{T}c)(\rho(\mathcal{T}a)b - (-1)^{\bar{b}\bar{a}}\rho(\mathcal{T}b)a + \lambda[a,b]_V)$$
$$- (-1)^{\bar{c}(\bar{a}+\bar{b})}\rho\mathcal{T}(\rho(\mathcal{T}a)b - (-1)^{\bar{b}\bar{a}}\rho(\mathcal{T}b)a + \lambda[a,b]_V)c$$

$$+ \lambda[c, \rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a,b]_V]_V)$$

$$= (-1)^{\bar{b}\bar{c}}\rho(\mathcal{T}c)\rho(\mathcal{T}a)b - (-1)^{(\bar{a}+\bar{c})\bar{b}}\rho(\mathcal{T}c)\rho(\mathcal{T}b)a$$

$$+ (-1)^{\bar{b}\bar{c}}\lambda\rho(\mathcal{T}c)[a,b]_V - (-1)^{\bar{c}\bar{a}}\rho\mathcal{T}(\rho(\mathcal{T}a)b$$

$$- (-1)^{\bar{b}\bar{a}}\rho(\mathcal{T}b)a + \lambda[a,b]_V)c + (-1)^{\bar{b}\bar{c}}\lambda[c, \rho(\mathcal{T}a)b]_V$$

$$- (-1)^{(\bar{a}+\bar{c})\bar{b}}\lambda[c, \rho(\mathcal{T}b)a]_V + \lambda^2(-1)^{\bar{b}\bar{c}}[c,[a,b]_V]_V.$$

Then based on the three results above we obtain

$$(-1)^{\bar{a}\bar{c}}[a,[b,c]_{\mathcal{T}}]_{\mathcal{T}} + (-1)^{\bar{a}\bar{b}}[b,[c,a]_{\mathcal{T}}]_{\mathcal{T}} + (-1)^{\bar{c}\bar{b}}[c,[a,b]_{\mathcal{T}}]_{\mathcal{T}} = 0.$$

This confirms that $(V, [\cdot,\cdot]_{\mathcal{T}})$ be a Lie superalgebra. ∎

Corollary 3.10. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie superalgebra and R be Rota-Baxter operator of weight λ. Then $(\mathfrak{g}, [\cdot,\cdot]_R)$ is a Lie superalgebra, where the bracket $[\cdot,\cdot]_R$ is given by

$$[x,y]_R := [Rx,y] + [x,Ry] + \lambda[x,y], \tag{3.6}$$

where x,y in $\mathfrak{g}$.

3.2   Homogeneous Rota-Baxter operators of weight 1 on the Witt superalgebra

The Witt superalgebra $\mathcal{W}$ is a Lie superalgebra written as $\mathcal{W} := \mathcal{W}_0 \oplus \mathcal{W}_1$, where $\mathcal{W}_0$ and $\mathcal{W}_1$ are the Lie algebras with the basis respectively $\{L_m | m \in \mathbb{Z}\}$ of parity 0 and $\{G_m | m \in \mathbb{Z}\}$ of parity 1, verifying the following relations:

$$[L_m, L_n] = (m-n)L_{m+n}, \tag{3.7}$$

$$[L_m, G_n] = (m-n-1)G_{m+n}, \tag{3.8}$$

$$\text{and } [G_m, G_n] = 0, \forall m,n \in \mathbb{Z}. \tag{3.9}$$

Definition 3.11. [4] *A homogeneous operator $\mathcal{F}$ with degree k on the Witt superalgebra $\mathcal{W}$ is a linear operator on $\mathcal{W}$ satisfying*

$$\mathcal{F}(\mathcal{W}_{i,m}) \subset \mathcal{W}_{i,m+k},$$

where $m,k \in \mathbb{Z}$ and $i \in \{0,1\}$. Hence, a homogeneous Rota-Baxter operator R_k with degree k on the Witt superalgebra $\mathcal{W}$ is a Rota-Baxter operator on $\mathcal{W}$ of the following form:

$$R_k(L_m) := f(m+k)L_{m+k}, \tag{3.10}$$

$$R_k(G_m) := g(m+k)G_{m+k}, \forall m \in \mathbb{Z}. \tag{3.11}$$

In contrast to the case of the Rota-Baxter operator with weight 0 [?, 29], it follows directly from its definition, that no homogeneous Rota-Baxter operator of weight 1 with a nonzero degree k exists on the Witt algebra $\mathcal{W}$[29].Furthermore, this result holds true even when $\mathcal{W}$ is a Witt superalgebra.

Let R_0 be a homogeneous Rota-Baxter operator of weight 1 with degree 0 on the Witt superalgebra $\mathcal{W}$, satisfying Eqs. (3.10) and (3.11), that is

$$\begin{cases} R_0(L_m) = f(m)L_m, \\ R_0(G_m) = g(m)G_m, \quad \forall m \in \mathbb{Z}. \end{cases}$$

Then by Eqs. (3.2), (3.7), (3.8), (3.10) and (3.11), we show that the function f, g satisfies the following equations:

$$f(m)f(n)(m-n) = f(m+n)(f(m)+f(n)+1)(m-n), \ \forall m,n \in \mathbb{Z}, \tag{3.12}$$

and

$$g(m+n)(f(m)+g(n)+1)(m-n-1) = f(m)g(n)(m-n-1), \ \forall m,n \in \mathbb{Z}. \tag{3.13}$$

Let $n = 0,\ m = 0$ respectively in Eqs.([3.12](#)) and ([3.13](#)), then we have:

$$\begin{cases} mf(m)(f(m)+1) = 0, & \forall m \in \mathbb{Z}, \\ (n+1)g(n)(g(n)+1) = 0, & \forall n \in \mathbb{Z}. \end{cases} \tag{3.14}$$

As in [29], we set:

$$\mathcal{I}_1 := \{m \in \mathbb{Z} | f(m) = 0\}, \quad \mathcal{I}_2 := \{m \in \mathbb{Z} | f(m) = -1\}.$$

and we define

$$\mathcal{J}_1 := \{m \in \mathbb{Z} | g(m) = 0\}, \quad \mathcal{J}_2 := \{m \in \mathbb{Z} | g(m) = -1\}.$$

Lemma 3.12. If $\mathcal{J}_1$ and $\mathcal{J}_2$ as in the above result, then:

(1) $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$.

(2) $\mathcal{J}_1 \cup \mathcal{J}_2 \cup \{-1\} = \mathbb{Z}$.

Proof. Let $\mathcal{J}_1$ and $\mathcal{J}_2$ as in the result above.

(i) Assume that $\mathcal{J}_1 \cap \mathcal{J}_2 \neq \emptyset$. Then, there exists $m_0 \in \mathcal{J}_1 \cap \mathcal{J}_2$, which implies that $g(m_0) = 0$ and $g(m_0) = -1$ (contradictory).

(ii) Let $m \in \mathbb{Z} \backslash \{-1\}$. Then, according to the system in Eq. ([3.14](#)), we have $g(m) = 0$ or $g(m) = -1$, leading to $m \in \mathcal{J}_1$ or $m \in \mathcal{J}_2$, respectively. This shows that $\mathbb{Z} \backslash \{-1\} \subset \mathcal{J}_1 \cup \mathcal{J}_2$, and consequently, $\mathbb{Z} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \{-1\}$.

∎

In the sequel, in this section, we denote by δ as the Kronecker delta. For all $a < b$ where a and b are real numbers, we define $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$, and

$$1_{[\![a,b]\!]}(m) := \begin{cases} 1, & m \in [\![a, b]\!], \\ 0, & Otherwise. \end{cases}$$

Let $k \in \mathbb{Z}$ and $\mathcal{I}$ any subset of $\mathbb{Z}$, we define the set

$$\mathcal{I} + k := \{m + k, m \in \mathcal{I}\}.$$

Proposition 3.13. [29]. Using the above notations, if there exists a nonzero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_1$, then $\mathcal{I}_1, \mathcal{I}_2$ are one of the following cases:

(i) $\mathcal{I}_1 =]\!] -\infty, 1]\!], \ \mathcal{I}_2 = [\![2, +\infty[\![$.

(ii) $\mathcal{I}_1 = [\![-1, +\infty[\![, \ \mathcal{I}_2 =]\!] -\infty, -2]\!]$.

(iii) $\mathcal{I}_1 = \mathbb{Z}, \ \mathcal{I}_2 = \emptyset$.

Proposition 3.14. [29]. With the notations as above, if there exists a nonzero integer m_0 such that $m_0, -m_0 \in I_2$, then I_1, I_2 are one of the following cases:

(i) $\mathcal{I}_1 = [\![2, +\infty[\![, \ \mathcal{I}_2 =]\!] -\infty, 1]\!]$.

(ii) $\mathcal{I}_1 =]\!] -\infty, -2]\!], \ \mathcal{I}_2 = [\![-1, +\infty[\![$.

(iii) $\mathcal{I}_1 = \emptyset, \ \mathcal{I}_2 = \mathbb{Z}$.

Lemma 3.15. Let f and g be $\mathbb{C}$-valued functions defined on $\mathbb{Z}$, satisfying Eqs. ([3.12](#)) and ([3.13](#)). If m and n are integers from $\mathbb{Z}$ such that $m \neq n + 1$ and $m \in \mathcal{I}_j$, for $j \in \{1, 2\}$, then the following holds: If $n \in \mathcal{J}_j$ implies that $m + n \in \mathcal{J}_j$.

Proof. For $m \in \mathcal{I}_1$, if n is $\in \mathcal{J}_1$, then according to Eq. ([3.13](#)), we have $g(m + n) = 0$, which implies $m + n \in \mathcal{J}_1$. The same applies to $m \in \mathcal{I}_2$; for $n \in \mathcal{J}_2$, we have $m + n \in \mathcal{J}_2$. ∎

Lemma 3.16. Let $i \in \{1, 2\}$. We Assume that there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_i$. If $1 \in \mathcal{J}_i$ then $-1 \in \mathcal{J}_i$.

Proof.   (i) If $i = 1$ and we take $n = 1 \in \mathcal{J}_1$, then by Lemma 3.12 for all $m \in \mathcal{I}_1 \backslash \{2\}$ we obtain $m + 1 \in \mathcal{J}_1$. This implies that

$$\{1\} \cup (\mathcal{I}_1 \backslash \{2\} + 1) \subset \mathcal{J}_1. \tag{3.15}$$

Now from Proposition 3.13 we have that $-1 \in \mathcal{I}_1$, then by the relation (3.15), we get that $0 \in \mathcal{J}_1$. Finally, if we take $m = -1$ and $n = 0$ in Eq (3.13) and by using again Proposition 3.13, we infer that $g(-1) = 0$.

  (ii) If $i = 2$ and we take $n = 1 \in \mathcal{J}_2$, then, according to the Lemma 3.12, we get :

$$\{1\} \cup (\mathcal{I}_2 \backslash \{2\} + 1) \subset \mathcal{J}_2. \tag{3.16}$$

Likewise from Proposition 3.14, we have that $-1 \in \mathcal{I}_2$. Applying the relation (3.16), we get that $0 \in \mathcal{J}_1$. Now, if we take $m = -1$ and $n = 0$ in Eq (3.13) and thanks to Proposition 3.14, we infer that $g(-1) = -1$.   ∎

Proposition 3.17. With the notations as above, if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_1$ and $1 \in \mathcal{J}_1$, then $\mathcal{J}_1 = \mathbb{Z}$ and $\mathcal{J}_2 = \emptyset$.

Proof. To simplify, we devise the proof on four setup.
First step: Combine Proposition 3.13, Lemma 3.16 and the inclusion (3.15), we get

$$\{-1, 0, 1\} \subset \mathcal{J}_1.$$

Second step: In this step we will show that $\{-2, 2, 3\} \subset \mathcal{J}_1$. i) If we take $m = n = 1$ in Eq (3.13) and we use Proposition 3.13, we get $g(2) = 0$.
ii) Replace m by 1 and n by 2 in Eq (3.13). By using i), we have $g(3) = 0$.
iii) Take $m = n = -1$ in Eq (3.13). Applying Proposition 3.13 and Lemma 3.16, we infer that $g(-2) = 0$.
Third step: We combine the inclusion 3.15, Proposition3.13 and second step we conclude that $\mathcal{I}_1 \subset \mathcal{J}_1$.
Fourth step: It remains to proof that $\mathcal{I}_2 \subset \mathcal{J}_1$. First by i) and ii) of second step, we have that $\{-2.2\} \subset \mathcal{J}_1$. We assume that if there is $m_1 \in \mathcal{I}_2$, such for all $m^{'} \in \mathcal{I}_2$ and $|m^{'}| \leq |m_1|$, we have $-|m^{'}|$ and $|m^{'}|$ belong to $\mathcal{J}_1$. Now we take $m = 1$ and $n = |m_1|$ (respectively, $m = -1$ and $n = -|m_1|$) in Eq (3.13) we deduce that $g(|m_1| + 1) = 0$ (respectively, $g(-|m_1| - 1) = 0$).   ∎

Proposition 3.18. Under the above notations, if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_1$ and $1 \in \mathcal{J}_2$, then $\mathcal{J}_1, \mathcal{J}_2$ are one of the following cases:

  (i) $\mathcal{J}_1 = \mathcal{I}_1 \backslash \{1\}, \quad \mathcal{J}_2 = \mathcal{J}_2 = \mathcal{I}_2 \cup \{1\}$.

  (ii) $\mathcal{J}_1 = \emptyset, \quad \mathcal{J}_2 = \mathbb{Z}$.

Proof.   (i) We assume that $0 \in \mathcal{J}_1$. If $m \in \mathcal{I}_1 \backslash \{1\}$, as $0 \in \mathcal{J}_1$ then by Lemma 3.15, we obtain $m \in \mathcal{J}_1$. Then $\mathcal{I}_1 \backslash \{1\} \subset \mathcal{J}_1$. In the author case, if $m \in \mathcal{I}_2 \backslash \{2\}$ and by Lemma 3.15, we have $m + 1 \in \mathcal{J}_2$. The same, we take $m = -1$ and $n = 2$ in Eq.(3.13), we obtain $2 \subset \mathcal{J}_2$, then $\mathcal{I}_2 \cup \{1\} \subset \mathcal{J}_2$. where $\mathcal{I}_2$ as in the Proposition 3.13.

  (ii) We assume that $0 \in \mathcal{J}_2$. If $m \in \mathcal{I}_2$, then by Lemma 3.15 we have $m \in \mathcal{J}_2$. What gives $\mathcal{I}_2 \subset \mathcal{J}_2$. Now, we will show that $\mathcal{I}_1 \subset \mathcal{J}_2$. First by hypothesis $\{1, 0\} \subset \mathcal{J}_2$, then by Proposition 3.13 if we take $m = 1$ and $n = -1$, we get $g(-1) = -1$. We suppose that if there is $m_1 \in \mathcal{I}_1$, such for all $m^{'} \in \mathcal{I}_1$ and $|m^{'}| \leq |m_1|$, we have $-|m^{'}|$ and $|m^{'}|$ belong to $\mathcal{J}_2$. Now we take $m = -1$ and $n = |m_1| + 1$ (respectively, $m = 1$ and $n = -|m_1| - 1$) in Eq (3.13), we deduce that $g(|m_1| + 1) = -1$ (respectively, $g(-|m_1| - 1) = -1$).   ∎

Corollary 3.19. Using the above notations if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_1$ and $1 \in \mathcal{J}_2$, then $\mathcal{J}_1, \mathcal{J}_2$ are one of the following cases:

  (i) $\mathcal{J}_1 =]-\infty, 0], \; \mathcal{J}_2 = [1, +\infty[$.

(ii) $\mathcal{J}_1 = \{-1, 0\} \cup [\![2, +\infty[\![, \ \mathcal{J}_2 =]\!]-\infty, -2]\!] \cup \{1\}$.

(iii) $\mathcal{J}_1 = \mathbb{Z}\backslash\{1\}, \ \mathcal{J}_2 = \{1\}$.

(iv) $\mathcal{J}_1 = \emptyset, \ \mathcal{J}_2 = \mathbb{Z}$.

Proposition 3.20. Using the above notations, if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_2$ and $1 \in \mathcal{J}_2$, then $\mathcal{J}_1 = \emptyset, \ \mathcal{J}_2 = \mathbb{Z}$.

Proof. To simplify, we devise the proof on four setup.
First step: Combine Proposition 3.14, Lemma 3.16 and the inclusion (3.16), we get

$$\{-1, 0, 1\} \subset \mathcal{J}_2.$$

Second step: In this step we will show that $\{-2, 2, 3\} \subset \mathcal{J}_1$. i) If we take $m = n = 1$ in Eq (3.13) and we use Proposition 3.14, we get $g(2) = -1$.
ii) Replace m by 1 and n by 2 in Eq (3.13). By using i), we have $g(3) = -1$.
iii) Take $m = n = -1$ in Eq (3.13). Thanks to Proposition 3.14 and Lemma 3.16, we infer that $g(-2) = -1$.
Third step: We combine the inclusion 3.16, Proposition3.14 and second step we conclude that $\mathcal{I}_2 \subset \mathcal{J}_2$.
Fourth step: Now, all that remains remains is to proof that $\mathcal{I}_1 \subset \mathcal{J}_2$. First i) and ii) of second step give that $\{-2.2\} \subset \mathcal{J}_2$. We assume that if there is $m_1 \in \mathcal{I}_1$, such for all $m' \in \mathcal{I}_1$ and $|m'| \leq |m_1|$, we have $-|m'|$ and $|m'|$ belong to $\mathcal{J}_2$. Next we take $m = 1$ and $n = |m_1|$ (respectively, $m = -1$ and $n = -|m_1|$) in Eq (3.13) we deduce that $g(|m_1|+1) = -1$ (respectively, $g(-|m_1|-1) = -1$). ∎

Proposition 3.21. Using the above notations, if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_2$ and $1 \in \mathcal{J}_1$, then $\mathcal{J}_1, \mathcal{J}_2$ are one of the following cases:

(i) $\mathcal{J}_1 = \mathcal{I}_1 \cup \{1\}, \quad \mathcal{J}_2 = \mathcal{I}_2\backslash\{1\}$.

(ii) $\mathcal{J}_1 = \mathbb{Z}, \quad \mathcal{J}_2 = \emptyset$.

Proof. (i) We suppose that $0 \in \mathcal{J}_2$. If $m \in \mathcal{I}_2\backslash\{1\}$, since $0 \in \mathcal{J}_2$ then by Lemma 3.15, we infer $m \in \mathcal{J}_2$. Hence $\mathcal{I}_2\backslash\{1\} \subset \mathcal{J}_2$. Alternatively, if $m \in \mathcal{I}_1\backslash\{1\}$ and by Lemma 3.15, we get $m + 1 \in \mathcal{J}_1$. The same we take $m = -1$ and $n = 2$ in Eq.(3.13), we obtain $2 \subset \mathcal{J}_1$, then $\mathcal{I}_1 \cup \{1\} \subset \mathcal{J}_1$. where $\mathcal{I}_1$ as in the Proposition 3.14.

(ii) We assume that $0 \in \mathcal{J}_1$. If $m \in \mathcal{I}_1$, then by Lemma 3.15 we have $m \in \mathcal{J}_1$. What gives $\mathcal{I}_1 \subset \mathcal{J}_1$. Next, we will show that $\mathcal{I}_2 \subset \mathcal{J}_1$. First by hypothesis $\{1, 0\} \subset \mathcal{J}_1$, then by Proposition 3.14 if we take $m = 1$ and $n = -1$, we get $g(-1) = 0$. We suppose that if there is $m_1 \in \mathcal{I}_2$, such for all $m' \in \mathcal{I}_2$ and $|m'| \leq |m_1|$, we have $-|m'|$ and $|m'|$ belong to $\mathcal{J}_1$. Now, In view of Proposition 3.14 we take $m = -1$ and $n = |m_1| + 1$ (respectively, $m = 1$ and $n = -|m_1| - 1$) in Eq (3.13), we deduce that $g(|m_1| + 1) = -0$ (respectively, $g(-|m_1| - 1) = 0$. ∎

Corollary 3.22. Using the above notations, if there exists a non-zero integer m_0 such that $m_0, -m_0 \in \mathcal{I}_2$ and $1 \in \mathcal{J}_1$, then $\mathcal{J}_1, \mathcal{J}_2$ are one of the following cases:

(i) $\mathcal{J}_1 = [\![1, +\infty[\![, \ \mathcal{J}_2 =]\!]-\infty, 0]\!]$.

(ii) $\mathcal{J}_1 =]\!]-\infty, -2]\!] \cup \{1\}, \ \mathcal{J}_2 = \{-1, 0\} \cup [\![2, +\infty[\![$.

(iii) $\mathcal{J}_1 = \{1\}, \ \mathcal{J}_2 = \mathbb{Z}\backslash\{1\}$.

(iv) $\mathcal{J}_1 = \mathbb{Z}, \ \mathcal{J}_2 = \emptyset$.

Proposition 3.23. [29]. ;Using the above notations, if there does not exist a nonzero integer m such that $m, -m \in \mathcal{I}_i$, $i = 1, 2$, then either

$$\mathbb{Z}_+ \subset \mathcal{I}_1, \ \mathbb{Z}_- \subset \mathcal{I}_2,$$

$$\mathbb{Z}_- \subset \mathcal{I}_1, \ \mathbb{Z}_+ \subset \mathcal{I}_2,$$

where $\mathbb{Z}_+$ (resp. $\mathbb{Z}_-$) denotes the set of positive (resp. negative) integers. Moreover, $f(0) \in \mathbb{C}$ is arbitrary.

$$[2]\begin{cases} \begin{cases} R_0^{\geq -1,\,\mathbb{Z}}(L_m) = -L_m 1_{\rrbracket -\infty,\,-2\rrbracket}(m). \\ R_0^{\geq -1,\,\mathbb{Z}}(G_m) = 0,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\geq -1,\,\emptyset}(L_m) = -L_m 1_{\rrbracket -\infty,\,-2\rrbracket}(m). \\ R_0^{\geq -1,\,\emptyset}(G_m) = -G_m,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\geq -1,\,\geq -1}(L_m) = -L_m 1_{\rrbracket -\infty,\,-2\rrbracket}(m). \\ R_0^{\geq -1,\,\geq -1}(G_m) = -G_m 1_{\rrbracket -\infty,\,-2\rrbracket \cup \{1\}}(m). \end{cases} \end{cases}$$

$$[3]\begin{cases} \begin{cases} R_0^{\mathbb{Z},\,\mathbb{Z}}(L_m) = 0,\ \forall m \in \mathbb{Z}. \\ R_0^{,\,\mathbb{Z}}(G_m) = 0,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\mathbb{Z},\,\emptyset}(L_m) = 0,\ \forall m \in \mathbb{Z}. \\ R_0^{\mathbb{Z},\,\emptyset}(G_m) = -G_m,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\mathbb{Z},\,\mathbb{Z}\setminus\{1\}}(L_m) = 0,\ \forall m \in \mathbb{Z}. \\ R_0^{\mathbb{Z},\,\mathbb{Z}\setminus\{1\}}(G_m) = -G_m \delta_{m,1}. \end{cases} \end{cases}$$

$$[4]\begin{cases} \begin{cases} R_0^{\geq 2,\,\emptyset}(L_m) = -L_m 1_{\rrbracket -\infty,\,1\rrbracket}(m). \\ R_0^{\geq 2,\,\emptyset}(G_m) = -G_m,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\geq 2,\,\mathbb{Z}}(L_m) = 1_{\rrbracket -\infty,\,1\rrbracket}(m). \\ R_0^{\geq 2,\,\mathbb{Z}}(G_m) = 0,\ \forall m \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} R_0^{\geq 2,\,\geq 1}(L_m) = -L_m 1_{\rrbracket -\infty,\,1\rrbracket}(m). \\ R_0^{\geq 2,\,\geq 1}(G_m) = -G_m 1_{\rrbracket -\infty,\,0\rrbracket}(m). \end{cases} \end{cases}$$

$$[5]\begin{cases} \begin{cases} R_0^{\emptyset,\,\emptyset}(L_m) = -L_m,\ \forall m \in \mathbb{Z}. \\ R_0^{\emptyset,\,\emptyset}(G_m) = -G_m,\ \forall m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\emptyset,\,\mathbb{Z}}(L_m) = -L_m.\ \forall m \in \mathbb{Z}, \\ R_0^{\emptyset,\,\mathbb{Z}}(G_m) = 0,\ \forall m \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} R_0^{\emptyset,\,\{1\}}(L_m) = -L_m,\ \forall m \in \mathbb{Z}. \\ R_0^{\emptyset,\,\{1\}}(G_m) = -G_m,\ \forall m \in \mathbb{Z}\setminus\{1\}. \end{cases} \end{cases}$$

$$[6]\begin{cases} \begin{cases} R_0^{\leq -2,\,\emptyset}(L_m) = -L_m 1_{\llbracket -1,\,+\infty\llbracket}(m). \\ R_0^{\leq -2,\,\emptyset}(G_m) = -G_m,\ \ m \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{\leq -2,\,\mathbb{Z}}(L_m)) = -L_m 1_{\llbracket -1,\,+\infty\llbracket}(m). \\ R_0^{\leq -2,\,\mathbb{Z}}(G_m) = 0,\ \forall m \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} R_0^{\leq -2,\,\leq -2}(L_m) = -L_m 1_{\llbracket -1,\,+\infty\llbracket}(m). \\ R_0^{\leq -2,\,\leq -2}(G_m) = -G_m 1_{\{-1,0\}\cup\llbracket 2,\,+\infty\llbracket}(m). \end{cases} \end{cases}$$

$$[7]\begin{cases} \begin{cases} R_0^{+,\,\alpha,\,>n_0}(L_m) = -L_m 1_{\rrbracket -\infty,\,-1\rrbracket}(m) + \alpha L_0 \delta_{m,0}. \\ R_0^{+,\,\alpha,\,>n_0}(G_m) = -G_m 1_{\rrbracket -\infty,\,n_0\rrbracket}(m),\ \alpha \in \mathbb{C},\ n_0 \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{+,\,\alpha,\,\geq 0}(L_m) = -L_m 1_{\rrbracket -\infty,\,-1\rrbracket}(m) + \alpha L_0 \delta_{m,0}. \\ R_0^{+,\,\alpha,\,\geq 0}(G_m) = -G_m 1_{\rrbracket -\infty,\,-2\rrbracket} + \beta \delta_{m,1} G_{-1},\ \alpha, \beta \in \mathbb{C}. \end{cases} \end{cases}$$

$$[8] \begin{cases} \begin{cases} R_0^{-,\,\alpha,\,\leq n_0}(L_m) = -L_m 1_{[\![1,\,+\infty[\![}(m) + \alpha L_0 \delta_{m,0}. \\ R_0^{-,\,\alpha,\,\leq n_0}(G_m) = -G_m 1_{[\![n_0+1,\,+\infty[\![}(m),\; \alpha \in \mathbb{C}, n_0 \in \mathbb{Z}. \end{cases} \\ \text{Otherwise} \\ \begin{cases} R_0^{-,\,\alpha,\,\leq -2}(L_m) = -L_m 1_{[\![1,\,+\infty[\![}(m) + \alpha L_0 \delta_{m,0}. \\ R_0^{-,\,\alpha,\,\leq -2}(G_m) = -G_m 1_{[\![0,\,+\infty[\![}(m) + \beta \delta_{m,1} G_{-1},\; \alpha, \beta \in \mathbb{C}. \end{cases} \end{cases}$$

Proof. We begin by recall that $\mathcal{W} = \mathcal{W}_0 \oplus \mathcal{W}_0$, where

$$\mathcal{W}_0 = \{L_m | m \in \mathbb{Z}\} \quad \text{and} \quad \mathcal{W}_1 = \{G_m | m \in \mathbb{Z}\}.$$

Therefore any homogenous Rota-Baxtor operator R_0 on $\mathcal{W}$ can be expressed as fellows:

$$R_0 = \begin{pmatrix} R_{0|\mathcal{W}_0} & 0 \\ 0 & R_{0|\mathcal{W}_1} \end{pmatrix},$$

where $R_{|0,\mathcal{W}_0}(L_m) = f(m)L_m$ and $R_{|0,\mathcal{W}_0}(G_m) = g(m)G_m$.

The explicit expressions of $R_{0|\mathcal{W}_0}$ are are provided in Theorem 2.22 in [29] (see also Propositions 3.13 3.14 and 3.23). Therefore, the only remaining task is to detail all the various forms of $R_{0|\mathcal{W}_1}$. If we combine i) of Proposition 3.13 (and correspondingly ii) and iii)), Proposition 3.17 and Corollary 3.19, we deduce immediately the system [1] (and in a similar fashion [3] and [3]). Likewise, by combining i) of Proposition 3.14 (and correspondingly ii) and iii)), Proposition 3.20 and Corollary 3.22, we prove the system [4] (respectively [5] and [6]). The systems [7] and [8] are obtaines by combining Proposition 3.23 and 3.24. ∎

Remark 3.26. In the above notation, the first and the second parts of the superscript respectively represent the zero set of the corresponding $\mathbb{C}$-valued functions f and g. The third part, if it exists, is the value $f(0)$ and $g(-1)$. We use $R_0^{\mathcal{I},0}$ and $R_0^{0,\mathcal{I}}$, respectively, instead of $R_0^{\mathcal{I},\mathbb{Z}}$ and $R_0^{\mathbb{Z},\mathcal{I}}$ since it is the same operator as $R_0^{\mathcal{I},0}$ and $R_0^{0,\mathcal{I}}(where, \mathcal{I} \subsetneq \mathbb{Z})$ in Theorem 3.25.

4 Induced post-Lie superalgebras on the Witt superalgebras

In this section, we shall present the construction of post-Lie superalgebras from the homogeneous $\mathcal{O}$-operators with a weight of 1.

4.1 Post-Lie superalgebras

In this section, we generalize post-Lie superalgebras in the $\mathbb{Z}_2$-graded case, as presented in [29], and provide constructions of Witt superalgebras using homogeneous Rota-Baxter operators with weight 1.

Definition 4.1. A post-Lie superalgebra is a Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot])$ with an even product $\circ$ that satisfies the following equations:

$$((x \circ y) \circ z - x \circ (y \circ z)) - (-1)^{\bar{x}\bar{y}}((y \circ x) \circ z - y \circ (x \circ z)) + [x, y] \circ z = 0, \qquad (4.1)$$

$$z \circ [x, y] - [z \circ x, y] - (-1)^{\bar{x}\bar{z}}[x, z \circ y] = 0. \qquad (4.2)$$

We denote it as $(\mathfrak{g}, [\cdot, \cdot], \circ)$. These equations hold for all x and y in $\mathfrak{g}$.

Proposition 4.2. Let $(\mathfrak{g}, [\cdot, \cdot], \circ)$ be a post-Lie superalgebra. Then the following operation:

$$\{x, y\} = x \circ y - (-1)^{\bar{x}\bar{y}} y \circ x + [x, y], \; x, y \in \mathfrak{g}, \qquad (4.3)$$

defines a Lie superalgebra structure on $\mathfrak{g}$ which is called the sub-adjacent Lie superalgebra.

Proof. It is easy to show that the bracket $\{\cdot, \cdot\}$ is super-skew-symmetric. Now, let $x, y, z \in \mathcal{H}(\mathfrak{g})$; then, by Eqs.(4.1), (4.2) and (4.3), we infer:

$$(-1)^{\bar{x}\bar{z}}\{x, \{y, z\}\} + (-1)^{\bar{y}\bar{z}}\{y, \{z, x\}\} + (-1)^{\bar{z}\bar{y}}\{z, \{x, y\}\}$$

$$= (-1)^{\bar{x}\bar{z}}\{x, y \circ z - (-1)^{\bar{y}\bar{z}}z \circ y + [y, z]\} + (-1)^{\bar{y}\bar{x}}\{y, z \circ x - (-1)^{\bar{z}\bar{x}}x \circ z + [z, x]\}$$

$$+ (-1)^{\bar{y}\bar{x}}\{y, z \circ x - (-1)^{\bar{z}\bar{x}}x \circ z + [z, x]\}$$

$$= (-1)^{\bar{x}\bar{z}}\big(x \circ (y \circ z - (-1)^{\bar{y}\bar{z}}z \circ y + [y, z]) - (-1)^{\bar{x}(\bar{y}+\bar{z})}(y \circ z - (-1)^{\bar{y}\bar{z}}z \circ y$$

$$+ [y, z]) \circ x + [x, y \circ z - (-1)^{\bar{y}\bar{z}}z \circ y + [y, z]]\big)$$

$$+ (-1)^{\bar{y}\bar{x}}\big(y \circ (z \circ x - (-1)^{\bar{z}\bar{x}}x \circ z + [z, x]) - (-1)^{\bar{y}(\bar{z}+\bar{x})}(z \circ x - (-1)^{\bar{z}\bar{x}}x \circ z + [z, x]) \circ y$$

$$+ [y, z \circ x - (-1)^{\bar{z}\bar{x}}x \circ z + [z, x]]\big)$$

$$+ (-1)^{\bar{z}\bar{y}}\big(z \circ (x \circ y - (-1)^{\bar{x}\bar{y}}y \circ x + [x, y]) - (-1)^{\bar{z}(\bar{x}+\bar{y})}(x \circ y - (-1)^{\bar{x}\bar{y}}y \circ x + [x, y]) \circ z$$

$$+ [z, x \circ y - (-1)^{\bar{x}\bar{y}}y \circ x + [x, y]]\big)$$

$$= (-1)^{\bar{x}\bar{z}}x \circ (y \circ z) - (-1)^{(\bar{x}+\bar{y})\bar{z}}x \circ (z \circ y) + (-1)^{\bar{x}\bar{z}}x \circ [y, z] - (-1)^{\bar{x}\bar{y}}(y \circ z)x$$

$$+ (-1)^{\bar{y}(\bar{x}+\bar{z})}(z \circ y) \circ x - (-1)^{\bar{x}\bar{y}}[y, z] \circ x + (-1)^{\bar{x}\bar{z}}x \circ (y \circ z) - (-1)^{\bar{z}(\bar{y}+\bar{x})}x \circ (z \circ y)$$

$$+ (-1)^{\bar{x}\bar{z}}x \circ [y, z] - (-1)^{\bar{x}\bar{y}}(y \circ z) \circ x + (-1)^{\bar{y}(\bar{x}+\bar{z})}(z \circ y) \circ x + (-1)^{\bar{x}\bar{y}}[y, z] \circ x$$

$$+ (-1)^{\bar{y}\bar{x}}y \circ (z \circ x) - (-1)^{(\bar{y}+\bar{z})\bar{x}}y \circ (x \circ z) + (-1)^{\bar{y}\bar{x}}y \circ [z, x] - (-1)^{\bar{y}\bar{z}}(z \circ x) \circ y$$

$$+ (-1)^{\bar{z}(\bar{y}+\bar{x})}(x \circ z) \circ y - (-1)^{\bar{y}\bar{z}}[z, x] \circ y + (-1)^{\bar{y}\bar{x}}y \circ (z \circ x) - (-1)^{\bar{x}(\bar{z}+\bar{y})}y \circ (x \circ z)$$

$$+ (-1)^{\bar{y}\bar{x}}y \circ [z, x] - (-1)^{\bar{y}\bar{z}}(z \circ x) \circ y + (-1)^{\bar{z}(\bar{y}+\bar{x})}(x \circ z) \circ y - (-1)^{\bar{y}\bar{z}}[z, x] \circ y$$

$$+ (-1)^{\bar{z}\bar{y}}z \circ (x \circ y) - (-1)^{(\bar{z}+\bar{x})\bar{y}}z \circ (y \circ x) + (-1)^{\bar{z}\bar{y}}z \circ [x, y] - (-1)^{\bar{z}\bar{x}}(x \circ y) \circ z$$

$$+ (-1)^{\bar{x}(\bar{z}+\bar{y})}(y \circ x) \circ z - (-1)^{\bar{z}\bar{x}}[x, y] \circ z + (-1)^{\bar{z}\bar{y}}z \circ (x \circ y) - (-1)^{\bar{y}(\bar{z}+\bar{x})}z \circ (y \circ x)$$

$$+ (-1)^{\bar{z}\bar{y}}z \circ [x, y] - (-1)^{\bar{z}\bar{x}}(x \circ y) \circ z + (-1)^{\bar{x}(\bar{z}+\bar{y})}(y \circ x) \circ z - (-1)^{\bar{z}\bar{x}}[x, y] \circ z$$

$$= 0.$$

Then $(\mathfrak{g}, \{\cdot, \cdot\})$ forms a Lie superalgebra. ∎

Remark 4.3. Let $(\mathfrak{g}, [\cdot, \cdot], \circ)$ be a post-Lie superalgebra. If $\circ$ is super-commutative, then the two brackets $[\cdot, \cdot]$ and $\{\cdot, \cdot\}$ coincide.

Corollary 4.4. If $(\mathfrak{g}, [\cdot, \cdot], \circ)$ is a post-Lie superalgebra, then $(\mathfrak{g}, \triangle)$ is an admissible Lie superalgebra with the product $\triangle$ defined as follows:

$$x \triangle y = x \circ y + \frac{1}{2}[x, y], \ \forall x, y \in \mathfrak{g}. \tag{4.4}$$

Proposition 4.5. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra, and let $\mathcal{T} : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator of weight λ with respect to the $\mathbb{K}$-module Lie superalgebra $(V, \rho, [\cdot, \cdot]_V)$. Define new operations as follows:

$$[a, b]_V^\lambda = \lambda[a, b]_V \quad \text{and} \quad a \circ b = \rho(\mathcal{T}a)b \quad \forall a, b \in V. \tag{4.5}$$

Then $(V, \circ, [\cdot, \cdot]_V^\lambda)$ forms a post-Lie superalgebra. Moreover, $\mathcal{T}$ is a Lie superalgebra morphism from the sub-adjacent Lie superalgebra $(V, \{\cdot, \cdot\})$ as given in the above proposition to $(\mathfrak{g}, [\cdot, \cdot])$.

Proof. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra, and let $\mathcal{T} : V \to \mathfrak{g}$ be an $\mathcal{O}$-operator. It is easy to show that $\mathcal{T}$ is a morphism of Lie superlgebras. On the other hand, for $a, b, c \in V$, using Eqs.(2.6) and (4.5), we have:

$$((a \circ b) \circ c - a \circ (b \circ c)) - (-1)^{\bar{a}\bar{b}}((b \circ a) \circ c - b \circ (a \circ c)) + [a, b] \circ c$$

$$= \big(\rho(\mathcal{T}a)b \circ c - \rho(\mathcal{T}a)(\rho(\mathcal{T}b)c) - (-1)^{\bar{a}\bar{b}}\big((\rho(\mathcal{T}b)a) \circ c - \rho(\mathcal{T}b)(\rho(\mathcal{T}a)c) + \rho(\mathcal{T}[a, b])c$$

$$
\begin{aligned}
&= \rho(\rho(\mathcal{T}a)b)c - \rho(\mathcal{T}a)(\rho(\mathcal{T}b)c) - (-1)^{\bar{a}\bar{b}}\big(\rho(\rho(\mathcal{T}b)a)c - \rho(\mathcal{T}b)(\rho(\mathcal{T}a)c)\big) + \rho(\mathcal{T}[a,b]_V)c \\
&= \rho\big((\rho(\mathcal{T}a)b)c - (-1)^{\bar{a}\bar{b}}(\rho(\mathcal{T}b)a)c + \mathcal{T}[a,b]_V c\big) - \big(\rho(\mathcal{T}a)(\rho(\mathcal{T}b)c - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)(\rho(\mathcal{T}a)c\big) \\
&= \lambda\big(\rho(\mathcal{T}[a,b]_V)c - [\rho(\mathcal{T}a),\rho(\mathcal{T}b)]_V c\big) \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
a \circ [b,c]_V^\lambda - [a \circ b, c]_V^\lambda - (-1)^{\bar{a}\bar{b}}[b, a \circ c]_V^\lambda &= \lambda\big(a \circ [b,c]_V - [a \circ b, c]_V - (-1)^{\bar{a}\bar{b}}[b, a \circ c]_V\big) \\
&= \lambda\big(\rho(\mathcal{T}a)[b,c]_V - [\rho(\mathcal{T}a)b,c]_V - (-1)^{\bar{a}\bar{b}}[b,\rho(\mathcal{T}a)c]_V\big) \\
&= 0.
\end{aligned}
$$

Then $(V, \circ, [\cdot,\cdot]_V^\lambda)$ be a post-Lie superalgebra. $\blacksquare$

Corollary 4.6. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie superalgebra, $(V, [\cdot,\cdot]_V, \rho)$ be a $\mathfrak{g}$-module Lie superalgebra and $\mathcal{T} : V \to \mathfrak{g}$ be a λ-weighted $\mathcal{O}$-operator associated to $(V, [\cdot,\cdot]_V, \rho)$. Then $(V, [\cdot,\cdot]_\mathcal{T})$ is a Lie superalgebra, where the bracket $[\cdot,\cdot]_\mathcal{T}$ is given by:

$$[a,b]_\mathcal{T} = \rho(\mathcal{T}a)b - (-1)^{\bar{a}\bar{b}}\rho(\mathcal{T}b)a + \lambda[a,b]_V, \quad \forall a,b \in V. \tag{4.6}$$

Corollary 4.7. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie superalgebra, and let $\mathcal{R} : \mathfrak{g} \to \mathfrak{g}$ be a Rota-Baxter operator of weight λ. Then $(\mathfrak{g}, \circ, \lambda[\cdot,\cdot])$ forms a post-Lie superalgebra, where

$$x \circ y = [\mathcal{R}x, y] \text{ for all } x,y \in \mathfrak{g}. \tag{4.7}$$

Corollary 4.8. Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie superalgebra and R be a Rota-Baxter operator of weight λ. Then $(\mathfrak{g}, [\cdot,\cdot]_R)$ is a Lie superalgebra, where the bracket $[\cdot,\cdot]_R$ is given by:

$$[x,y]_R = [Rx,y] + [x,Ry] + \lambda[x,y], \tag{4.8}$$

where x,y in $\mathfrak{g}$.

Proposition 4.9. If $(\mathfrak{g}, [\cdot,\cdot], \circ)$ be a post-Lie superalgebra, then $(\mathfrak{g}, -[\cdot,\cdot], \bullet)$ is also a post-Lie superalgebra, where for all $x,y \in \mathfrak{g}$,

$$x \bullet y = x \circ y + [x,y]. \tag{4.9}$$

Proof. By construction, we have $(\mathfrak{g}, \triangle)$ is a Lie superalgebra. Moreover, using Eqs. (4.9), (4.1), and (4.2), for all x, y, and z in $\mathfrak{g}$, we obtain:

$$
\begin{aligned}
&\big((x \bullet y) \bullet z - x \bullet (y \bullet z)\big) - (-1)^{\bar{x}\bar{y}}\big((y \bullet x) \bullet z - y \bullet (x \bullet z)\big) - [x,y] \bullet z \\
&= \big((x \circ y + [x,y]) \bullet z - x \bullet (y \circ z + [y,z])\big) \\
&\quad - (-1)^{\bar{x}\bar{y}}\big((y \circ x + [y,x]) \bullet z - y \bullet (x \circ z + [x,z])\big) - [x,y] \circ z - [[x,y],z] \\
&= (x \circ y) \circ z + [x,y] \circ z + [x \circ y, z] + [[x,y],z] - x \circ (y \circ z) - x \circ [y,z] \\
&\quad - [x, y \circ z] - [x,[y,z]] - (-1)^{\bar{x}\bar{y}}(y \circ x) \circ z - (-1)^{\bar{x}\bar{y}}[y,x] \circ z - (-1)^{\bar{x}\bar{y}}[y \circ x, z] \\
&\quad - (-1)^{\bar{x}\bar{y}}[y,x],z] + (-1)^{\bar{x}\bar{y}}y \circ (x \circ z) \\
&\quad + (-1)^{\bar{x}\bar{y}}y \circ [x,z] + (-1)^{\bar{x}\bar{y}}[y,x \circ z] + (-1)^{\bar{x}\bar{y}}[y,[x,z]] - [x,y] \circ z - [[x,y],z] \\
&= \big((x \circ y) \circ z - x \circ (y \circ z) - (-1)^{\bar{x}\bar{y}}(y \circ x) \circ z + (-1)^{\bar{x}\bar{y}}y \circ (x \circ z) + [x,y] \circ z\big) \\
&\quad + \big([[x,y],z] - [x,[y,z]] + (-1)^{\bar{x}\bar{y}}[y,[x,z]]\big) \\
&\quad + \big(-x \circ [y,z] + [x \circ y, z] + (-1)^{\bar{x}\bar{y}}[y, x \circ z]\big) \\
&\quad + \big(-(-1)^{\bar{x}\bar{y}}[[y,x],z] - [[x,y],z]\big) + \big(-(-1)^{\bar{x}\bar{y}}[y,x] \circ z - [x,y] \circ z\big) \\
&\quad + \big(-[x, y \circ z] - (-1)^{\bar{x}\bar{y}}[y \circ x, z] + (-1)^{\bar{x}\bar{y}}y \circ [x,z]\big)
\end{aligned}
$$

$$= 0$$

and

$$- z \bullet [x, y] + [z \bullet x, y] + (-1)^{\bar{x}\bar{z}}[x, z \bullet y]$$
$$= -z \circ [x, y] + [z, [x, y]] + [z \circ x - [z, x], y] + (-1)^{\bar{x}\bar{z}}[x, z \circ y - [z, y]]$$
$$= -z \circ [x, y] + [z, [x, y]] - [[z, x], y] + [z \circ x, y] + (-1)^{\bar{x}\bar{z}}[x, z \circ y]$$
$$- (-1)^{\bar{x}\bar{z}}[x, [z, y]]$$
$$= 0.$$

Therefore $(\mathfrak{g}, -[\cdot, \cdot], \bullet)$ is a post-Lie superalgebra. ∎

Proposition 4.10. Let $(\mathfrak{g}, [\cdot, \cdot], \circ)$ be a post-Lie superalgebra. Define $\varrho : \mathfrak{g} \to \mathfrak{g}$ by $\varrho(x)y = x \circ y$, for any $x, y \in \mathfrak{g}$. Then $(\mathfrak{g}; [\cdot, \cdot], \varrho)$ is an $\mathfrak{g}$-module Lie superalgebra of $(\mathfrak{g}, \{\cdot, \cdot\})$. given in Proposition 4.2.

Proof. We know that both $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{g}, \cdot, \cdot)$ are Lie superalgebras. By Equation (4.1), it follows that ϱ is a representation of $(\mathfrak{g}, \{\cdot, \cdot\})$. To illustrate this, consider $x, y, z \in \mathfrak{g}$:

$$\varrho\big(\{x, y\}\big)(z) = \{x, y\} \circ z$$
$$= (x \circ y) \circ z - (-1)^{\bar{x}\bar{y}}(y \circ x) \circ z + [x, y] \circ z.$$

Furthermore, we can show:

$$\varrho(x)\varrho(y)(z) - (-1)^{\bar{x}\bar{y}}\varrho(y)\varrho(x)(z) = x \circ (y \circ z) - (-1)^{\bar{x}\bar{y}}y \circ (x \circ z)$$
$$= (x \circ y) \circ z - (-1)^{\bar{x}\bar{y}}(y \circ x) \circ z + [x, y] \circ z.$$

Additionally, utilizing both Equation (4.2) and (2.6), we find:

$$\varrho(z)(\{x, y\}) - \big(\{\varrho(z)x, y\} - (-1)^{\bar{x}\bar{z}}\{x, \varrho(z)y\}\big) = z \circ \{x, y\} - \{z \circ x, y\} + (-1)^{\bar{x}\bar{z}}\{x, z \circ y\}$$
$$= z \circ \big(x \circ y - (1)^{\bar{x}\bar{y}}y \circ x + [x, y]\big)$$
$$- (z \circ x) \circ y + (-1)^{\bar{x}\bar{y}+\bar{y}\bar{z}}y \circ (z \circ x)$$
$$- [z \circ x, y] + (-1)^{\bar{x}\bar{z}}\big(x \circ (z \circ y)$$
$$- (-1)^{\bar{x}\bar{z}+\bar{x}\bar{y}}(z \circ y) \circ x + [x, z \circ y].$$

By further simplifying this expression using the properties of operations in a post-Lie superalgebra, especially Equation (4.1), we can demonstrate that:

$$\varrho(z)(x, y) - \big(\{\varrho(z)x, y\} - (-1)^{\bar{x}\bar{z}}\{x, \varrho(z)y\}\big) = 0.$$

Therefore, we can conclude that $(\mathfrak{g}, [\cdot, \cdot], \varrho)$ forms an $\mathfrak{g}$-module of $(\mathfrak{g}, \{\cdot, \cdot\})$. ∎

4.2 Induced post-Lie superalgebras on the Witt superalgebras

Theorem 4.11. The homogeneous Rota-Baxtor operator Lie superalgebra of weight 1 on the Witt superalgebra $\mathcal{W}$ provided in Theorem 3.25 gives rise to the following post-Lie superalgebras $(\mathcal{W}, [\cdot, \cdot], \circ)$, where $(\mathcal{W}, [\cdot, \cdot])$ is the Witt superalgebra.

$$1\begin{cases}\begin{cases}L_m\circ^{\le1,\mathbb{Z}}L_n=-(m-n)L_{m+n}1_{[\![2,+\infty[\![}(m).\\L_m\circ^{\le1,\mathbb{Z}}G_n=-(m-n-1)G_{m+n}1_{[\![2,+\infty[\![}(m).\\G_n\circ^{\le1,\mathbb{Z}}L_m=0,\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\le1,\emptyset}L_n=-(m-n)L_{m+n}1_{[\![2,+\infty[\![}(m).\\L_m\circ^{\le1,\emptyset}G_n=-(m-n-1)G_{m+n}1_{[\![2,+\infty[\![}(m).\\G_n\circ^{\le1,\emptyset}L_m=(m-n-1)G_{m+n},\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\le1,<1}L_n=-(m-n)L_{m+n}1_{[\![2,+\infty[\![}(m).\\L_m\circ^{\le1,<1}G_n=-(m-n-1)G_{m+n}1_{[\![2,+\infty[\![}(m).\\G_n\circ^{\le1,<1}L_m=(m-n-1)G_{m+n}1_{[\![1,+\infty[\![}(n).\end{cases}\end{cases}$$

$$2\begin{cases}\begin{cases}L_m\circ^{\ge-1,\mathbb{Z}}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\L_m\circ^{\ge-1,\mathbb{Z}}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\G_n\circ^{\ge-1,\mathbb{Z}}L_m=0,\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\ge-1,\emptyset}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\L_m\circ^{\ge-1,\emptyset}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\G_n\circ^{\ge-1,\emptyset}L_m=(m-n-1)G_{m+n},\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\ge-1,\ge-1}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\L_m\circ^{\ge-1,\ge-1}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,-2]\!]}(m).\\G_n\circ^{\ge-1,\ge-1}L_m=(m-n-1)G_{m+n}1_{]\!]-\infty,-2]\!]\cup\{1\}}(n).\end{cases}\end{cases}$$

$$3\begin{cases}\begin{cases}L_m\circ^{\mathbb{Z},\mathbb{Z}}L_n=0,\forall m\in\mathbb{Z}.\\L_m\circ^{\mathbb{Z},\mathbb{Z}}G_n=0,\forall m\in\mathbb{Z}.\\G_n\circ^{\mathbb{Z},\mathbb{Z}}L_m=0,\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\mathbb{Z},\emptyset}L_n=0,\forall m\in\mathbb{Z}.\\L_m\circ^{\mathbb{Z},\emptyset}G_n=0,\forall m\in\mathbb{Z}.\\G_n\circ^{\mathbb{Z},\emptyset}L_m=(m-n-1)G_{m+n},\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\mathbb{Z},\mathbb{Z}\setminus\{1\}}L_n=0,\forall m\in\mathbb{Z}.\\L_m\circ^{\mathbb{Z},\mathbb{Z}\setminus\{1\}}G_n=0,\forall m\in\mathbb{Z}.\\G_n\circ^{\mathbb{Z},\mathbb{Z}\setminus\{1\}}L_n=(m-n-1)\delta_{n,1}G_{m+n}.\end{cases}\end{cases}$$

$$4\begin{cases}\begin{cases}L_m\circ^{\ge2,\emptyset}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,1]\!]}(m).\\L_m\circ^{\ge2,\emptyset}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,1]\!]}(m).\\G_n\circ^{\ge2,\emptyset}L_m=(m-n-1)G_{m+n}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\ge2,\mathbb{Z}}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,1]\!]}(m).\\L_m\circ^{\ge2,\mathbb{Z}t}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,1]\!]}(m).\\G_n\circ^{\ge2,\mathbb{Z}}L_m=0,\forall n\in\mathbb{Z}.\end{cases}\\Otherwise\\\begin{cases}L_m\circ^{\ge2,\ge1}L_n=-(m-n)L_{m+n}1_{]\!]-\infty,1]\!]}(m).\\L_m\circ^{\ge2,\ge1}G_n=-(m-n-1)G_{m+n}1_{]\!]-\infty,1]\!]}(m).\\G_n\circ^{\ge2,\ge1}L_m=(m-n-1)G_{m+n}1_{]\!]-\infty,0]\!]}(n).\end{cases}\end{cases}$$

$$5 \begin{cases} \begin{cases} L_m \circ^{\emptyset,\,\emptyset} L_n = -(m-n)L_{m+n},\ \forall m \in \mathbb{Z}. \\ L_m \circ^{\emptyset,\,\emptyset} G_n = -(m-n-1)G_{m+n},\ \forall m \in \mathbb{Z}. \\ G_n \circ^{\emptyset,\,\emptyset} L_m = (m-n-1)G_{m+n},\ \forall n \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{\emptyset,\,\mathbb{Z}} L_n = -(m-n)L_{m+n},\ \forall m \in \mathbb{Z}. \\ L_m \circ^{\emptyset,\,\mathbb{Z}} G_n = -(m-n-1)G_{m+n},\ \forall m \in \mathbb{Z}. \\ G_n \circ^{\emptyset,\,\mathbb{Z}} L_m = 0,\ \forall n \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{\emptyset,\,\{1\}} L_n = -(m-n)L_{m+n}. \\ L_m \circ^{\emptyset,\,\{1\}} G_n = -(m-n-1)G_{m+n}. \\ G_n \circ^{\emptyset,\,\{1\}} L_m = (m-n-1)G_{m+n},\ \forall n \in \mathbb{Z}\backslash\{1\}. \end{cases} \end{cases}$$

$$6 \begin{cases} \begin{cases} L_m \circ^{\leq-2,\,\emptyset} L_n = -(m-n)L_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ L_m \circ^{\leq-2,\,\emptyset} G_n = -(m-n-1)G_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ G_n \circ^{\leq-2,\,\emptyset} L_m = (m-n-1)G_{m+n},\ \forall n \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{\leq-2,\,\mathbb{Z}} L_n = -(m-n)L_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ L_m \circ^{\leq-2,\,\mathbb{Z}} G_n = -(m-n-1)G_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ G_n \circ^{\leq-2,\,\mathbb{Z}} L_m = 0,\ \forall n \in \mathbb{Z}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{\leq-2,\,\leq-2} L_n = -(m-n)L_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ L_m \circ^{\leq-2,\,\leq-2} G_n = -(m-n-1)G_{m+n}\mathbf{1}_{[\![-1,+\infty[\![}(m). \\ G_n \circ^{\leq-2,\,\leq-2} L_m = (m-n-1)G_{m+n}\mathbf{1}_{\{-1,0\}\cup[\![2,+\infty[\![}(n). \end{cases} \end{cases}$$

$$7 \begin{cases} \begin{cases} L_m \circ^{+,\,\alpha,\,>n_0} L_n = -(m-n)\mathbf{1}_{]\!]-\infty,-1]\!]}(m)L_{m+n} - n\alpha\delta_{m,0}L_n. \\ L_m \circ^{+,\,\alpha,\,>n_0} G_n = -(m-n-1)\mathbf{1}_{]\!]-\infty,-1]\!]}(m)G_{m+n} - (n+1)\alpha\delta_{m,0}G_n. \\ G_n \circ^{+,\,\alpha,\,>n_0} L_m = (m-n-1)\mathbf{1}_{]\!]-\infty,n_0]\!]}(n)G_{m+n}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{+,\,\alpha,\,\geq0} L_n = -(m-n)\mathbf{1}_{]\!]-\infty,-1]\!]}(m)L_{m+n} - n\alpha\delta_{m,0}L_n. \\ L_m \circ^{+,\,\alpha,\,\geq0} G_n = -(m-n-1)\mathbf{1}_{]\!]-\infty,-1]\!]}(m)G_{m+n} - (n+1)\alpha\delta_{m,0}G_n. \\ G_n \circ^{+,\,\alpha,\,\geq0} L_m = (m-n-1)\mathbf{1}_{]\!]-\infty,-2]\!]}(n)G_{m+n} + m\beta\delta_{n,-1}G_{m-1}. \end{cases} \end{cases}$$

$$8 \begin{cases} \begin{cases} L_m \circ^{-,\,\alpha,\,\leq n_0} L_n = -(m-n)\mathbf{1}_{[\![1,+\infty[\![}(m)L_{m+n} - n\alpha\delta_{m,0}L_n. \\ L_m \circ^{-,\,\alpha,\,\leq n_0} G_n = -(m-n-1)\mathbf{1}_{[\![1,+\infty[\![}(m)G_{m+n} - (n+1)\alpha\delta_{m,0}G_n. \\ G_n \circ^{-,\,\alpha,\,\leq n_0} L_m = (m-n-1)\mathbf{1}_{[\![n_0+1,+\infty[\![}(n)G_{m+n}. \end{cases} \\ Otherwise \\ \begin{cases} L_m \circ^{-,\,\alpha,\,\leq-2} L_n = -(m-n)\mathbf{1}_{[\![1,+\infty[\![}(m)L_{m+n} - n\alpha\delta_{m,0}L_n. \\ L_m \circ^{-,\,\alpha,\,\leq-2} G_n = -(m-n-1)\mathbf{1}_{[\![1,+\infty[\![}(m)G_{m+n} - (n+1)\alpha\delta_{m,0}G_n. \\ G_n \circ^{-,\,\alpha,\,\leq-2} L_m = (m-n-1)\mathbf{1}_{[\![0,+\infty[\![}(n)G_{m+n} + m\beta\delta_{n,-1}G_{m-1}. \end{cases} \end{cases}$$

Where $n_0 \in \mathbb{Z}$ and $\alpha,\,\beta \in \mathbb{C}$.

Proof. We simply apply Expression $\circ$ given in Corollary 4.7 to the various forms of the Rota-Baxter operator presented in 3.25. ∎

Proposition 4.12. The post-Lie superalgebras in Theorem 4.11 give rise to the following Lie superalgebras under the bracket $\{\cdot,\cdot\}$ defined in Eq.(4.3):

$$1 \begin{cases} \begin{cases} \{L_m, L_n\}_{R_0^{\leq 1, \, \mathbb{Z}}} = \begin{cases} -(m-n)L_{m+n}, \, , \; \forall m \text{ and } n \geq 2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \leq 1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\leq 1, \, \mathbb{Z}}} = \begin{cases} (m-n-1)G_{m+n}, \; \forall m \leq 1 \text{ and } n \in \mathbb{Z}. \\ 0, \; otherwise. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R_0^{\leq 1, \, \emptyset}} = \begin{cases} -(m-n)L_{m+n}, \; \forall m \text{ and } n \geq 2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \leq 1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\leq 1, \, \emptyset}} = \begin{cases} -(m-n-1)G_{m+n}, \; m \geq 2 \text{ and } n \in \mathbb{Z}. \\ 0, \; otherwise. \end{cases} \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R_0^{\leq 1, \, <1}} = \begin{cases} -(m-n)L_{m+n}, \; \forall m \text{ and } n \geq 2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \leq 1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\leq 1, \, <1}} = \begin{cases} -(m-n-1)G_{m+n}, \; \text{if } m \geq 2 \text{ and } n \geq 1. \\ (m-n-1)G_{m+n}, \; \text{if } m \leq 1, \text{ and } n \leq 0. \\ 0, \; otherwise. \end{cases} \end{cases} \end{cases}$$

$$2 \begin{cases} \begin{cases} \{L_m, L_n\}_{R^{\mathbb{Z}, \, \mathbb{Z}}} = (m-n)L_{m+n}, \\ \{L_m, G_n\}_{R^{\mathbb{Z}, \, \mathbb{Z}}} = (m-n-1)G_{m+n}, \end{cases} \; \forall m, n \in \mathbb{Z}. \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\mathbb{Z}, \, \emptyset}} = (m-n)L_{m+n}, \\ \{L_m, G_n\}_{R^{\mathbb{Z}, \, \emptyset}} = 0, \end{cases} \; \forall m, n \in \mathbb{Z}. \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\mathbb{Z}, \, \mathbb{Z}\backslash\{1\}}} = (m-n)L_{m+n}, \; \forall; m, n \in \mathbb{Z}. \\ \{L_m, G_n\}_{R^{\mathbb{Z}, \, \mathbb{Z}\backslash\{1\}}} = \begin{cases} (m-n-1)G_{m+n}, \; \forall m \in \mathbb{Z} \text{ and } ; n \in \mathbb{Z}\backslash\{1\}. \\ 0, \; otherwise. \end{cases} \end{cases} \end{cases}$$

$$3 \begin{cases} \begin{cases} \{L_m, L_n\}_{R_0^{\geq -1, \, \mathbb{Z}}} = \begin{cases} -(m-n)L_{m+n}, \; \forall m \text{ and } n \leq -2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \geq -1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\geq -1, \, \mathbb{Z}}} = \begin{cases} (m-n-1)G_{m+n}, \; \forall m \geq -1 \text{ and } n \in \mathbb{Z}. \\ 0, \; otherwise. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R_0^{\geq -1, \, \emptyset}} = \begin{cases} -(m-n)L_{m+n}, \; \forall m \text{ and } n \leq -2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \geq -1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\geq -1, \, \emptyset}} = \begin{cases} -(m-n-1)G_{m+n}, \; m \leq -2 \text{ and } n \in \mathbb{Z}. \\ 0, \; otherwise. \end{cases} \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R_0^{\geq -1, \, \geq -1}} = \begin{cases} -(m-n)L_{m+n}, \; \forall m \text{ and } n \leq -2. \\ (m-n)L_{m+n}, \; \forall m \text{ and } n \geq -1. \\ 0, \; otherwise. \end{cases} \\ \{L_m, G_n\}_{R_0^{\geq -1, \, \geq -1}} = \begin{cases} -(m-n-1)G_{m+n}, \; \text{if } m \leq -2, \text{ and } n \in \rrbracket -\infty, -2 \rrbracket \cup \{1\}. \\ (m-n-1)G_{m+n}, \; \text{if } m \geq -1, \text{ and } n \in \rrbracket -1, +\infty \rrbracket \backslash\{1\}. \\ 0, \; otherwise. \end{cases} \end{cases} \end{cases}$$

$$4\begin{cases} \begin{cases} \{L_m, L_n\}_{R^{\geq 2, \mathbb{Z}}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \leq 1. \\ (m-n)L_{m+n}, \ \forall m \text{ and } n \geq 2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\geq 2, \mathbb{Z}}} = \begin{cases} (m-n-1)G_{m+n}, \ \forall m \leq 2 \text{ and } n \in \mathbb{Z}. \\ 0, \quad otherwise. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\geq 2, \emptyset}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \leq 1. \\ (m-n)L_{m+n}, \ \forall m, n \geq 2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\geq 2, \emptyset}} = \begin{cases} -(m-n-1)G_{m+n}, \ \forall m \leq 1 \text{ and } n \in \mathbb{Z}. \\ 0, \quad otherwise. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\geq 2, \geq 1}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \leq 1. \\ (m-n)L_{m+n}, \ \forall m \text{ and } n \geq 2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\geq 2, \geq 1}} = \begin{cases} -(m-n-1)G_{m+n}, \ \forall m \leq 1 \text{ and } n \leq 0. \\ (m-n-1)G_{m+n}, \ \forall m \geq 2 \text{ and } n \geq 1. \\ 0, \quad otherwise. \end{cases} \end{cases} \end{cases} \end{cases}$$

$$5\begin{cases} \begin{cases} \{L_m, L_n\}_{R^{\emptyset, \mathbb{Z}}} = -(m-n)L_{m+n}, \ \forall m, n \in \mathbb{Z}. \\ \{L_m, G_n\}_{R^{\emptyset, \mathbb{Z}}} = 0, \ \forall m, n \in \mathbb{Z}. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\emptyset, \emptyset}} = -(m-n)L_{m+n}, \ \forall m, n \in \mathbb{Z}. \\ \{L_m, G_n\}_{R^{\emptyset, \emptyset}} - (m-n-1)G_{m+n}, \ \forall m, n \in \mathbb{Z}. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\emptyset, \{1\}}} = -(m-n)L_{m+n}, \ \forall m, n \in \mathbb{Z}. \\ \{L_m, G_n\}_{R^{\emptyset, \leq -2}} = -(m-n-1)G_{m+n}, \ \forall m \in \mathbb{Z} \text{ and } n \in \mathbb{Z} \backslash \{1\}. \end{cases} \end{cases}$$

$$6\begin{cases} \begin{cases} \{L_m, L_n\}_{R^{\leq -2, \mathbb{Z}}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \geq -1. \\ (m-n)L_{m+n}, \ \forall m \text{ and } n \leq -2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\leq -2, \mathbb{Z}}} = \begin{cases} (m-n-1)G_{m+n}, \ \forall m \leq -2. \text{ and } n \in \mathbb{Z}. \\ 0, \quad otherwise \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\leq -2, \emptyset}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \geq -1. \\ (m-n)L_{m+n}, \ \forall m \text{ and } n \leq -2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\leq -2, \emptyset}} = \begin{cases} -(m-n-1)G_{m+n}, \ \forall m \geq -1 \text{ and } n \in \mathbb{Z}. \\ 0, \quad otherwise. \end{cases} \\ \text{Or else} \\ \begin{cases} \{L_m, L_n\}_{R^{\leq -2, \leq -2}} = \begin{cases} -(m-n)L_{m+n}, \ \forall m \text{ and } n \geq -1. \\ (m-n)L_{m+n}, \ \forall m \text{ and } n \leq -2. \\ 0, \quad otherwise. \end{cases} \\ \{L_m, G_n\}_{R^{\leq -2, \leq -2}} = \begin{cases} -(m-n-1)G_{m+n}, \ \forall m \geq -1 \text{ and } n \in \{-1, 0\} \cup \rrbracket 2, +\infty \rrbracket. \\ (m-n-1)G_{m+n}, \ \forall m \leq -2 \text{ and } n \in \rrbracket -\infty, -2 \rrbracket \cup \{1\}. \\ 0, \quad otherwise. \end{cases} \end{cases} \end{cases} \end{cases}$$

$$7\begin{cases}
\begin{cases}
\{L_m, L_n\}_{R_0^{+,\,\alpha,\,>n_0}} = \begin{cases}
-(m-n)L_{m+n}, \text{ if } \quad m < 0 \text{ and } n < 0. \\
-(\alpha+1)nL_n, \text{ if } \quad m = 0 \text{ and } n > 0. \\
(m-n)L_{m+n}, \text{ if } \quad m > 0 \text{ and } n > 0. \\
-n\alpha L_n, \text{ if } \quad m = 0 \text{ and } , n < 0. \\
0, \text{ if } \quad m > 0 \text{ and } n > 0.
\end{cases} \\[4em]
\{L_m, G_n\}_{R_0^{+,\,\alpha,\,>n_0}} = \begin{cases}
(m-n-1)G_{m+n}, \quad m > 0, n > n_0. \\
-(m-n-1)G_{m+n}, \quad m < 0, n \le n_0. \\
-\alpha(n+1)G_n, \quad m = 0, n \le n_0. \\
-(1+\alpha)(1+n)\alpha n G_n, \quad m = 0, n > n_0. \\
0, \quad otherwise.
\end{cases}
\end{cases} \\
\text{Or else} \\
\begin{cases}
\{L_m, L_n\}_{R_0^{+,\,\alpha,\,>n_0}} = \begin{cases}
-(m-n)L_{m+n}, \text{ if } \quad m < 0 \text{ and } n < 0. \\
-(1+\alpha)nL_n, \text{ if } \quad m = 0 \text{ and } n > 0. \\
(m-n)L_{m+n}, \text{ if } \quad m > 0 \text{ and } n > 0. \\
-n\alpha L_n, \text{ if } \quad m = 0 \text{ and } , n < 0. \\
0, \text{ if } \quad m < 0 \text{ and } n < 0.
\end{cases} \\[4em]
\{L_m, G_n\}_{R_0^{+,\,\alpha,\,\geq 0}} = \begin{cases}
(m-n-1)G_{m+n}, \text{ if } \quad m > 0 \text{ and } n > -1. \\
-(m-n-1)G_{m+n}, \text{ if } \quad m < 0 \text{ and } n < -1 \\
-\alpha(n+1)G_n, \text{ if } \quad m = 0 \text{ and } n < -1. \\
-(1+\alpha)(1+n)G_n, \text{ if } \quad m = 0 \text{ and } n > -1. \\
\beta m G_{m-1}, \text{ if } \quad m < 0 \text{ and } n = -1. \\
(1+\beta)m G_{m-1}, \text{ if } \quad m > 0 \text{ and } n = -1. \\
0, \quad otherwise.
\end{cases}
\end{cases}
\end{cases}$$

$$8 \begin{cases} \begin{cases} \{L_m, L_n\}_{R_0^-, \alpha, \leq n_0} = \begin{cases} -(m-n)L_{m+n}, \text{ if } m > 0 \text{ and } n > 0. \\ -\alpha n L_n, \text{ if } m = 0 \text{ and } n > 0. \\ (m-n)L_{m+n}, \text{ if } m < 0 \text{ and } n < 0. \\ -n(1+\alpha)L_n, \text{ if } m = 0 \text{ and }, n < 0. \\ 0, \text{ if } m < 0 \text{ and } n > 0. \end{cases} \\ \{L_m, G_n\}_{R_0^-, \alpha, \leq n_0} = \begin{cases} (m-n-1)G_{m+n}, \; m < 0, n \leq n_0. \\ -(m-n-1)G_{m+n}, \; m > 0, n > n_0. \\ -\alpha(n+1)G_n, \; m = 0, n > n_0. \\ -(1+\alpha)(1+n)\alpha n G_n, \; m = 0, n \leq n_0. \\ 0, \; otherwise. \end{cases} \\ \text{Or else} \\ \{L_m, L_n\}_{R_0^-, \alpha, \leq -2} = \begin{cases} -(m-n)L_{m+n}, \text{ if } m > 0 \text{ and } n > 0. \\ -\alpha n L_n, \text{ if } m = 0 \text{ and } n > 0. \\ (m-n)L_{m+n}, \text{ if } m < 0 \text{ and } n < 0. \\ -n(1+\alpha)L_n, \text{ if } m = 0 \text{ and }, n < 0. \\ 0, \text{ if } m < 0 \text{ and } n > 0. \end{cases} \\ \{L_m, G_n\}_{R_0^-, \alpha, \leq -2} = \begin{cases} (m-n-1)G_{m+n}, \text{ if } m < 0 \text{ and } n < -1. \\ -(m-n-1)G_{m+n}, \text{ if } m > 0 \text{ and } n > -1 \\ -\alpha(n+1)G_n, \text{ if } m = 0 \text{ and } n > -1. \\ -(1+\alpha)(1+n)G_n, \text{ if } m = 0 \text{ and } n < -1. \\ \beta m G_{m-1}, \text{ if } m > 0 \text{ and } n = -1. \\ (1+\beta)m G_{m-1}, \text{ if } m < 0 \text{ and } n = -1. \\ 0, \; otherwise. \end{cases} \end{cases} \end{cases}$$

Proof. An immediate consequence of the result in Corollary 4.8. Its various systems are obtained through straightforward computation by applying Equation 4.8 to the different systems presented in Theorem 4.11 (Alternately Theorem 3.25). ∎

5 Cohomology of weighted $\mathcal{O}$-operators on Lie superalgebras

In this section, we present a cohomology theory for $\mathcal{O}$-operators on Lie superalgebras. Let $\mathcal{T}$ be a λ-weighted $\mathcal{O}$-operator on the Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot])$ with the representation $(V, [\cdot, \cdot]_V, \rho)$.

Proposition 5.1. Let $\mathcal{T}$ be an λ-weighted $\mathcal{O}$-operator on a Lie superalgebra $(\mathfrak{g}, [\cdot, \cdot])$ with respect to a $\mathfrak{g}$-module Lie superalgebra $(V, [\cdot, \cdot]_V, \rho)$. Define $\tilde{\rho} : V \to gl(\mathfrak{g})$ by, for $x \in \mathfrak{g}$ and $a \in V$,

$$\tilde{\rho}(a)(x) := [\mathcal{T}a, x] + \mathcal{T}(\rho(x)a). \tag{5.1}$$

Then $(\mathfrak{g}, [\cdot, \cdot], \tilde{\rho})$ is a $V_{\mathcal{T}}$-module Lie superalgebra.

Proof. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra and (V, ρ) their representation and apply Eqs. (2.4) on (5.1), we obtain:

$$\begin{aligned} [\tilde{\rho}(a), \tilde{\rho}(b)](z) &= \tilde{\rho}(a)([\mathcal{T}b, z] + \mathcal{T}\rho(z)(b)) - (-1)^{\bar{a}\bar{b}}\tilde{\rho}(b)([\mathcal{T}a, z] + \mathcal{T}\rho(z)(a)) \\ &= [\mathcal{T}a, [\mathcal{T}b, z] + \mathcal{T}\rho(z)(b)] + \mathcal{T}(\rho([\mathcal{T}b, z] + \mathcal{T}\rho(z)(b))a) \\ &\quad - (-1)^{\bar{a}\bar{b}}([\mathcal{T}b, [\mathcal{T}a, z] + \mathcal{T}\rho(z)(a)] + \mathcal{T}(\rho([\mathcal{T}a, z] + \mathcal{T}\rho(z)(a))b) \\ &= [\mathcal{T}a, [\mathcal{T}b, z]] + [\mathcal{T}a, \mathcal{T}\rho(z)(b)] + \mathcal{T}(\rho([\mathcal{T}b, z] + \mathcal{T}\rho(z)(b))a) \\ &\quad - (-1)^{\bar{a}\bar{b}}([\mathcal{T}b, [\mathcal{T}a, z]] + [\mathcal{T}b, \mathcal{T}\rho(z)(a)] + \mathcal{T}(\rho([\mathcal{T}a, z] + \mathcal{T}\rho(z)(a))b)) \\ &= [\mathcal{T}a, [\mathcal{T}b, z]] + [\mathcal{T}a, \mathcal{T}\rho(z)(b)] - (-1)^{\bar{a}\bar{b}}[\mathcal{T}b, [\mathcal{T}a, z]] - (-1)^{\bar{a}\bar{b}}[\mathcal{T}b, \mathcal{T}\rho(z)(a)] \end{aligned}$$

and

$$
\begin{aligned}
\tilde{\rho}([a,b]_{\mathcal{T}})(z) &= [\mathcal{T}[a,b]_{\mathcal{T}}, z] + \mathcal{T}\rho(z)[a,b]_{\mathcal{T}} \\
&= [[\mathcal{T}a, \mathcal{T}b], z] + \mathcal{T}\rho(z)[a,b]_{\mathcal{T}} \\
&= [[\mathcal{T}a, \mathcal{T}b], z] + \mathcal{T}\big([\rho(z)a, b]_{\mathcal{T}} - (-1)^{\bar{z}\bar{a}}[a, \rho(z)b]_{\mathcal{T}}\big).
\end{aligned}
$$

Therefore, $\tilde{\rho}$ is a representation of the sub-adjacent Lie superalgebra $(V, [\cdot, \cdot]_{\mathcal{T}})$ on the vector space $\mathfrak{g}$. ∎

Let $x_1, \ldots, x_k$ be k homogeneous elements of V, and $\overline{x_1, \ldots, x_k} := \bar{x}_1 + \ldots + \bar{x}_k$. The parity of an element $(x_1, \ldots, x_k)$ in V^k is determined by whether it contains even or odd elements.

The set $C^k(\mathfrak{g}, V)$ of k-cochains on $\mathfrak{g}$ with values in V is the set of k-linear maps $f : \otimes^k \mathfrak{g} \to V$ satisfying

$$
f(x_1, ..., x_i, x_{i+1}, ..., x_k) := -(-1)^{x_i x_{i+1}} f(x_1, ..., x_{i+1}, x_i, ..., x_k),
$$

where $1 < i < k+1$. For $k = 0$, we have $C^0(\mathfrak{g}, V) = V$. The map f is called even (resp. odd) when $f(x_1, \ldots, x_k) \in V_{\bar{0}}$ (resp. $f(x_1, \ldots, x_k) \in V_{\bar{1}}$) for all even (resp. odd) elements $(x_1, \ldots, x_k) \in \mathfrak{g}^k$.

The coboundary map $\delta^k : C^k(\mathfrak{g}, V) \to C^{k+1}(\mathfrak{g}, V)$ is given by:

$$
\begin{aligned}
\delta^k(f)(x_0, ..., x_k) &:= \sum_{1 \le s < t \le k} (-1)^{t + \bar{x}_t(\bar{x}_{s+1} + ... + \bar{x}_{t-1})} f(x_0, ..., x_{s-1}, [x_s, x_t], x_{s+1}, ..., \hat{x}_t, ..., x_k) \\
&\quad + \sum_{s=0}^{k} (-1)^{s + \bar{x}_s(\bar{f} + \bar{x}_0 + ... + \bar{x}_{s-1})} [x_s, f(x_0, ..., \hat{x}_s, ..., x_k)]_V,
\end{aligned} \tag{5.2}
$$

where $f \in C^k(\mathfrak{g}, V)$, $\bar{f}$ is the parity of f, $x_0, ..., x_k \in \mathfrak{g}$ and $\hat{x}_i$ means that x_i is omitted.

Theorem 5.2. Let $(\mathfrak{g}, [\cdot, \cdot])$ be a Lie superalgebra and $(V, [\cdot, \cdot])$ be a $\mathfrak{g}$-module. The pair:

$$
\Big(\bigoplus_{k>0} C^k(\mathfrak{g}, V), \{\delta^k\}_{k>0} \Big)
$$

defines a cohomology complex, that is

$$
\delta^k \circ \delta^{k-1} = 0.
$$

Let $d_{\tilde{\rho}} : Hom(\wedge^k V, \mathfrak{g}) \to Hom(\wedge^k V, \mathfrak{g})$ be the corresponding Chevalley-Eilenberg coboundary operator. More precisely, for all $f \in Hom(\wedge^k V, \mathfrak{g})$ and $x_1, ..., x_{k+1} \in V$, we have:

$$
\begin{aligned}
d_{\tilde{\rho}} f(x_1, ... x_{k+1}) &:= \sum_{i=1}^{k+1} (-1)^{\bar{x}_i(\bar{f} + \bar{x}_1 + ... + \bar{x}_{i-1})} \tilde{\rho}(x_i)(f(x_1, ..., \hat{x}_i, ..., x_{k+1})) \\
&\quad + \sum_{1 \le i < j \le k+1} (-1)^{j + \bar{x}_j(\bar{x}_{i+1} + ... + \bar{x}_{j-1})} f(x_1, ..., [x_i, x_j]_{\mathcal{T}}, ..., \hat{x}_j, ..., x_{k+1}) \\
&= \sum_{i=1}^{k+1} (-1)^{\bar{x}_i(\bar{f} + \bar{x}_1 + ... + \bar{x}_{i-1})} [\mathcal{T}x_i, f(x_1, ..., \hat{x}_i, ..., x_{k+1})] \\
&\quad + \sum_{i=1}^{k+1} (-1)^{\bar{x}_i(\bar{f} + \bar{x}_1 + ... + \bar{x}_{i-1})} \mathcal{T}\rho(f(x_1, ..., \hat{x}_i, ..., x_{k+1})) \\
&\quad + \sum_{1 \le i < j \le k+1} (-1)^{j + \bar{x}_j(\bar{x}_{i+1}, ..., \bar{x}_{j-1})} f(x_1, ..., \rho(\mathcal{T}x_i)x_j - (-1)^{\bar{x}\bar{y}}\rho(\mathcal{T}x_j)x_i \\
&\quad + \lambda[x_i, x_j], ..., \hat{x}_j, ..., x_{k+1}),
\end{aligned} \tag{5.3}
$$

where $f \in C^k(V, \mathfrak{g})$, $\bar{f}$ is the parity of f, $x_1, ..., x_{k+1} \in \mathfrak{g}$ and $\hat{x}_i$ means that x_i is omitted. Let T be an $\mathcal{O}$-operator on a Lie superalgebra $\mathfrak{g}$ with respect to a representation (V, ρ).

Let $(C^*(V, \mathfrak{g}))) := \oplus_{k \geq 0} C^k(V, \mathfrak{g}))$ *denote the cochain complex defined above. Define the set of* k-*cocycles by* $Z_{\mathcal{T}}^k(V, \mathfrak{g})$ *and the set of* k-*coboundaries by* $B_{\mathcal{T}}^k(V, \mathfrak{g})$. *Then, the* k-*th cohomology group ssociated with the the* $\mathcal{O}$-*operator* $\mathcal{T}$ *is given by:*

$$H_{\mathcal{T}}^k(V, \mathfrak{g}) := Z_{\mathcal{T}}^k(V, \mathfrak{g}) / B_{\mathcal{T}}^k(V, \mathfrak{g}), \ k \geq 0, \tag{5.4}$$

and is referred to as the k-*th cohomology group corresponding to the* $\mathcal{O}$-*operator* $\mathcal{T}$.
We need the following statement to prove the functoriality of our cohomology theory.

Definition 5.3. Let $\mathcal{T}$ and $\mathcal{T}'$ be two $\mathcal{O}$-operators on a Lie superalgebra $\mathfrak{g}$ with respect to the action $(V, \rho, [\cdot, \cdot]_V)$. A homomorphism from $\mathcal{T}'$ to $\mathcal{T}$ consists of a Lie superalgebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{g}$ and a Lie superalgebra homomorphism $\psi : V \to V$ such that:

$$\phi \circ \mathcal{T}' = \mathcal{T} \circ \psi, \tag{5.5}$$

$$\psi(\rho(x)a) = \rho(\phi(x))\psi(a), \ \ \forall x \in \mathfrak{g}, a \in V. \tag{5.6}$$

In particular, if both ϕ and ψ are invertible, (ϕ, ψ) is called an isomorphism from $\mathcal{T}'$ to $\mathcal{T}$.

Proposition 5.4. Let $\mathcal{T}$ and $\mathcal{T}'$ be two $\mathcal{O}$-operators on a Lie superalgebra $\mathfrak{g}$ with respect to an action $(V; \rho, [\cdot, \cdot]_V)$ and (ϕ, ψ) be a homomorphism from $\mathcal{T}'$ to $\mathcal{T}$.

 (i) ψ is also a Lie superalgebra homomorphism from the descendent Lie superalgebra $(V, [\cdot, \cdot]_{\mathcal{T}'})$ of $\mathcal{T}'$ to the descendent Lie superalgebra $(V, [\cdot, \cdot]_{\mathcal{T}})$ of $\mathcal{T}$. Furthermore, in the event that ψ is invertible, its inverse, denoted as $\psi^{-1} : V \to V$, also upholds the properties of a Lie superalgebra homomorphism.

(ii) The induced representation $(\mathfrak{g}, \tilde{\rho})$ of the Lie superalgebra $(V, [\cdot, \cdot]_{\mathcal{T}})$ and the induced representation $(\mathfrak{g}, \tilde{\rho}')$ of the Lie superalgebra $(V, [\cdot, \cdot]_{\mathcal{T}'})$ satisfy the following relation

$$\phi \circ \tilde{\rho}'(a) = \tilde{\rho}(\psi(a)) \circ \phi, \quad \forall a \in V.$$

That is, for all $a \in V$, the following diagram commutes:

Proof. Let $a, b \in V$, according to Eqs.(5.5), (5.6) and since ψ is a Lie superalgebra homomorphism, we have

$$
\begin{aligned}
\psi([a, b]_{\mathcal{T}'}) &= \psi\big(\tilde{\rho}'(\mathcal{T}'(a))b - (-1)^{\bar{a}\bar{b}}\tilde{\rho}'(\mathcal{T}'(b))a + \lambda[a, b]_V\big) \\
&= \psi\big(\tilde{\rho}'(\mathcal{T}'(a))b\big) - (-1)^{\bar{a}\bar{b}}\psi\big(\tilde{\rho}'(\mathcal{T}'(b))a\big) + \lambda\psi\big([a, b]_V\big) \\
&= \tilde{\rho}\big(\phi(\mathcal{T}'(a))\big)\psi(b) - (-1)^{\bar{a}\bar{b}}\tilde{\rho}\big(\phi(\mathcal{T}'(b))\big)\psi(a) + \lambda[\psi(a), \psi(b)]_V \\
&= \tilde{\rho}\big(\mathcal{T}(\psi(a))\big)\psi(b) - (-1)^{\bar{a}\bar{b}}\tilde{\rho}\big(\mathcal{T}(\psi(b))\big)\psi(a) + \lambda[\psi(a), \psi(b)]_V \\
&= [\psi(a), \psi(b)]_{\mathcal{T}}.
\end{aligned}
$$

Let a and b belong to the set V. If ψ is invertible, then $\psi^{-1} : V \to V$ is a Lie superalgebra morphism. Indeed,

$$
\begin{aligned}
\psi^{-1}[a, b]_V &= \psi^{-1}\left([\psi \circ \psi^{-1}(a), \psi \circ \psi^{-1}(b)]_V\right) \\
&= \psi^{-1} \circ \psi[\psi^{-1}(a), \psi^{-1}(b)]_V \\
&= [\psi^{-1}(a), \psi^{-1}(b)]_V.
\end{aligned}
$$

This implies that ψ_V is a homomorphism between the descendent Lie algebras. By Eqs.(5.5), (5.6) and (5.1), let $x \in \mathfrak{g}$, $a \in V$, we get:

$$
\begin{aligned}
\phi(\tilde{\rho}'(a)x) &= \phi\big(\mathcal{T}'(\rho(x)a) + [\mathcal{T}'(a), x]_{\mathfrak{g}}\big) \\
&= \mathcal{T}(\psi(\rho(x)a)) + [\phi(\mathcal{T}'(a)), \phi(x)]_{\mathfrak{g}} \\
&= \mathcal{T}(\psi(\rho(x)a)) + [\mathcal{T}(\psi(a)), \phi(x)]_{\mathfrak{g}} \\
&= \mathcal{T}\big(\rho(\phi(x))\psi(a)\big) + [\mathcal{T}(\psi(a)), \phi(x)]_{\mathfrak{g}} \\
&= \tilde{\rho}(\psi(a))\phi(x).
\end{aligned}
$$

∎

Let $\mathcal{T}$ and $\mathcal{T}'$ be two $\mathcal{O}$-operators on $\mathfrak{g}$ with respect to an action $(V, \rho, [\cdot, \cdot]_V)$, and (ϕ, ψ) a homomorphism from $\mathcal{T}'$ to $\mathcal{T}$ in which ψ is invertible. Define a map $\gamma : C^k(V, \mathfrak{g}) \to C^k(V, \mathfrak{g})$ by

$$
\gamma(\omega)(a_1, \cdots, a_k) = \phi\big(\omega(\psi^{-1}(a_1), \cdots, \psi^{-1}(a_k))\big), \quad \forall a_i \in V.
$$

Theorem 5.5. With above notation, γ is a cochain map from the cochain complex $(C^\star(V, \mathfrak{g}), d_{\tilde{\rho}}^{\mathcal{T}})$ to the cochain complex $(C^\star(V, \mathfrak{g}), d_{\tilde{\rho}'}^{\mathcal{T}'})$. Consequently, it induces a homorphism $\gamma_\star$ from the cohomology group $H^k(V, \mathfrak{g})$ to $H^k(V, \mathfrak{g})$ for all $k \geq 1$.

Proof. For all $\omega \in C^k(V, \mathfrak{g})$, by Proposition 5.4, we have:

$$
\begin{aligned}
& d_{\tilde{\rho}}^{\mathcal{T}}(\gamma(\omega))(a_1, ..., a_k) \\
={} & \sum_{i=1}^{k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \tilde{\rho}(a_i)\gamma(\omega)(a_1, ..., \hat{a}_i, ..., a_k) \\
& + \sum_{1 \leq i < j \leq k} (-1)^{j+a_j(a_{i+1}+...+a_{j-1})} \gamma(\omega)\big(a_1, ..., [a_i, a_j]_{\mathcal{T}}, ..., \hat{a}_j, ..., a_k\big) \\
={} & \sum_{i=1}^{k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \tilde{\rho}(a_i)\phi\big(\omega(\psi^{-1}(a_1), ..., \psi^{-\hat{1}}(a_i), ..., \psi^{-1}(a_k))\big) \\
& + \sum_{1 \leq i < j \leq k} (-1)^{j+a_j(a_{i+1}+...+a_{j-1})} \phi\big(\omega(\psi^{-1}(a_1), ..., \psi^{-1}(\tilde{\rho}(\mathcal{T}a_i)a_j - (-1)^{\bar{a}\bar{b}}\tilde{\rho}(\mathcal{T}a_j)a_i \\
& + \lambda[a_i, a_j]), ..., \psi^{-\hat{1}}(a_j), ..., \psi^{-1}(a_k))\big) \\
={} & \sum_{i=1}^{k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \phi\big(\tilde{\rho}(\psi^{-1}(a_i))\omega(\psi^{-1}(a_1), ..., \psi^{-\hat{1}}(a_i), ..., \psi^{-1}(a_k))\big) \\
& + \sum_{1 \leq i < j \leq k} (-1)^{j+a_j(a_{i+1}+...+a_{-1})} \phi\big(\omega(\psi^{-1}(a_1), ..., \psi^{-1}(\tilde{\rho}'(\mathcal{T}a_i)a_j - (-1)^{\bar{a}\bar{b}}\tilde{\rho}'(\mathcal{T}a_j)a_i \\
& + \lambda[a_i, a_j]), ..., \psi_V^{-\hat{1}}(a_j), ..., \psi^{-1}(a_k))\big) \\
={} & \sum_{i=1}^{k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \phi\big(\tilde{\rho}(\psi^{-1}(a_i))\omega(\psi^{-1}(a_1), ..., \psi^{-\hat{1}}(a_i), ..., \psi^{-1}(a_k))\big) \\
& + \sum_{1 \leq i < j \leq k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \phi\big(\omega(\psi^{-1}(a_1), ..., \psi^{-1}([a_i, a_j]_{\mathcal{T}'}), ..., \psi^{-\hat{1}}(a_j), ..., \psi^{-1}(a_k))\big) \\
={} & \sum_{i=1}^{k} (-1)^{a_i(\gamma(\omega)+a_1+...+a_{i-1})} \phi\big(\tilde{\rho}'(\psi^{-1}(a_i))\omega(\psi^{-1}(a_1), ..., \psi^{-\hat{1}}(a_i), ..., \psi^{-1}(a_k))\big) \\
& + \sum_{1 \leq i < j \leq k} (-1)^{a_i(\gamma(\omega)+a_1+..+a_{i-1})} \phi\big(\omega(\psi^{-1}(a_1), .., [\psi^{-1}(a_i), \psi^{-1}(a_j)]_{\mathcal{T}'}, .., \psi^{-\hat{1}}(a_j), .., \psi^{-1}(a_k))\big) \\
={} & \phi\big(d_{\tilde{\rho}'}^{\mathcal{T}'}\omega(\psi^{-1}(a_1), ..., \psi^{-1}(a_k))\big) = \gamma(d_{\tilde{\rho}'}^{\mathcal{T}'}\omega)(a_1, ..., a_k), \ \ a_i \in V.
\end{aligned}
$$

Thus, γ is a cochain map. Consequently, it induces a homomorphism $\gamma_\star$ from the cohomology group $H^k(V, \mathfrak{g})$ to $H^k(V, \mathfrak{g})$ for all $k \geq 1$. ∎

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