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REPRESENTATION NUMBERS BY QUATERNARY QUADRATIC FORMS OF LEVEL 28

A. Bouchikhi and S. Mezroui

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Corresponding Author: A. Bouchikhi

Abstract. In this paper, we present some examples of representation of integers by a positive-definite, integral, non-diagonal quaternary quadratic form of level 28 whose representation numbers can be determined explicitly using the theory of modular forms.

1 Introduction

Research on quadratic and quaternary forms has been active in recent years, with many contributions appearing in the literature. For example, problems concerning primitive elements and quadratic forms have been studied using analytic tools such as Kloosterman sums [8]. On the other hand, related investigations into sums of special sequences, like bi-periodic Fibonacci and Lucas numbers, have also been pursued with number-theoretic techniques [20]. These works illustrate the variety of approaches in modern number theory, from analytic to combinatorial, and provide context for our present study on explicit formulas for the representation numbers of certain positive-definite integral quaternary quadratic forms.

Let $\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{C}$, and $\mathcal{H} := \{z \in \mathbb{C} \ / \ \Im(z) > 0\}$, denote the sets of positive integers, nonnegative integers, integers, complex numbers and the Poincaré upper-half plan, respectively. The modular group $\mathrm{SL}_2(\mathbb{Z})$ is defined as follows:

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ \mid a, b, c, d \in \mathbb{Z}, \ ad - bc = 1 \right\}.$$

For $N \in \mathbb{N}$, the modular congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \ \mid c \equiv 0 \pmod{N} \right\}.$$

The index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p \mid N} \left(1 + \frac{1}{p}\right),$$

where p runs through the prime dividing N, see [10].

Let $n \in \mathbb{N}$, we define as :

$$\sigma(n) = \sum_{d \mid n} d,$$

where d runs through the positive divisors of n. If $n \notin \mathbb{N}$, we set $\sigma(n) = 0$. Let χ and ψ be a Dirichlet characters of modulus dividing N. For $n \in \mathbb{N}$, the generalized divisor sum $\sigma_{\chi,\psi}(n)$ is defined as:

$$\sigma_{\chi,\psi}(n) := \sum_{1\leq d|n} \chi(n)\psi\left(\frac{n}{d}\right) d.$$

If $n \notin \mathbb{N}$, we set $\sigma_{\chi,\psi}(n) = 0$. If $\chi = \psi = 1$, then $\sigma_{\chi,\psi}(n) = \sigma(n)$.
For positive integers k and N such that $(-1)^k N \equiv 0$ or $1 \pmod 4$, let $\chi((-1)^k N)$ be the Dirichlet character defined by the Kronecker symbol :

$$\chi_{(-1)^k N}(*) = \left(\frac{(-1)^k N}{*}\right).$$

Then $(-1)^k N$ is a fundamental discriminant if and only if $\chi((-1)^k N)$ is a primitive Dirichlet character modulo N (see [18, Theorem 9.13]).

Let $k \in \mathbb{Z}$ and χ be a Dirichlet character of modulus dividing N. Let $M_k(\Gamma_0(N), \chi)$ denote the space of modular forms of weight k with multiplier system χ for $\Gamma_0(N)$, and $E_k(\Gamma_0(N), \chi)$ and $S_k(\Gamma_0(N), \chi)$ denote the subspaces of Eisenstein forms and cusp forms of $M_k(\Gamma_0(N), \chi)$, respectively. It is known that

$$M_k(\Gamma_0(N), \chi) = E_k(\Gamma_0(N), \chi) \oplus S_k(\Gamma_0(N), \chi).$$

For $z \in \mathcal{H}$, let $q := e^{2\pi i z}$ so that $|q| < 1$. The Dedekind eta function $\eta(z)$ is defined by

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n).$$

An eta quotient $f(z)$ is a function of the form

$$f(z) := \prod_{\delta\in I} \eta^{r_\delta}(\delta z), \tag{1.1}$$

where I is a finite set of positive integers and $r_\delta > 0$ for all $\delta \in I$.

For $(x, y, z, t) \in \mathbb{Z}^4$, let $Q := Q(x, y, z, t)$ a positive-definite integral quadratic form and let $n \in \mathbb{N}_0$. We define

$$N(Q(x, y, z, t) = n) := \text{Card}\left\{(x, y, z, t) \in \mathbb{Z}^4 \mid Q(x, y, z, t) = n\right\}$$

The theta function of f is defined by

$$\theta_Q(z) = \sum_{(x,y,z,t)\in\mathbb{Z}^4} q^{Q(x,y,z,t)} = 1 + \sum_{n=1}^{\infty} N\left(Q(x, y, z, t) = n\right) q^n,\ z \in \mathcal{H}.$$

If $Q(x, y, z, t) = ax^2 + by^2 + cz^2 + dt^2$. A formula for $N\left(Q(x, y, z, t) = n\right)$ for the diagonal quaternary quadratic forms

$$(a, b, c, d) \in \{(1, 1, 1, 1), (1, 1, 1, 2), (1, 2, 2, 2), (1, 1, 5, 5), (1, 1, 5, 5), (1, 5, 5, 5)\},$$

are in the literature see ([7, 5, 3, 4, 1, 2, 9, 11, 12, 13, 14, 15, 16]).

For positive-definite integral nondiagonal quaternary quadratic forms of the type

$$ax^2 + by^2 + cz^2 + dt^2 + exy + fxz + gxt + hyz + jyt + kzt.$$

Alaca et al. [6] explicitly determined the representation numbers for forms belonging to a genus containing exactly two classes.

In the present paper, we explicitly compute the representation numbers for the following positive-definite, integral quaternary quadratic forms

$$x^2 + y^2 + z^2 + 2t^2 + xt,$$
$$x^2 + y^2 + z^2 + 4t^2 + xy + xz + xt,$$

and

$$x^2 + y^2 + 2z^2 + 2t^2 + xy + xz + yt + zt$$

These quadratic forms were selected because they belong to the same genus (see [19]), allowing their representation numbers to be derived from one another.

Now, we proceed to state our main theorem.

Theorem 1.1. *Let $n \in \mathbb{N}$. Then*

$$N(x^2+y^2+z^2+2t^2+xt=n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) - \frac{3}{2}\sigma_{(\chi_4,\chi_7)}(n) + \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ \frac{13}{4}\sigma_{(\chi_{28},\chi_1)}(n) - \frac{5}{2}a_{f_1}(n) + 2a_{f_2}(n).$$

$$N(x^2+y^2+z^2+4t^2+xy+xz+xt=n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) + \frac{1}{2}\sigma_{(\chi_4,\chi_7)}(n) + \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ \frac{17}{2}\sigma_{(\chi_{28},\chi_1)}(n) + \frac{15}{2}a_{f_1}(n) - 6a_{f_2}(n).$$

and

$$N(x^2+y^2+2z^2+2t^2+xy+xz+yt+zt=n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) + \sigma_{(\chi_4,\chi_7)}(n) - \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ 7\sigma_{(\chi_{28},\chi_1)}(n).$$

where the integers $a_{f_i}(n)$ with $1 \le i \le 2$ are given by

$$f_1(z) := \frac{\eta(z)\eta(4z)\eta(14z)^9}{\eta(2z)\eta(7z)^3\eta(28z)^3} = \sum_{n=1}^{\infty} a_{f_1}(n)q^n,$$
$$f_2(z) := \frac{\eta(2z)^2\eta(14z)^{12}}{\eta(z)\eta(7z)^5\eta(28z)^4} = \sum_{n=1}^{\infty} a_{f_2}(n)q^n.$$

2 Preliminaries

Let χ and ψ be primitive Dirichlet characters with conductors L and R, respectively. The Eisenstein series $E_{k,\chi,\psi}(q)$ is defined as :

$$E_{k,\chi,\psi}(q) = b_0 + \sum_{m\ge 1}\left(\sum_{n|m}\psi(n)\chi\left(\frac{m}{n}\right)n^{k-1}\right)q^m \in \mathbb{Q}(\chi,\psi)[[q]],$$

where

$$b_0 = \begin{cases} 0 & \text{if } L > 1,\\ -\frac{B_{k,\psi}}{2k} & \text{if } L = 1.\end{cases}$$

with $B_{k,\psi}$ denoting the k-th generalized Bernoulli number associated with ψ.
Note that if $\chi = \psi = 1$ and $k \ge 4$, then $E_{k,\chi,\psi} = E_k$, where E_k is the classical Esienstien series.

Miyake prove statements that imply the following in [17].

Theorem 2.1. *Suppose t is a positive integer and χ, ψ be as above and that k is a positive integer such that $\chi(-1)\psi(-1) = (-1)^k$. Except when $k = 2$ and $\chi = \psi = 1$, the power series $E_{k,\chi,\psi}(q^t)$ defines an element of $M_k(RLt, \chi\psi)$. If $\chi = \psi = 1$, $k = 2$, $t > 1$ and $E_2(q) = E_{k,\chi,\psi}(q)$, then $E_2(q) - tE_2(q^t)$ is a modular form in $M_2(\Gamma_0(t))$.*

Theorem 2.2. *The Eisenstein series in $M_k(N,\varepsilon)$ coming from [21, Theorem 5.8] with RLt divide N and $\chi\psi = \varepsilon$ form a basis for the Eisenstein subspace $E_k(N,\varepsilon)$.*

For the space $M_2(\Gamma_0(28), (\frac{28}{*}))$ the dimension is 12. To compute a basis for this space, we first determine a basis for the Eisenstein subspace $E_2(\Gamma_0(28), (\frac{28}{*}))$ and the cusp subspace $S_2(\Gamma_0(28), (\frac{28}{*}))$. Let's calculate a basis for the space $E_2(\Gamma_0(28), (\frac{28}{*}))$. For the space $E_2(\Gamma_0(28), (\frac{28}{*}))$ the dimension is 4. Hence,

Proposition 2.3. *A basis of the space $E_2(\Gamma_0(28), (\frac{28}{*}))$ is given by :*

$$\{E_{\chi_1,\chi_{28}}(q), E_{\chi_4,\chi_7}(q), E_{\chi_7,\chi_4}(q), E_{\chi_{28},\chi_1}(q)\}$$

Proof. See 2.1 and 2.2. □

The first twenty-five coefficients of the Eisenstien subspace is given as:

$$E_{\chi_1,\chi_{28}}(q) := -4 + q + q^2 + 4q^3 + q^4 - 4q^5 + 4q^6 + q^7 + q^8 + O(q^9)$$
$$E_{\chi_4,\chi_7}(q) := q + 2q^2 - 4q^3 + 4q^4 - 4q^5 - 8q^6 - q^7 + 8q^8 + O(q^9)$$
$$E_{\chi_7,\chi_4}(q) := q + q^2 - 4q^3 + q^4 + 4q^5 - 4q^6 - 7q^7 + q^8 + O(q^9)$$
$$E_{\chi_{28},\chi_1}(q) := q + 2q^2 + 4q^3 + 4q^4 + 4q^5 + 8q^6 + 7q^7 + 8q^8 + O(q^9)$$

We use the Sturm bound given in Theorem 2.4 to determine when two modular forms are equal.

Theorem 2.4 ([10]). *Let $n \in \mathbb{N}$. If $f(z) = \sum_{n=0}^{\infty} a_f(n)q^n \in M_k(\Gamma_0(N),\chi)$ and $g(z) = \sum_{n=0}^{\infty} a_g(n)q^n \in M_k(\Gamma_0(N),\chi)$ with $a_f(n) = a_g(n)$ for all $n = 0, 1, \dots, \frac{k}{12}\,[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)]$. Then*

$$f = g.$$

We use the following lemma to determine if certain eta- quotients are modular forms.

Lemma 2.5 ([10]). *Let $f(z)$ be an eta quotient given by (1.1), $k = \frac{1}{2}\sum_{\delta\in I} r_\delta$ and $s = \prod_{\delta\in I}\delta^{r_\delta}$. Suppose that the following conditions are satisfied:*

(i) $\sum_{\delta\in I}\delta r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{\delta\in I}\frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$,

(iii) $\sum_{\delta\in I}\frac{\gcd(d,\delta)^2}{\delta} r_\delta \geq 0$ *for each positive integers $d \in I$,*

(iv) k is an integer.

Then $f(z) \in M_k(\Gamma_0(N),\chi)$, where the Dirichlet character χ is given by

$$\chi(m) = \left(\frac{(-1)^k s}{m}\right).$$

Next, the following theorem identifies the modular space to which the theta function of a positive-definite integral quadratic form belongs, (see [22, Theorem 10.1]).

Theorem 2.6. *Let $f := f(x_1,\dots,x_k)$ be a positive-definite integral quadratic form in k variables and $M(f)$ be the matrix of $f(x_1,\dots,x_k)$. Let N be the level of $f(x_1,\dots,x_k)$, that is, the least positive integer such that $NM(f)^{-1}$ is an integral matrix with even diagonal entries. Then*

$$\theta_f(z) \in M_{\frac{k}{2}}(\Gamma_0(N),\chi),$$

where the character χ is given by

$$\begin{cases} \left(\frac{2\det(M(f))}{*}\right) & \text{if } k \text{ is odd},\\ \left(\frac{(-1)^{k/2}\det(M(f))}{*}\right) & \text{if } k \text{ is even}.\end{cases}$$

The following theorem identifies a difference of theta functions as a cusp form

Theorem 2.7 ([22]). *Let $f := f(x_1,\dots,x_k)$ and $g := g(x_1,\dots,x_k)$ be two positive-definite integral quadratic forms in k variables which belong to the same genus. Also, let $\theta_f(z)$ and $\theta_g(z)$ be the theta functions of f and g, respectively. Then $\theta_f(z) - \theta_g(z)$ is a cusp form.*

Let's calculate a basis for the space $S_2(\Gamma_0(28),(\frac{28}{*}))$. Since $\dim S_2(\Gamma_0(28),(\frac{28}{*})) = 2$. Thus,

Proposition 2.8. *A basis of the space $S_2(\Gamma_0(28),(\frac{28}{*}))$ space is given by*

$$f_1(z) := \frac{\eta(z)\eta(4z)\eta(14z)^9}{\eta(2z)\eta(7z)^3\eta(28z)^3} = \sum_{n=1}^{\infty} a_{f_1}(n)q^n,$$
$$f_2(z) := \frac{\eta(2z)^2\eta(14z)^{12}}{\eta(z)\eta(7z)^5\eta(28z)^4} = \sum_{n=1}^{\infty} a_{f_2}(n)q^n.$$

Proof. To prove this result, we employ Lemma 2.5 to identify all eta-quotients in the space $S_2(\Gamma_0(28),(\frac{28}{*}))$. Subsequently, we consider the family of functions above and verify their linear independence. □

3 Proof of the main theorem

Proof. Let

$$Q_1 := Q_1(x,y,z,t) = x^2 + y^2 + z^2 + 2t^2 + xt.$$

The matrix $M(Q_1)$ of Q_1 is

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

so $\det(M(Q_1)) = 28$ and

$$28M(Q_1)^{-1} = \begin{bmatrix} 16 & 0 & 0 & -4 \\ 0 & 14 & 0 & 0 \\ 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

Thus, the level of Q_1 is 28 and the character associated with Q_1 is given by

$$\left(\frac{\det(M(Q_1))}{*}\right) = \left(\frac{28}{*}\right)$$

So by Theorem 2.6 we have

$$\theta_{Q_1}(z) \in M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right).$$

The Hecke bound of Theorem 2.4 for $M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right)$ is

$$\left[\frac{2 \cdot 28}{12} \prod_{p|28}\left(1 + \frac{1}{p}\right)\right] = 8,$$

The first nine terms of θ_{Q_1} are given by

$$\theta_{Q_1}(z) = 1 + 6q + 16q^2 + 24q^3 + 26q^4 + 40q^5 + 56q^6 + 34q^7 + 44q^8 + O(q^9) \tag{3.1}$$

By Lemma 2.5, for all $1 \le i \le 2$, we have

$$f_i(z) = \sum_{n=1}^{\infty} a_{f_i}(n)q^n \in M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right).$$

Using Pari/GP [24], we compute the first nine coefficients of $f_i(z)$

$$\begin{aligned} f_1(z) &= q - q^2 - q^4 + q^7 + 3q^8 + O(q^9), \\ f_2(z) &= q + q^2 + q^4 + q^7 + 5q^8 + O(q^9). \end{aligned} \tag{3.2}$$

In identifying the coefficients using a SageMath program, agree up to the Sturm bound. Hence, by using Proposition 2.3, 2.8 and Theorem 2.4 with (3.2) and 3.1, we obtain that

$$\theta_{Q_1}(q) = -\frac{1}{4}E_{(\chi_1,\chi_{28})}(q) - \frac{3}{2}E_{(\chi_4,\chi_7)}(q) + \frac{7}{4}E_{(\chi_7,\chi_4)}(q) + \frac{13}{2}E_{(\chi_{28},\chi_1)}(q) - \frac{5}{2}f_1(q) + 2f_2(q). \tag{3.3}$$

By equating coefficients of q^n $(n \in \mathbb{N})$ in 3.3, we derive

$$N(x^2 + y^2 + z^2 + 2t^2 + xt = n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) - \frac{3}{2}\sigma_{(\chi_4,\chi_7)}(n) + \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ \frac{13}{4}\sigma_{(\chi_{28},\chi_1)}(n) - \frac{5}{2}a_{f_1}(n) + 2a_{f_2}(n).$$

Let

$$Q_2 := Q_2(x, y, z, t) = x^2 + y^2 + z^2 + 4t^2 + xy + xz + xt.$$

The matrix $M(Q_2)$ of Q_2 is

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 8 \end{bmatrix}$$

so $\det(M(Q_2)) = 28$ and

$$28M(Q_2)^{-1} = \begin{bmatrix} 32 & -16 & -16 & -4 \\ -16 & 22 & 8 & 2 \\ -16 & 8 & 22 & 2 \\ -4 & 2 & 2 & 4 \end{bmatrix}$$

Hence, the level of Q_2 is 28 and the character associated with Q_2 is

$$\left(\frac{\det(M(Q_2))}{*}\right) = \left(\frac{28}{*}\right).$$

Thus by Theorem 2.6 we have

$$\theta_{Q_2}(z) \in M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right).$$

The first nine terms of θ_{Q_2} are given by

$$\theta_{Q_2}(z) = 1 + 12q + 6q^2 + 24q^3 + 24q^4 + 40q^5 + 56q^6 + 48q^7 + 66q^8 + O(q^9) \tag{3.4}$$

In identifying the coefficients using a SageMath program, agree up to the Sturm bound. Hence, by using Proposition 2.3, 2.8 and Theorem 2.4 with (3.2) and 3.4, we obtain that

$$\theta_{Q_2}(q) = -\frac{1}{4}E_{(\chi_1,\chi_{28})}(q) + \frac{1}{2}E_{(\chi_4,\chi_7)}(q) + \frac{7}{4}E_{(\chi_7,\chi_4)}(q) + \frac{17}{2}E_{(\chi_{28},\chi_1)}(q) + \frac{15}{2}f_1(q) - 6f_2(q). \tag{3.5}$$

Equating coefficients of q^n for $n \in \mathbb{N}$ in (3.5), we obtain

$$N(x^2 + y^2 + z^2 + 4t^2 + xy + xz + xt = n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) + \frac{1}{2}\sigma_{(\chi_4,\chi_7)}(n) + \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ \frac{17}{2}\sigma_{(\chi_{28},\chi_1)}(n) + \frac{15}{2}a_{f_1}(n) - 6a_{f_2}(n).$$

Let

$$Q_3 := Q_3(x, y, z, t) = x^2 + y^2 + z^2 + 4t^2 + xy + xz + xt.$$

The matrix $M(Q_3)$ of Q_3 is

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

so $\det(M(Q_3)) = 28$ and

$$28M(Q_3)^{-1} = \begin{bmatrix} 26 & -16 & -8 & 6 \\ -16 & 26 & 6 & -8 \\ -8 & 6 & 10 & -4 \\ 6 & -8 & -4 & 10 \end{bmatrix}.$$

Therefore, the level of Q_3 is 28 and the character associated with Q_3 is

$$\left(\frac{\det(M(Q_3))}{*}\right) = \left(\frac{28}{*}\right).$$

So by Theorem 2.6 we have

$$\theta_{Q_3}(z) \in M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right).$$

The first nine terms of θ_{Q_3} are given by

$$\theta_{Q_3}(z) = 1 + 6q + 14q^2 + 30q^3 + 30q^4 + 18q^5 + 54q^6 + 60q^7 + 62q^8 + O(q^9) \tag{3.6}$$

In identifying the coefficients using a SageMath program, agree up to the Sturm bound. Hence, by using Proposition 2.3, 2.8 and Theorem 2.4 with (3.6), we obtain that

$$\theta_{Q_1}(q) = -\frac{1}{4}E_{(\chi_1,\chi_{28})}(q) + E_{(\chi_4,\chi_7)}(q) - \frac{7}{4}E_{(\chi_7,\chi_4)}(q) + 7E_{(\chi_{28},\chi_1)}(q). \tag{3.7}$$

Equating coefficients of q^n for $n \in \mathbb{N}$ in (3.7), we get

$$N(x^2 + y^2 + 2z^2 + 2t^2 + xy + xz + yt + zt = n) = -\frac{1}{4}\sigma_{(\chi_1,\chi_{28})}(n) + \sigma_{(\chi_4,\chi_7)}(n) - \frac{7}{4}\sigma_{(\chi_7,\chi_4)}(n)$$
$$+ 7\sigma_{(\chi_{28},\chi_1)}(n).$$

□

4 Conclusion remarks

In this paper, we have explicitly determined the representation numbers for three positive-definite, integral, non-diagonal quaternary quadratic forms of level 28:

$$Q_1 = x^2 + y^2 + z^2 + 2t^2 + xt,$$
$$Q_2 = x^2 + y^2 + z^2 + 4t^2 + xy + xz + xt,$$
$$Q_3 = x^2 + y^2 + 2z^2 + 2t^2 + xy + xz + yt + zt.$$

Using the theory of modular forms, we expressed the theta functions associated with these forms in terms of Eisenstein series and cusp forms in the space $M_2\left(\Gamma_0(28), \left(\frac{28}{*}\right)\right)$. The formulas for $N(Q_i = n)$ $(i = 1, 2, 3)$ involve generalized divisor sums $\sigma_{\chi,\psi}(n)$ and Fourier coefficients of specific eta quotients $f_1(z)$ and $f_2(z)$. The results demonstrate the power of modular forms in solving classical representation problems for quadratic forms.

References

[1] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, *Nineteen quaternary quadratic forms*, Acta Arith., **130**, 277–310, (2007).

[2] A. Alaca, Ş. Alaca, M. F. Lemire and K. S. Williams, *The number of representations of a positive integer by certain quaternary quadratic forms*, Int. J. Number Theory, **5**, 13–40, (2009).

[3] Ş. Alaca, L. Pehlivan and K. S. Williams, *On the number of representations of a positive integer as a sum of two binary quadratic forms*, Int. J. Number Theory, **10** (6), 1395–1420, (2014).

[4] A. Alaca, Ş. Alaca and K. S. Williams, *On the quaternary forms* $x^2 + y^2 + z^2 + 5t^2$, $x^2 + y^2 + 5z^2 + 5t^2$ *and* $x^2 + 5y^2 + 5z^2 + 5t^2$, JP J. Algebra Number Theory Appl., **9** (1), 37–53, (2007).

[5] A. Alaca and A. Altiary, *Representations by quaternary quadratic forms with coefficients 1, 2, 5 or 10*, Commun. Korean Math. Soc., **34**, 27–41, (2019).

[6] K. S. Alaca, Ş. Alaca and K. S. Williams, *Representation numbers of seven quaternary quadratic forms each in a genus consisting of only two classes*, Turkish J. Math., **44** (6), 1955–1981, (2020).

[7] A. Alaca, S. Alaca and Z. S. Aygin, *Theta products and eta quotients of level* 24 *and weight* 2, Funct. Approx. Comment. Math., **57** (2), 205–234, (2017).

[8] A. Kh. Al-Shalabi and M. Al-Qudah, *Pair of Primitive Elements in Quadratic Form*, Palestine Journal of Mathematics, **13** (4), 905–917, (2024).

[9] C. G. J. Jacobi, *Fundamenta Nova Theoriae Functionum Ellipticarum*, in: Gesammelte Werke, Erster Band, Chelsea Publishing Co., New York, 49–239, 285–290, (1969).

[10] L. J. P. Kilford, *Modular Forms: A Classical and Computational Introduction*, 2nd edition, Imperial College Press, London, (2015).

[11] H. D. Kloosterman, *On the representation of numbers in the form* $ax^2 + by^2 + cz^2 + dt^2$, Proc. London Math. Soc., **25** (1–2), 143–173, (1926).

[12] J. Liouville, *Sur les deux formes* $x^2 + y^2 + 2(z^2 + t^2)$, J. Math. Pures Appl., **5**, 269–272, (1860).

[13] J. Liouville, *Sur les deux formes* $x^2 + y^2 + z^2 + 2t^2$ *and* $x^2 + 2(y^2 + z^2 + t^2)$, J. Math. Pures Appl., **6**, 225–230, (1861).

[14] J. Liouville, *La forme* $x^2 + y^2 + z^2 + 5t^2$, J. Math. Pures Appl., **9**, 1–12, (1864).

[15] J. Liouville, *La forme* $x^2 + 5(y^2 + z^2 + t^2)$, J. Math. Pures Appl., **9**, 17–22, (1864).

[16] J. Liouville, *La forme* $x^2 + y^2 + 5z^2 + 5t^2$, J. Math. Pures Appl., **10**, 1–8, (1865).

[17] T. Miyake, *Modular Forms*, translated from the Japanese by Y. Maeda, Springer, (1989).

[18] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I: Classical Theory*, Cambridge Studies in Advanced Mathematics, **97**, Cambridge Univ. Press, (2007).

[19] G. L. Nipp, *Quaternary Quadratic Forms*, Springer-Verlag, New York, (1991).

[20] S. Soykan, *Sums of the Powers of Bi-Periodic Fibonacci and Lucas Numbers*, Palestine Journal of Mathematics, **13** (3), 673–686, (2024).

[21] W. A. Stein, *Modular Forms: A Computational Approach*, Amer. Math. Soc., **79**, (2007).

[22] X. Wang and D. Pei, *Modular Forms with Integral and Half-Integral Weights*, Springer, Heidelberg, (2012).

[23] W. A. Stein et al., *Sage Mathematics Software (Version 10.4)*, The Sage Development Team, 2023. Available at: `http://www.sagemath.org`.

[24] The PARI Group, *PARI/GP, Version 2.15.4*, Bordeaux, 2023. Available at: `https://pari.math.u-bordeaux.fr`.

Author information

A. Bouchikhi, Mathematics and Intelligent Systems. National School of Applied Sciences of Tangier. Abdelmalek Essaadi University, Morocco.
E-mail: `bouchikhi.abdelmonaim@gmail.com`

S. Mezroui, Mathematics and Intelligent Systems. National School of Applied Sciences of Tangier. Abdelmalek Essaadi University, Morocco.
E-mail: `smezroui@uae.ac.ma`

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