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Existence and continuous dependence results of stochastic pantograph integro-differential equation with Hilfer fractional derivative

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Abstract. In this paper, we look at the existence and continuous dependence of solutions for stochastic pantograph integro-differential equation with Hilfer fractional derivative. The paper analyzes the existence of mild solutions through fixed-point theory while Picard operator theory demonstrates solution continuous dependence on initial conditions. The theoretical results are validated using an application.

1 Introduction

Fractional derivatives provide a flexible tool for modeling intricate processes in a variety of fields by extending classical differentiation to non-integer orders. There are several approaches to define fractional derivatives, such as using the (R-L) and Caputo derivatives. The R-L derivative provides a foundational approach for extending the idea of differentiation to fractional orders [18]. On the other hand, the Caputo derivative is frequently used in practical applications because it aligns better with standard initial conditions [18]. Hilfer introduced a generalized fractional differential operator by merging various existing formulations, including the Caputo and R-L operators. This new operator, known as the fractional Hilfer derivative, is particularly effective for modeling systems involving time delays and intricate boundary conditions, as highlighted in works such as [4, 25, 28, 35].

Fractional differential equations represent a substantial advancement over traditional mathematical models, particularly in the domains of signal processing, biology, and engineering. By integrating non-integer order derivatives, these equations provide a framework for capturing more complex system dynamics. A notable category within this field is fractional pantograph delay differential equations, which incorporate delays and have been successfully explored in various studies, as referenced in [8, 13, 19, 36, 37, 40, 41].

Stochastic fractional differential equations serve essential modeling purposes for systems showing random variations because they find widespread practical uses in financial and climate science applications. The introduction of temporal delays in stochastic fractional differential equations results in stochastic pantograph integro-differential equation (SPIDEs) with Hilfer fractional derivative, where the pantograph delay refers to a specific type of delay characterized by the non-linear dependence of the delay term on the past states of the system. Studies focuse on both stochastic impact and time delay complexities that complicate stability and well-posedness analysis of SPIDEs, refer to [2, 3, 6, 7, 9, 10, 11, 12, 24, 29, 34, 44]. SPIDEs modeling represents a powerful solution because it incorporates both memory effects alongside random elements. It proves highly advantageous to use this model for biological and ecological applications involving systems with combined environmental noise and delayed responses. For instance, our model effectiveness becomes clear through its application to a generalized Hutchinson's equation with diffusion and delay which represents essential population dynamics features under uncertain conditions.

The continuous dependence of Stochastic fractional differential equations is crucial for ensuring that small changes in initial conditions or coefficients lead to proportionally small variations in the solutions. Research has indicated that mild solutions of mean-field stochastic functional differential equations exhibit sensitivity to initial data and coefficients in an appropriate topological framework; see [5, 15, 45] and the references therein. Additionally, generalized Cauchy-type problems involving Hilfer fractional derivatives demonstrate continuous dependence on the fractional order, supported by a generalization of Gronwall's inequality [1, 14, 15, 33, 39, 45]. Similarly, solutions to random fractional-order differential equations with nonlocal conditions also maintain continuous dependence on initial conditions [16]. The existence of both strong and weak solutions for stochastic differential equations, including those involving fractional Brownian motion, reinforces this continuity [21, 31, 43]. Lastly, stochastic evolution equations with multiplicative Lévy noise exhibit similar properties, establishing conditions for asymptotic stability [26, 30].

This paper investigates the existence and continuous dependence of solutions to the following SPIDEs

$$\begin{cases} {}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}x(\iota) = Ax(\iota) + f\left(\iota, x(\iota), x(\kappa\iota), \int_0^{\iota} g(\iota, s, x(s), x(\kappa s))ds\right) \\ \qquad + \sigma(\iota, x(\iota), x(\kappa\iota)) \dfrac{dW(\iota)}{d\iota}, \quad \iota \in J := (0, b], \\ \mathfrak{I}^{1-\gamma}_{0^+,\iota}x(0) = x_0, \quad \gamma = p + q - pq, \end{cases} \tag{1.1}$$

where $\mathfrak{I}^{1-\gamma}_{0^+,\iota}$ and ${}^{H}\mathfrak{D}^{p,q}_{0^+,\iota}$ are the fractional integral of order $1-\gamma$ and the Hilfer fractional derivative of order p and type q, respectively. Here, $0 < p < 1, \frac{1}{2} < q \leq 1$. Let A be the generator of strongly continuous semigroup $\{\mathcal{S}(\iota) : \iota \geq 0\}$ on a Hilbert space $\mathcal{E}$, $(W(\iota))_{\iota \geq 0}$ denotes the Q-Wiener process defined in the complete probability space $(\Omega, \mathcal{F}_\iota, P)$. $f : J \times \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, $\sigma : J \times \mathcal{E} \times \mathcal{E} \to \mathcal{L}_2^0$ are continuous functions and $0 < \kappa < 1$. The space $\mathcal{L}_2^0$ will be defined subsequently.

This paper investigate the existence and continuous dependence of solutions on initial conditions for a class of Hilfer fractional stochastic pantograph integro-differential equations. The combined use of fractional derivatives, stochastic elements, and proportional delays creates complex analytical circumstances that model real-world memory-driven and probabilistic systems. Hereditary phenomena attain better representation through the Hilfer derivative system rather than classical models.

The main contributions of this work are outlined as follows:

- The existence of solutions for stochastic pantograph integro-differential equation with Hilfer fractional derivative is investigated using fixed-point theory.
- Pantograph-type delays and stochastic effects are incorporated into the study to produce accurate modeling real-world temporal dynamics.
- Sufficient conditions are established to guarantee the existence and continuous dependence of solutions under appropriate assumptions.
- A numerical example with graphical analysis is provided to validate the model's applicability.

This paper follows this structure: Section 2 presents preliminary definitions and notations. Section 3 presents the existence of solutions for our problem. Section 4 proves the continuous dependence on initial data. An application is given in Section 5 to demonstrate the theoretical findings.

2 Preliminaries

Let $(\Omega, \mathcal{F}_\iota, P)$ be a complete probability space with a normal filtration $(\mathcal{F}_\iota)_{\iota\geq 0}$. Let $\mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E})$ is the Hilbert space of real-valued random variables that are square-integrable with respect to the probability measure on $(\Omega, \mathcal{F}_\iota)$. Let $C(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))$ is the space of continuous time stochastic processes that are square-integrable with the norm $\|x\|^2 = \sup_{\iota\in J} \mathbb{E}\|x(\iota)\|^2$, where $\mathbb{E}$ is the mathematical expectation. On the other hand, define the Banach space

$$C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E})) = \left\{x : J \to \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}) : \iota^{1-\gamma}x(\iota) \in C(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))\right\}, \quad 0 < \gamma \leq 1,$$

using the norm

$$\|x\|^2_{1-\gamma} = \sup_{\iota\in J} \mathbb{E}\left\|\iota^{1-\gamma}x(\iota)\right\|^2.$$

Consider $W : J \times \Omega \to K$ as a standard Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}_\iota, P)$, with Q being a linear bounded covariance operator such that $\mathrm{Tr}\, Q < \infty$. This process is associated with the normal filtration $(\mathcal{F}_\iota)_{\iota\geq 0}$. Suppose there exists a complete orthonormal basis $\{e_n\}_{n\geq 1}$ in K and a sequence of nonnegative real numbers $\{\lambda_n\}_{n\in\mathbb{N}}$ satisfying

$$Qe_n = \lambda_n e_n, \quad \lambda_n \geq 0,\ n = 1, 2, \ldots,$$

as well as a set of independent real-valued Brownian motions $\{\beta_n\}_{n\geq 1}$ such that

$$\langle W(\iota), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\langle e_n, e\rangle\beta_n(\iota), \quad e \in K,\ \iota \in J.$$

Define the Hilbert space

$$\mathcal{L}^0_2 = \{f \mid f \text{ is a Hilbert-Schmidt operator from } Q^{\frac{1}{2}}(K) \text{ to } X\},$$

with the inner product defined as

$$\langle\psi, \phi\rangle_{\mathcal{L}^0_2} = \mathrm{tr}[\psi Q\phi^*], \quad \psi, \phi \in \mathcal{L}^0_2.$$

Definition 2.1. ([18]) For $p > 0$, the fractional R-L integral of order p for a function $x : [a, \infty) \to \mathbb{R}$ is given by

$$\mathfrak{I}^p_{a^+,\iota}x(\iota) = \frac{1}{\Gamma(p)}\int_a^\iota (\iota - s)^{p-1}x(s)ds. \tag{2.1}$$

Definition 2.2. ([18]) For $n - 1 < p \leq n$, the fractional R-L derivative of order p for a function x is represented as

$$\mathfrak{D}^p_{a^+,\iota}x(\iota) = D^n\mathfrak{I}^{n-p}_{a^+,\iota}x(\iota) = \frac{1}{\Gamma(n-p)}\left(\frac{d}{d\iota}\right)^n \int_a^\iota (\iota - s)^{n-p-1}x(s)\mathrm{d}s. \tag{2.2}$$

Definition 2.3. ([25]) For $n - 1 < p \leq n$, the Hilfer fractional derivative of order p and type $0 \leq q \leq 1$ of x is given by

$${}^H\mathfrak{D}^{p,q}_{a^+,\iota}x(\iota) = \mathfrak{I}^{q(n-p)}_{a^+,\iota}D^n\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x(\iota) = \mathfrak{I}^{q(n-p)}_{a^+,\iota}\mathfrak{D}^{\varrho}_{a^+,\iota}x(\iota),\ \varrho = p + q\,(n - p)\,. \tag{2.3}$$

Lemma 2.4. *([25]) For $n - 1 < p \leq n$, $f \in \mathbb{L}^1(a, b)$, $0 \leq \beta \leq 1$, and $\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x \in AC^k[a, b]$. Then,*

$$\mathfrak{I}^p_{a^+,\iota}\,{}^H\mathfrak{D}^{p,q}_{a^+,\iota}x(\iota) = x(\iota) - \sum_{k=1}^{n}\frac{(\iota - a)^{\varrho-k}}{\Gamma(\varrho + 1 - k)} \cdot \lim_{\iota\longrightarrow+a}\frac{d^k}{d\iota^k}\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}x(\iota). \tag{2.4}$$

Lemma 2.5. *([25]) Let $p > 0$ and $q > 0$. Then, for all $\iota \in J$, we have*

$$\left[\mathfrak{I}^p_{a^+,\iota}(\iota)^{q-1}\right](\iota) = \frac{\Gamma(q)}{\Gamma(q+p)}\iota^{q+p-1},$$

and

$$\left[\mathfrak{D}^p_{a^+,\iota}(\iota)^{p-1}\right](\iota) = 0, \quad 0 < p < 1.$$

Definition 2.6. ([46]) Let the metric space $(\mathcal{E}, d)$. If there is a $x^* \in \mathcal{E}$ such that
(i) $\mathcal{Y}_{\mathcal{T}} = \{x^*\}$, where $\mathcal{Y}_{\mathcal{T}} = \{x \in \mathcal{X} : \mathcal{T}(x) = x\}$;
(ii) $\{\mathcal{T}^n(x_0)\}_{n\in\mathbb{N}}$ converges to x^* for each $x_0 \in \mathcal{E}$. Then, the operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Picard operator.

Lemma 2.7. *([38]) Let $\mathcal{T} : \mathcal{E} \to \mathcal{E}$ an increasing Picard operator with $\mathcal{Y}_{\mathcal{T}} = \{x^*_{\mathcal{T}}\}$ and let $(\mathcal{E}, d, \leq)$ an ordered metric space. Then, for each $x \in \mathcal{E}, x \leq \mathcal{T}(x)$ shows $x \leq x^*_{\mathcal{T}}$.*

Lemma 2.8. *(Jensen's inequality, [27]). Let $m \in \mathbb{N}$ and $\iota_1, \iota_2, \ldots, \iota_m$ be nonnegative real numbers, then*

$$\left(\sum_{i=1}^{m} \iota_i\right)^p \leq m^{p-1} \sum_{i=1}^{m} \iota_i^p, \quad \text{for } p > 1.$$

Lemma 2.9. *([23]) A stochastic process $x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))$ is called a mild solution of problem (1.1) if x satisfies the following stochastic integral equation*

$$\begin{aligned} x(\iota) =& \mathcal{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) f\left(s, x(s), x(\kappa s), \int_0^s g(s, \tau, x(\tau), x(\kappa\tau)) d\tau\right) ds \\ &+ \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \sigma(s, x(s), x(\kappa s)) dW(s), \quad \iota \in J, \end{aligned} \tag{2.5}$$

where

$$\mathcal{S}_{p,q}(\iota) = \mathfrak{I}_{0^+,\iota}^{p(1-q)} T_q(\iota), \quad T_q(\iota) = \iota^{q-1} P_q(\iota), \quad P_q(\iota) = \int_0^\infty q\theta M_q(\theta) \mathcal{S}(\iota^q \theta)\, d\theta.$$

Lemma 2.10. *([23]) Assume that $\mathcal{S}(\iota)$ is continuous in the uniform operator topology for $\iota > 0$ and $\{\mathcal{S}(\iota)\}_{\iota\geq 0}$ is uniformly bounded (i.e., there exists $M > 1$ such that $\sup_{\iota\in[0,\infty)} \|\mathcal{S}(\iota)\| < M$), we have the following properties.*

(i) $P_q(\iota)$, $T_q(\iota)$, and $\mathcal{S}_{p,q}(\iota)$ are linear and bounded operators, that is, for $\forall \iota \geq 0, x \in \mathcal{X}$,

$$\|P_q(\iota)x\| \leq \frac{M\|x\|}{\Gamma(q)}, \quad \|T_q(\iota)x\| \leq \frac{M\iota^{q-1}\|x\|}{\Gamma(q)} \quad \text{and}$$

$$\|\mathcal{S}_{p,q}(\iota)x\| \leq \frac{M\iota^{\gamma-1}\|x\|}{\Gamma(\gamma)}.$$

(ii) Operators $P_q(\iota)$, $T_q(\iota)$, and $S_{p,q}(\iota)$ are strongly continuous.

Lemma 2.11. *([32]) For any $\iota \in [a, b]$.*

$$x(\iota) \leq a(\iota) + \int_a^\iota b(\iota)x(s).$$

Then,

$$x(\iota) \leq a(\iota) + \int_a^\iota a(s)b(s) exp\left[\int_s^\iota b(u)du\right] ds, \; \iota \in [a, b].$$

Moreover, if $a(\iota)$ is a nondecreasing function on $[\iota_0, b]$. Then,

$$x(\iota) \leq a(\iota) exp\left[\int_a^\iota b(s)ds\right].$$

Theorem 2.12. *([22]) Let $\mathcal{M}$ be a closed, bounded, convex, and nonempty subset of a Banach space. Let $\mathcal{A}_1, \mathcal{A}_2$ a pair of operators such that*
(i) $\mathcal{A}_1 x + \mathcal{A}_2 y \in \mathcal{M}$ once $x, y \in \mathcal{M}$;
(ii) $\mathcal{A}_1$ is compact and continuous;
(iii) $\mathcal{A}_2$ is contraction mapping.
Hence, the equation $x = \mathcal{A}_1 x + \mathcal{A}_2 x$ has a solution on $x \in \mathcal{M}$.

3 Existence results

In this section, we prove the criteria for the existence of mild solution to problem (1.1). Let us state the following hypotheses to get the desired result.
(H_1): $f : J \times \mathcal{E} \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is continuous and there exist positive constants L, L_1 such that for $\iota \in J$, $x_1, x_2, y_1, y_2 \in \mathcal{E}$

$$\|f(\iota, x_1, x_2, x_3) - f(\iota, y_1, y_2, y_3)\|^2 \leq L\iota^{2(1-\gamma)} \left(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_3 - y_3\|^2\right),$$
$$\|f(\iota, x_1, x_2, x_3)\|^2 \leq L_1 \left(1 + \iota^{2(1-\gamma)} \|x_1\|^2 + \iota^{2(1-\gamma)} \|x_2\|^2 + \iota^{2(1-\gamma)} \|x_3\|^2\right).$$

(H_2): $\sigma : J \times \mathcal{E} \times \mathcal{E} \to \mathcal{L}_2^0$ is continuous and there exist positive constants K, K_1 such that for each $\iota \in J$, we have

$$\|\sigma(\iota, x_1, x_2) - \sigma(\iota, y_1, y_2)\|^2 \leq K\iota^{2(1-\gamma)} \left(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2\right),$$
$$\|\sigma(\iota, x_1, x_2)\|^2 \leq K_1 \left(1 + \iota^{2(1-\gamma)} \|x_1\|^2 + \iota^{2(1-\gamma)} \|x_2\|^2\right).$$

(H_3): $g : J \times J \times \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is continuous, and there exist positive constants m, m_1 such that for each $\iota \in J$, we have

$$\|g(\iota, s, x_1, x_2) - g(\iota, s, y_1, y_2)\|^2 \leq m \left(\|x_1 - y_1\|^2 + \|x_2 - y_2\|^2\right),$$
$$\|g(\iota, s, x_1, x_2)\|^2 \leq m_1 \left(1 + \|x_1\|^2 + \|x_2\|^2\right).$$

(H_4): Assume that the following inequality holds:

$$\Theta := \frac{6M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1} L_1 + \frac{2M^2 b^{4-2\gamma+2q} m_1}{\Gamma^2(q)2q-1} L_1 + \frac{24M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1} K_1 < 1. \tag{3.1}$$

We define the operator $\mathcal{A} : C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E})) \to C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))$ given by

$$(\mathcal{A}x)(\iota) = \begin{cases} \mathcal{S}_{p,q}(\iota)x_0 + \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) f\left(s, x(s), x(\kappa s), \int_0^s g(s, \tau, x(\tau), x(\kappa\tau)) d\tau\right) ds \\ + \displaystyle\int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \sigma(s, x(s), x(\kappa s))\, dW(s), \quad \iota \in J. \end{cases} \tag{3.2}$$

As we can see, the existence of an operator's $\mathcal{A}$ fixed point ensures that problem (1.1) has a mild solution.

Theorem 3.1. *Assume that $(H_1) - (H_3)$ are satisfied and $\Theta < 1$. Then the problem (1.1) has a mild solution on $C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))$.*

Proof. Define $\mathcal{B}_p = \{x \in C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E})) : \|x\|_{1-\gamma}^2 \leq p, p > 0\}$. Clearly, the set $\mathcal{B}_p$ is a bounded, closed, and convex subset of $C_{1-\gamma}(J, \mathbb{L}^2(\Omega, \mathcal{F}_\iota, \mathcal{E}))$. Moreover, on the bounded set $\mathcal{B}_p$ we divide $\mathcal{A}$ into two operators $\mathcal{A}_1$ and $\mathcal{A}_2$ where $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ by

$$(\mathcal{A}_1 x)(\iota) = \mathcal{S}_{p,q}(\iota)x_0 + \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) f\left(s, x(s), x(\kappa s), \int_0^s g(s, \tau, x(\tau), x(\kappa\tau)) d\tau\right) ds \tag{3.3}$$

and

$$(\mathcal{A}_2 x)(\iota) = \int_0^\iota (\iota - s)^{q-1} P_q(\iota - s) \sigma(s, x(s), x(\kappa s)) dW(s). \tag{3.4}$$

We will now proceed to verify, step by step, that the operator $\mathcal{A}_1 + \mathcal{A}_2$ has a fixed point on $\mathcal{B}_p$, which serves as the solution to problem (1.1).

Step 1: $\mathcal{A}$ maps $\mathcal{B}_p$ into itself.
We show that there exists a positive number p such that $\mathcal{A}\mathcal{B}_p \subset \mathcal{B}_p$. If it is not true, then for each positive number p, there exists $x \in \mathcal{B}_p$, but $x\mathcal{B}_p$, for some $\iota = \iota(p) \in J$, using Lemma 2.8, we have

$$\begin{aligned}
p <& \sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}x)(\iota)\right\|^2 \\
\leq& 3\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\mathcal{S}_{p,q}(\iota)x_0\right\|^2 \\
+& 3\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)f\left(s,x(s),x(\kappa s),\int_0^s g(s,\tau,x(\tau),x(\kappa\tau))d\tau\right)ds\right\|^2 \\
+& 3\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\int_0^\iota (\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2.
\end{aligned}$$

By applying the Cauchy-Schwartz (C-S) inequality, the Doob's martingale inequality, a simple calculation gives us

$$\begin{aligned}
p <& \sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}x)(\iota)\right\|^2 \leq \frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|x_0\right\|^2 \\
&+ \frac{3M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}\int_0^\iota \sup_{0<s_1\leq s}\mathbb{E}\left\|f\left(s_1,x(s_1),x(\kappa s_1),\int_0^{s_1} g(s_1,\tau,x(\tau),x(\kappa\tau))d\tau\right)\right\|^2 ds \\
&+ \frac{12M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}\int_0^\iota \sup_{0<s_1\leq s}\mathbb{E}\left\|\sigma(s_1,x(s_1),x(\kappa s_1))\right\|^2 ds.
\end{aligned}$$

Using the hypotheses (H_1) and (H_2), we arrive at

$$\begin{aligned}
p <& \sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}x)(\iota)\right\|^2 \\
\leq& \frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|x_0\right\|^2 + \frac{3M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}L_1\left[b+\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 ds\right. \\
&+ \int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)\right\|^2 ds \\
&\left.+ \int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_1} g(s_1,\tau,x(\tau),x(\kappa\tau))d\tau\right\|^2 ds\right] \\
&+ \frac{12M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}K_1\left[b+\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 ds + \int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)\right\|^2 ds\right].
\end{aligned} \tag{3.5}$$

Additionally, C-S inequality and assumption (H_3) gives us

$$\begin{aligned}
\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_1} g(\tau,x(\tau),x(\kappa\tau))d\tau\right\|^2 ds \leq& m_1\int_0^\iota s\int_0^s \sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}d\tau ds \\
&+ m_1\int_0^\iota s\int_0^s \sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 d\tau ds \\
&+ m_1\int_0^\iota s\int_0^s \sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)\right\|^2 d\tau ds.
\end{aligned} \tag{3.6}$$

Here, $\sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2$ is non-negative, non-decreasing in τ. Therefore, we can get

$$\sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 \leq \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2.$$

Then,

$$\int_0^\iota s \int_0^s \sup_{0<s_1\leq\tau} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 d\tau ds \leq \int_0^b s^2 \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 ds.$$

So, Eq. (3.6) simplifies to

$$\begin{aligned}\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|\int_0^{s_1} g(\tau, x(\tau), x(\kappa\tau))d\tau\right\|^2 ds \leq m_1 \int_0^\iota s^{4-2\gamma} ds \\ + m_1 \int_0^\iota s^2 \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 ds \\ + m_1 \int_0^\iota s^2 \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)\right\|^2 ds.\end{aligned} \tag{3.7}$$

Substituting (3.7) into (3.5), we obtain

$$\begin{aligned}&p < \sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}x)(\iota)\right\|^2 \\ &\leq \frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|x_0\right\|^2 + \frac{3M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}L_1\left(b + 2b\|x\|_{1-\gamma}^2 + m_1\frac{b^{5-2\gamma}}{5-2\gamma} + m_1\frac{2b^3}{3}\|x\|_{1-\gamma}^2\right) \\ &+ \frac{12M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}K_1\left(b + 2b\|x\|_{1-\gamma}^2\right).\end{aligned}$$

Therefore, it follows that

$$p < \sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}x)(\iota)\right\|^2 \leq \Lambda + \Theta\|x\|_{1-\gamma}^2 \leq \Lambda + \Theta p, \tag{3.8}$$

where

$$\Lambda = \frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|x_0\right\|^2 + \frac{3M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1}L_1 + \frac{3M^2 b^{6-4\gamma+2q}m_1}{5-2\gamma\Gamma^2(q)2q-1}L_1 + \frac{12M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1}K_1.$$

Dividing both sides of (3.8) by p and taking $p \longrightarrow \infty$, we get that $\Theta \geq 1$, which is a contradiction by referring (H_4), hence, the operator $\mathcal{A}$ maps $\mathcal{B}_p$ into itself, for some $p > 0$.

Step 2: $\mathcal{A}_1$ is a contraction.

Let $x, y \in \mathcal{B}_p$. By C-S inequality, for each $\iota \in J$, we have

$$\begin{aligned}&\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}_1x)(\iota) - (\mathcal{A}_1y)(\iota)\right\|^2 \\ &\leq \frac{2M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}\int_0^\iota \sup_{0<s_1\leq s}\mathbb{E}\left\|f\left(s_1, x(s_1), x(\kappa s_1), \int_0^{s_1} g(s_1,\tau,x(\tau),x(\kappa\tau))d\tau\right)\right. \\ &\left. - f\left(s_1, y(s_1), y(\kappa s_1), \int_0^{s_1} g(s_1,\tau,y(\tau),y(\kappa\tau))d\tau\right)\right\|^2 ds.\end{aligned}$$

By using C-S inequality, (H_1) and (H_3), we have

$$\begin{aligned}&\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|(\mathcal{A}_1x)(\iota) - (\mathcal{A}_1y)(\iota)\right\|^2 \\ &\leq \frac{2M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}L\left[\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)-y(s_1)\right\|^2 ds\right. \\ &+ \int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)-y(\kappa s_1)\right\|^2 ds \\ &+ m\int_0^\iota s^2 \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)-y(s_1)\right\|^2 ds \\ &\left.+ m\int_0^\iota s^2 \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)-y(\kappa s_1)\right\|^2 ds\right].\end{aligned}$$

It follows that

$$\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\left(\mathcal{A}_1 x\right)(\iota)-\left(\mathcal{A}_1 y\right)(\iota)\right\|^2 \leq \frac{2M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1} L\left(2b\|x-y\|_{1-\gamma}^2+m\frac{2b^3}{3}\|x-y\|_{1-\gamma}^2\right).$$

This leads to

$$\begin{aligned}\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\left(\mathcal{A}_1 x\right)(\iota)-\left(\mathcal{A}_1 y\right)(\iota)\right\|^2 \leq&\left(\frac{4M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1}L+\frac{4M^2 b^{4-2\gamma+2q}m}{3\Gamma^2(q)2q-1}\right)\|x-y\|_{1-\gamma}^2,\\ \leq&\Delta\|x-y\|_{1-\gamma}^2,\end{aligned}$$

where

$$\Delta=\frac{4M^2 b^{2-2\gamma+2q}}{\Gamma^2(q)2q-1}L+\frac{4M^2 b^{4-2\gamma+2q}m}{3\Gamma^2(q)2q-1}<1.$$

We conclude that $\mathcal{A}_1$ is a contraction mapping.
Now, we show that $\mathcal{A}_2$ is completely continuous. To demonstrate this, we continue with the next step by proving

- $\mathcal{A}_2$ maps bounded sets to bounded sets in $\mathcal{B}_p$.
- $\mathcal{A}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}_p$.
- $\mathcal{A}_2$ maps $\mathcal{B}_p$ into a precompact set in $\mathcal{B}_p$.

Step 3: $\mathcal{A}_2$ maps bounded sets to bounded sets in $\mathcal{B}_p$.
By Doob's martingale inequality and (H_2), for each $x\in\mathcal{B}_\tau$ and $\iota\in J$, we have

$$\begin{aligned}&\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\left(\mathcal{A}_2 x\right)(\iota)\right\|^2\\ &\leq\frac{12M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}K_1\left[b+\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(s_1)\right\|^2 ds+\int_0^\iota \sup_{0<s_1\leq s} s_1^{2(1-\gamma)}\mathbb{E}\left\|x(\kappa s_1)\right\|^2 ds\right].\end{aligned}$$

It follows that

$$\sup_{0<\iota\leq b} \iota^{2(1-\gamma)}\mathbb{E}\left\|\left(\mathcal{A}_2 x\right)(\iota)\right\|^2 \leq \frac{12M^2 b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}K_1\left(b+2b\|x\|_{1-\gamma}^2\right)=\mathcal{M}.$$

So, this implies that there exists a positive constant $\mathcal{M}$ such that for $x\in\mathcal{B}_p$, we have $\|\mathcal{A}_2 x\|_{1-\gamma}^2\leq\mathcal{M}$. Therefore, $\mathcal{A}_2$ maps bounded sets to bounded sets in $\mathcal{B}_p$.
Step 4: $\mathcal{A}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}_p$.
Let $0\leq\iota_1<\iota_2\leq b$, $\epsilon>0$ and $x\in\mathcal{B}_p$, then

$$\begin{aligned}&\mathbb{E}\left\|\iota_2^{1-\gamma}\left(\mathcal{A}_2 x\right)(\iota_2)-\iota_1^{1-\gamma}\left(\mathcal{A}_2 x\right)(\iota_1)\right\|^2\\ \leq&\mathbb{E}\left\|\iota_2^{2(1-\gamma)}\int_0^{\iota_2}(\iota_2-s)^{q-1}P_q(\iota_2-s)\sigma(s,x(s),x(\kappa s))dW(s)\right.\\ &\left.-\iota_1^{2(1-\gamma)}\int_0^{\iota_1}(\iota_1-s)^{q-1}P_q(\iota_1-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2.\end{aligned}$$

Using Doob's martingale inequality and (H_2), we get

$$\begin{aligned}
&\mathbb{E}\left\|\iota_2^{1-\gamma}\left(\mathcal{A}_2 x\right)\left(\iota_2\right)-\iota_1^{1-\gamma}\left(\mathcal{A}_2 x\right)\left(\iota_1\right)\right\|^2 \\
&\leq 3\mathbb{E}\left\|\int_0^{\iota_1}\left[\iota_2^{2(1-\gamma)}(\iota_2-s)^{q-1}-\iota_1^{2(1-\gamma)}(\iota_1-s)^{q-1}\right]P_q(\iota_2-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2 \\
&+3\mathbb{E}\left\|\int_0^{\iota_1}\iota_1^{2(1-\gamma)}(\iota_1-s)^{q-1}\left[P_q(\iota_2-s)-P_q(\iota_1-s)\right]\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2 \\
&+3\mathbb{E}\left\|\int_{\iota_1}^{\iota_2}\iota_2^{2(1-\gamma)}(\iota_2-s)^{q-1}P_q(\iota_2-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2 \\
&\leq\frac{12M^2\left(1+2\|x\|_{1-\gamma}^2\right)}{\Gamma^2(q)}\int_0^{\iota_1}\left[\iota_2^{2(1-\gamma)}(\iota_2-s)^{q-1}-\iota_1^{2(1-\gamma)}(\iota_1-s)^{q-1}\right]^2ds \\
&+\frac{24\iota_1^{2(1-\gamma}\left(\iota_1^{2q-1}-\epsilon^{2q-1}\right)}{2q-1}\left(1+2\|x\|_{1-\gamma}^2\right)\times\left(\sup_{s\in[0,\iota_1-\varepsilon]}\left\|P_q\left(\iota_2-s\right)-P_q\left(\iota_1-s\right)\right\|\right)^2 \\
&+\frac{96M^2}{\Gamma^2(q)}\left[\frac{\epsilon^{2q-1}}{2q-1}+\frac{2\epsilon^{2q-1}\|x\|_{1-\gamma}^2}{2q-1}\right] \\
&+\frac{12M^2\iota_2^{2(1-\gamma}}{\Gamma^2(q)2q-1}K_1\left(1+2\|x\|_{1-\gamma}^2\right)\left(\iota_2-\iota_1\right)^{2q-1}\longrightarrow 0,\quad\text{as}\quad\iota_1\longrightarrow\iota_2,\epsilon\longrightarrow 0.
\end{aligned}$$

Therefore, we find that

$$\mathbb{E}\left\|\iota_2^{1-\gamma}\left(\mathcal{A}_2 x\right)\left(\iota_2\right)-\iota_1^{1-\gamma}\left(\mathcal{A}x\right)\left(\iota_1\right)\right\|^2$$

approaches zero as $\iota_1\to\iota_2$, independently of $x\in\mathcal{B}_p$. This implies that $\{\mathcal{A}_2 x, x\in\mathcal{B}_p\}$ is equicontinuous.

Step 5: $\mathcal{A}_2$ maps $\mathcal{B}_p$ into a precompact set in $\mathcal{B}_p$.

Let $\iota\in J$ is fixed and ϵ is a real number such that $\epsilon\in(0,\iota)$. For $x\in\mathcal{B}_p$, we define

$$\left(\mathcal{A}_2^\epsilon x\right)(\iota)=\int_0^{\iota-\epsilon}(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s).$$

Since the operator $P_q(\iota-s)$ is compact, the set $\{\left(\mathcal{A}_2^\epsilon x\right)(\iota):x\in\mathcal{B}_p\}$ is precompact in $\mathcal{E}$, for any ϵ, $\epsilon<\iota$. Further, for any $x\in\mathcal{B}_p$, we get

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma}\left(\mathcal{A}_2 x\right)(\iota)-\iota^{1-\gamma}\left(\mathcal{A}_2^\epsilon x\right)(\iota)\right\|^2 \\
&=\mathbb{E}\left\|\int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s)\right. \\
&\left.-\int_0^{\iota-\epsilon}(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s)\right\|^2 \\
&\leq 4\int_{\iota-\epsilon}^{\iota}(\iota-s)^{q-1}\left\|P_q(\iota-s)\right\|^2\mathbb{E}\left\|\sigma(s,x(s),x(\kappa s))\right\|^2ds \\
&\leq\frac{4M^2b^{2(1-\gamma)}}{\Gamma^2(q)2q-1}K_1\left(1+2\|x\|_{1-\gamma}^2\right)\epsilon^{2q-1}\to 0\text{ as }\epsilon\to 0.
\end{aligned}$$

So, there are relatively compact sets arbitrarily close to $\{\left(\mathcal{A}_2 x\right)(\iota):x\in\mathcal{B}_p\}$. Hence, the set $\{\left(\mathcal{A}_2^\epsilon x\right)(\iota):x\in\mathcal{B}_p\}$ is precompact in $\mathcal{E}$.

As a result, the Arzela-Ascoli hypothesis is fulfilled, $\mathcal{A}_2$ is completly continuous. Hence, our hypothesis for Theorem 2.12 holds, which leads the problem (1.1) has at least one mild solution.

□

4 Continuous dependence of mild solutions

Now, we study the continuous dependence of the solution for problem (1.1).

Theorem 4.1. *Assume hypotheses (H_1)-(H_3) are fulfilled, then the solution of the problem (1.1) is continuously dependent on x_0.*

Proof. Let

$$\begin{aligned} x(\iota) =& \mathcal{S}_{p,q}(\iota)x_0 + \int_0^{\iota} (\iota-s)^{q-1}P_q(\iota-s)f\left(s, x(s), x(\kappa s), \int_0^s g(s,\tau,x(\tau),x(\kappa\tau))d\tau\right)ds \\ &+ \int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,x(s),x(\kappa s))dW(s), \quad \iota\in J, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} y(\iota) =& \mathcal{S}_{p,q}(\iota)y_0 + \int_0^{\iota} (\iota-s)^{q-1}P_q(\iota-s)f\left(s, y(s), y(\kappa s), \int_0^s g(s,\tau,y(\tau),y(\kappa\tau))d\tau\right)ds \\ &+ \int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\sigma(s,y(s),y(\kappa s))dW(s), \quad \iota\in J. \end{aligned} \tag{4.2}$$

By means of Lemma 2.8, for any $\iota \in J$, we get

$$\begin{aligned} &\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma)}\|y(\iota)-x(\iota)\|^2\right) \\ \le& 3\,\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma)}\|\mathcal{S}_{p,q}(\iota)y_0-\mathcal{S}_{p,q}(\iota)x_0\|^2\right) \\ &+3\,\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma)}\left\|\int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\right.\right. \\ &\quad\times\left[f\left(s,y(s),y(\kappa s),\int_0^s g(s,\tau,y(\tau),y(\kappa\tau))\,d\tau\right)\right. \\ &\quad\left.\left.\left.-f\left(s,x(s),x(\kappa s),\int_0^s g(s,\tau,x(\tau),x(\kappa\tau))\,d\tau\right)\right]ds\right\|^2\right) \\ &+3\,\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma)}\left\|\int_0^{\iota}(\iota-s)^{q-1}P_q(\iota-s)\right.\right. \\ &\quad\left.\left.\times\,[\sigma(s,y(s),y(\kappa s))-\sigma(s,x(s),x(\kappa s))]\,dW(s)\right\|^2\right) \\ =& I_1+I_2+I_3. \end{aligned}$$

By applying Lemma 2.10, we obtain

$$I_1 \le \frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|y_0-x_0\|^2 \tag{4.3}$$

By C-S inequality and hypothesis (H_1), we derive

$$\begin{aligned} I_2 \le& \frac{3M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}L\left[\int_0^{\iota}\mathbb{E}\left(\sup_{0<s_1\le s}s_1^{2(1-\gamma}\|y(s_1)-x(s_1)\|^2\right)ds\right. \\ &+\int_0^{\iota}\mathbb{E}\left(\sup_{0<s_1\le s}s_1^{2(1-\gamma}\|y(\kappa s_1)-x(\kappa s_1)\|^2\right)ds \\ &\left.+\int_0^{\iota}\mathbb{E}\left(\sup_{0<s_1\le s}s_1^{2(1-\gamma}\left\|\int_0^{s_1}[g(\tau,y(\tau),y(\kappa\tau))-g(\tau,x(\tau),x(\kappa\tau))]\,d\tau\right\|^2\right)ds\right]. \end{aligned} \tag{4.4}$$

Moreover, (H_2) lends us

$$\begin{aligned}&\int_0^\iota \mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\left\|\int_0^{s_1}\left[g(\tau,y(\tau),y(\kappa\tau))-g(\tau,x(\tau),x(\kappa\tau))\right]d\tau\right\|^2\right)ds\\&\le m\int_0^\iota s^2\mathbb{E}\left(\sup_{0<s_1\le\tau} s_1^{2(1-\gamma)}\|y(s_1)-x(s_1)\|^2\right)ds\\&+m\int_0^\iota s^2\mathbb{E}\left(\sup_{0<s_1\le\tau} s_1^{2(1-\gamma)}\|y(\kappa s_1)-x(\kappa s_1)\|^2\right)ds.\end{aligned}\tag{4.5}$$

Substituting (4.5) into (4.4), we obtain

$$\begin{aligned}I_2\le&\Lambda_1\int_0^\iota\left(1+ms^2\right)\\&\times\left\{\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(s_1)-x(s_1)\|^2\right)+\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(\kappa s_1)-x(\kappa s_1)\|^2\right)\right\}ds,\end{aligned}\tag{4.6}$$

where $\Lambda_1=\frac{3M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}L$.
The third term I_3 can be obtained using Doob's martingale inequality and (H_2),

$$\begin{aligned}I_3\le&\Lambda_2\int_0^\iota\left\{\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(s_1)-x(s_1)\|^2\right)\right.\\&\left.+\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(\kappa s_1)-x(\kappa s_1)\|^2\right)\right\}ds,\end{aligned}\tag{4.7}$$

where $\Lambda_2=\frac{12M^2b^{1-2\gamma+2q}}{\Gamma^2(q)2q-1}K$.
It follows from (4.3), (4.6) and (4.7)

$$\begin{aligned}&\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma)}\|y(\iota)-x(\iota)\|^2\right)\\\le&\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|y_0-x_0\|^2+\int_0^\iota\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\\&\times\left\{\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(s_1)-x(s_1)\|^2\right)+\mathbb{E}\left(\sup_{0<s_1\le s} s_1^{2(1-\gamma)}\|y(\kappa s_1)-x(\kappa s_1)\|^2\right)\right\}ds.\end{aligned}\tag{4.8}$$

Now, for every $z\in C\left(J,\mathbb{L}^2\left(\Omega,\mathcal{F}_\iota,\mathcal{E}\right)\right)$, we define $\mathcal{T}:C\left(J,\mathbb{L}^2\left(\Omega,\mathcal{F}_\iota,\mathcal{E}\right)\right)\to C\left(J,\mathbb{L}^2\left(\Omega,\mathcal{F}_\iota,\mathcal{E}\right)\right)$ as

$$(\mathcal{T}z)(\iota)=\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\|y_0-x_0\|^2+\int_0^\iota\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\left(z(s)+z(\kappa s)\right)ds.\tag{4.9}$$

We prove that $\mathcal{T}$ is a Picard operator. Let $z_1,z_2\in C\left(J,\mathbb{L}^2\left(\Omega,\mathcal{F}_\iota,\mathcal{E}\right)\right)$, for any $\iota\in J$, we have

$$\begin{aligned}&\mathbb{E}\|(\mathcal{T}z_1)(\iota)-(\mathcal{T}z_2)(\iota)\|^2\\\le&\mathbb{E}\left\|\int_0^\iota\left[\Omega_1\left(1+ms^2\right)+\Omega_2\right]\left\{z_1(s)-z_2(s)+z_1(\kappa s)-z_2(\kappa s)\right\}ds\right\|^2.\end{aligned}$$

By applying the C-S inequality, we get

$$\begin{aligned}
&\mathbb{E}\left\|(\mathcal{T}z_1)(\iota)-(\mathcal{T}z_2)(\iota)\right\|^2\\
\le& 2b\int_0^\iota\left[\Omega_1^2\left(1+ms^2\right)^2+\Omega_2^2\right]\mathbb{E}\left\|z_1(s)-z_2(s)+z_1(\kappa s)-z_2(\kappa s)\right\|^2 ds\\
\le& 4b\int_0^\iota\left[\Omega_1^2\left(1+ms^2\right)^2+\Omega_2^2\right]\left\{\mathbb{E}\left\|z_1(s)-z_2(s)\right\|^2+\mathbb{E}\left\|z_1(\kappa s)-z_2(\kappa s)\right\|^2\right\} ds\\
\le&\left[8b^2\Omega_1^2+\frac{16mb^4\Omega_1^2}{3}+\frac{8m^2b^6\Omega_1^2}{5}+8b^2\Omega_2^2\right]\left\|z_1-z_2\right\|^2.
\end{aligned}\tag{4.10}$$

We assume that $8b^2\Omega_1^2+\frac{16mb^4\Omega_1^2}{3}+\frac{8m^2b^6\Omega_1^2}{5}+8b^2\Omega_2^2<1$, the operator $\mathcal{T}$ is a contraction mapping. Consequently, by [42, Theorem 2.1], $\mathcal{T}$ is a Picard operator with $\mathcal{F}_{\mathcal{T}}=z^*$. Therefore, for all $\iota\in J$

$$z^*(\iota)=\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|y_0-x_0\right\|^2+\int_0^\iota\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\left(z^*(s)+z^*(\kappa s)\right)ds.\tag{4.11}$$

Further, we show that z^* is increasing.
Let $\iota_1,\iota_2\in J$ be such that $\iota_1<\iota_2$. Define $N=\min_{s\in[0,b]}\left(z^*(s)+z^*(\sigma(s))\right)\in\mathbb{R}^+$. Then, we have

$$\begin{aligned}
z^*(\iota_2)-z^*(\iota_1)=&\int_0^{\iota_2}\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\left(z^*(s)+z^*(\kappa s)\right)ds\\
&-\int_0^{\iota_1}\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\left(z^*(s)+z^*(\kappa s)\right)ds\\
=&\int_{\iota_1}^{\iota_2}\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]\left(z^*(s)+z^*(\kappa s)\right)ds.
\end{aligned}$$

Then

$$\begin{aligned}
z^*(\iota_2)-z^*(\iota_1)\ge& N\int_{\iota_1}^{\iota_2}\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]ds\\
\ge& N\left[\left(\Lambda_1+\Lambda_2\right)\left(\iota_2-\iota_1\right)+\Lambda_1 m\left(\frac{\iota_2^3}{3}-\frac{\iota_1^3}{3}\right)\right]\\
>&0.
\end{aligned}$$

Thus, the operator z^* is increasing. Given that $\kappa\iota\le\iota$, it follows that $z^*(\kappa\iota)\le z^*(\iota)$ for $\iota\in J$. Then, Eq. (4.11) simplifies to

$$z^*(\iota)\le\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|y_0-x_0\right\|^2+2\int_0^\iota\left[\Lambda_1\left(1+ms^2\right)+\Lambda_2\right]z^*(s)ds.$$

By applying Lemma 2.11, for $\iota\in J$, we arrive at

$$z^*(\iota)\le\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|y_0-x_0\right\|^2 e^{\left[2(\Lambda_1+\Lambda_2)b+\frac{2\Lambda_1 mb^3}{3}\right]}.$$

In particular, for $z(\iota)=\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma}\left\|y(\iota)-x(\iota)\right\|^2\right)$, we use Eq. (4.8) to derive $z\le\mathcal{T}(z)$, where $\mathcal{T}$ is an increasing Picard operator. Subsequently, applying Lemma 2.7 results in $z\le z^*$. Then, it follows that

$$\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma}\left\|y(\iota)-x(\iota)\right\|^2\right)\le\frac{3M^2}{\Gamma^2(q)}\mathbb{E}\left\|y_0-x_0\right\|^2 e^{\left[2(\Lambda_1+\Lambda_2)b+\frac{2\Lambda_1 mb^3}{3}\right]},$$

which yields that

$$\lim_{y_0\to x_0}\mathbb{E}\left(\sup_{0<\iota\le b}\iota^{2(1-\gamma}\left\|y(\iota)-x(\iota)\right\|^2\right)=0.$$

We conclude that the solution of the problem (1.1) is continuously dependent on x_0. □

5 An application

We consider the model of population dynamics with time delays and diffusion, known as the Hutchinson's equation with diffusion and delay [12, 13, 43]. The system is governed by a delay differential equation, which captures the impact of delayed reproduction, coupled with diffusion terms that represent the dispersal of individuals across space. This model is useful for analyzing species whose growth and reproduction exhibit delayed responses, and where the spatial spread of the population is an important factor, particularly in ecological systems where both of these dynamics interact.
Consider the following Diffusive Hutchinson's equation

$$\begin{cases} \frac{\partial}{\partial \iota} u(\iota,\xi) = \frac{\partial^2}{\partial \xi^2} u(\iota,\xi) + \beta u(\iota,\xi)\left(1 - u(\iota-\tau,\xi)\right), & \iota,\xi \in [0,1], \\ u(\iota,0) = u(\iota,1) = 0, \\ u(\iota,\xi) = u_0(\iota,\xi), \quad \iota \in [-\tau,0], \quad \tau > 0, \end{cases} \tag{5.1}$$

where β is positive constant.
A memory effect, stochasticity, and pantograph terms are added to the classical equation to formulate the SPIDEs as follows.

$$\begin{cases} {}^{H}\mathfrak{D}^{0.5,0.85}_{0^+,\iota} x(\iota,\xi) = \frac{\partial^2}{\partial \xi^2} x(\iota,\xi) + \beta_1 x(\iota,\xi)\left(1 - x(\kappa\iota,\xi)\right) \\ \qquad + \beta_2 \int_0^{\iota} x(s,\xi)\left(1 - x(\kappa s,\xi)\right) ds + \left[\alpha_1 x(\iota,\xi) + \alpha_2 x(\kappa\iota,\xi)\right] \frac{dW(\iota)}{d\iota}, \quad \iota,\xi \in [0,1], \\ x(\iota,0) = x(\iota,1) = 0, \\ \mathfrak{I}^{1-\gamma}_{0+,\iota} x(\iota,\xi)\big|_{\iota=0} = x_0(\xi). \end{cases} \tag{5.2}$$

Define

$$A\varsigma = \frac{\partial^2 \varsigma}{\partial \xi^2},$$

where,

$$D(A) = \left\{\varsigma \in \mathbb{L}^2([0,1],\mathbb{R}) : \varsigma, \frac{\partial \varsigma}{\partial \xi} \text{ are absolutely continuous}, \frac{\partial^2 \varsigma}{\partial \xi^2} \in \mathbb{L}^2([0,1],\mathbb{R}),\ \varsigma(0) = \varsigma(1) = 0\right\}.$$

It is easy to check that A generates a strongly continuous semigroup $\{\mathcal{S}(\iota)\}_{\iota\geq 0}$ which is compact, analytic, and self-adjoint.
Here,

$$f(\iota, x(\iota,\xi), x(\kappa\iota,\xi), x_3) = \beta_1 x(\iota,\xi)\left(1 - x(\kappa\iota,\xi)\right) + \beta_2 x_3,$$
$$x_3 = \int_0^{\iota} x(s,\xi)\left(1 - x(\kappa s,\xi)\right) ds,$$

and

$$\sigma(\iota, x(\iota,\xi), x(\kappa\iota,\xi)) = \alpha_1 x(\iota,\xi) + \alpha_2 x(\kappa\iota,\xi).$$

Let $p = 0.5$, $q = 0.85$, $\kappa = 0.5$, $\beta_1 = 0.285$, $\beta_2 = 0.111$, $\alpha_1 = 0.0354$, $\alpha_2 = 0.0224$.
So, the hypotheses of Theorem 3.1 are satisfied. Thus, there exists a mild solution for problem (5.2).
Moreover, we plot the solutions $x(\iota)$ and $y(\iota)$ with $x_0 = 1$ and y_0 perturbed by 0.01 to Eq. (5.2) in Figure 1. Figure 2 shows the solutions $x(\iota)$ and $y(\iota)$ with $x_0 = 1$ and y_0 perturbed by 0.0001. In Figure 3, for the same parameters, $x(\iota)$ and $y(\iota)$ are displayed with $x_0 = 0.5$ and y_0 perturbed by 0.01, while Figure 4 shows the solutions $x(\iota)$ and $y(\iota)$ with $x_0 = 0.5$ and y_0 perturbed by 0.0001.

Table 1 shows the error values between solutions $x(\iota)$ and $y(\iota)$ that use initial values $x_0 = 1$ and y_0 with perturbation 0.0001 using the parameter setup of Figure 2. Table data reveals convergence between x and y solutions during initial value conditions where $x_0 = 1$ and $y_0 = 1.0001$. This demonstrates the continuous dependence on initial values.

Figure 5 presents the numerical solutions $x(\iota)$ and $y(\iota)$ for different values of the fractional order q, while Table 2 shows the corresponding error. The maximum error value decreases when the parameter q increases from $1.15931e-02$ at $q = 0.70$ to $1.11395e-02$ at $q = 0.99$.

As q approaches 1, the solutions accelerate, which is due to the Hilfer derivative approaching the classical Caputo derivatives. The smaller q-values produces solutions that move at a slower pace because the Hilfer derivative maintains stronger memory effect. This highlights the robustness and accuracy of the Hilfer derivative, especially in systems that rely on past states whereas Caputo derivatives hold simpler memory calculation and allow the solution go fast.

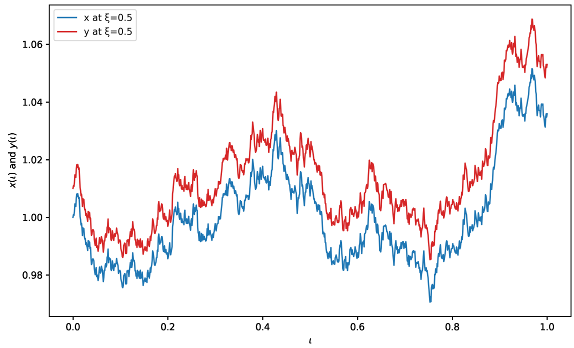

Figure 1. Plot of $x(\iota)$ and $y(\iota)$ with $x_0 = 1$ and y_0 perturbed by 0.01.

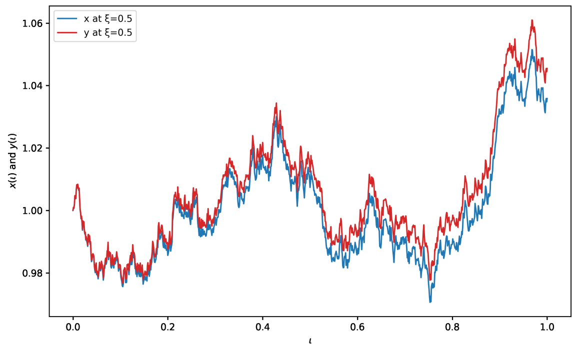

Figure 2. Plot of $x(\iota)$ and $y(\iota)$ with $x_0 = 1$ and y_0 perturbed by 0.0001.

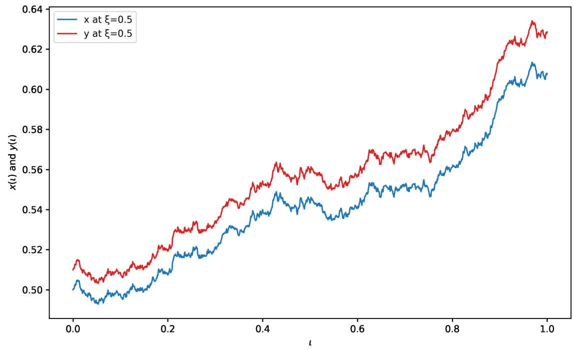

Figure 3. Plot of $x(\iota)$ and $y(\iota)$ with $x_0 = 0.5$ and y_0 perturbed by 0.01.

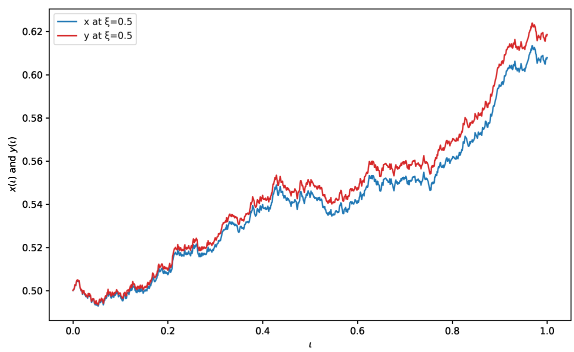

Figure 4. Plot of $x(\iota)$ and $y(\iota)$ with $x_0 = 0.5$ and y_0 perturbed by 0.0001.

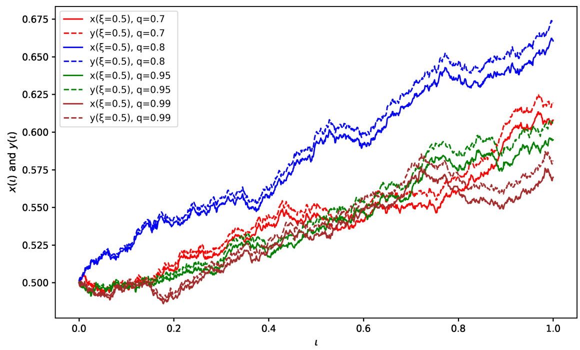

Figure 5. Plot of $x(\iota)$ and $y(\iota)$ for different values of orders.

6 Conclusion

In this paper, we studied a class of Hilfer fractional stochastic pantograph integro-differential equations. By using the fixed-point theory, we prove the existence of mild solutions through established suitable conditions. Furthermore, the analysis proved the continuous dependence of initial conditions through Picard operator theory. Hilfer derivatives together with stochastic perturbations and pantograph-type delays significantly enhance the model because they improve its ability to handle memory effects and random fluctuations observed in real-world systems. To support the theoretical findings, a numerical simulation was presented.

Time	x	y	Error
0.1	0.981201	0.982283	8.301139e-07
0.2	0.985635	0.987701	3.351554e-06
0.3	0.997798	1.000847	7.764187e-06
0.4	1.017709	1.021763	1.432695e-05
0.5	1.007402	1.012350	2.206578e-05
0.6	0.986478	0.992258	3.095056e-05
0.7	0.990577	0.997280	4.259363e-05
0.8	0.989172	0.996763	5.573204e-05
0.9	1.030941	1.039688	7.531397e-05
1.0	1.035843	1.045480	9.286877e-05

Table 1. Error between x and y for $x_0 = 1$ and y_0 perturbed by 0.0001.

q	Max error	Mean error	Std
0.70	1.15931e-02	6.13241e-03	3.02328e-03
0.80	1.17273e-02	6.30348e-03	3.10020e-03
0.95	1.14213e-02	6.15807e-03	3.04609e-03
0.99	1.11395e-02	6.09384e-03	2.96207e-03

Table 2. Error for different values of order.

Appendix A. Numerical Integration

To solve our problem, we give the following algorithm using quadrature method approximation:

Algorithm 1 Numerical Approximation

Step 1: Time interval discretization
Divide the time domain into n subintervals with uniform step size $h = 1/n$.
Divide the spatial domain into m subintervals with step size $k = 1/m$.
Let: $\iota_i = i \cdot h$, $\xi_j = j \cdot k$, for $i = 0, \ldots, n$, $j = 0, \ldots, m$.
Step 2: Initialize the solution arrays
Let: $x_0^j = x_0(\xi_j)$, $y_0^j = x_0(\xi_j) + \varepsilon$, for $j = 0, \ldots, m$, and $x_i^0 = x_i^m = y_i^0 = y_i^m = 0$, for $i = 0, \ldots, n$.
Step 3: Approximate noise terms
Compute $\Delta W_i = \sqrt{h} \cdot X_i$, where $X_i \sim \mathcal{N}(0, 1)$, for $i = 0, \ldots, n-1$.
Step 4: Main part scheme
for $i = 0$ to $n-1$ **do**
  for $j = 1$ to $m-1$ **do**
    (i)

$$\frac{\partial^2 x}{\partial \xi^2}(\iota_i, \xi_j) \approx \frac{x_i^{j+1} - 2x_i^j + x_i^{j-1}}{k^2}$$

(ii) Use the quadrature formula based on the Hilfer operator definition:

$${}^H\mathfrak{D}_{0^+,\iota}^{p,q} x[i, j] \approx \sum_{k=1}^{i} \omega_{i,k}^{(p,q)} (x_k^j - x_{k-1}^j), \text{ where}$$

$$\omega_{i,k}^{(p,q)} = \frac{h^{-p}}{\Gamma(2-p)} \left((i-k+1)^{1-p} - (i-k)^{1-p} \right)$$

(iii) Evaluate $x_3 \approx \sum_{k=0}^{i} x_k^j (1 - x_{0.5k}^j) \cdot h$ using numerical integration.
(iv) Evaluate x and y:

$$x_{i+1}^j = x_i^j + h \cdot \left(\frac{\partial^2 x}{\partial \xi^2} + {}^H\mathfrak{D}_{0^+,\iota}^{p,q} x[i, j] + \beta_1 x_i^j (1 - x_{0.5i}^j) + \beta_2 x_3 \right) + \left(\alpha_1 x_i^j + \alpha_2 x_{0.5i}^j \right) \cdot \Delta W_i$$

$$y_{i+1}^j = y_i^j + h \cdot \left(\frac{\partial^2 y}{\partial \xi^2} + {}^H\mathfrak{D}_{0^+,\iota}^{p,q} x[i, j] + \beta_1 y_i^j (1 - y_{0.5i}^j) + \beta_2 y_3 \right) + \left(\alpha_1 y_i^j + \alpha_2 y_{0.5i}^j \right) \cdot \Delta W_i + \varepsilon$$

  end for
end for
Step 5: Plot $x(\iota)$ and $y(\iota)$ at fixed $\xi = 0.5$

```
ax.plot(ι, x, label='x at ξ=0.5')
ax.plot(ι, y, label='y at ξ=0.5')
```

Label axes and add legend.

Declaration of competing interest

There are no conflicts of interest, according to the authors.

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Authors' contributions

All authors contributed equally to this work. The final manuscript has been read and approved by all authors.

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